SU(6) classification for "in" and "out" states*

Norman H. Fuchs
Department of Physics, Purdue University, West Lafayette, Indiana 47907
(Received 7 April 1975)

Within the context of the quark model with interaction, SU(6) algebras classifying "in" and "out" states are constructed and the unitary transformation relating these algebras to the $SU(6)_{W,\text{currents}}$ algebra generated by integrals of local currents is given simply. The formulation is general so that one may study kinematic effects, which arise even in the absence of interaction from the angular requirements on the generators of the classification algebras, separate from dynamical effects, which are peculiar to the nature of the interaction and are also induced by these angular constraints.

I. INTRODUCTION

We have previously investigated[1] the possible relation, in the context of interacting-quark models, between the algebra of $SU(6)_{W,\text{currents}}$, whose generators F_i^α are integrals of the local currents which describe the electromagnetic and weak interactions of hadrons, and the algebra of $SU(6)_{W,\text{strong}}$ whose generators W_i^α are supposed to classify hadrons into approximately degenerate multiplets. The generators F_i^α are bilinear forms in field operators $\varphi_+(x;\text{current})$ while the W_i^α are bilinear forms in field operators $\varphi_+(x;\text{constituent})$. These fields separately satisfy canonical anticommutation relations on the null plane and are unitarily equivalent. We constructed a unitary transformation relating them in such a way that the W_i^α classify states in a momentum-independent manner when the states for arbitrary momentum are obtained from rest states by means of K boosts, while the F_i^α classify states in a momentum-independent manner when the states for arbitrary momentum are obtained from rest states by means of E boosts.[2] We then found that the W_i^α have simple angular momentum properties, so that their action on hadronic states is just that expected in the naive quark model. The results of Ref. 1 were not restricted, in general, to a specific choice of interaction. In that regard we expressed the hope that further work along those lines might answer some of the questions we have dodged, such as why the free-quark structure abstracted from the model works so well for some matrix elements and not for others.

Unfortunately, our results in the interacting-quark model were not expressed in a form convenient for further phenomenological investigation of the effects of interaction. In the present work we will obtain an expression for the transformation connecting current- and constituent-quark bases which displays the dependence on interaction in a rather transparent manner. Thus we may understand the relation of the quark-parton picture to the traditional quark model for hadronic structure.

In the next section we review some of the formalism of Ref. 1 which is relevant to the present work, while in Sec. III we develop it in a different way so as to exhibit the simple form that the transformation from one set of fields, $\varphi_+(x;\text{current})$, to the other, $\varphi_+(x;\text{constituent})$, in fact has. In Sec. IV we derive expressions for W_i^α, generators of an $SU(6)_W$ algebra, and for analogous generators of $SU(6)_W$ algebras which may be used to classify "in" and "out" physical states. Finally, in the last section we discuss in detail the current-constituent quark transformation in terms of its action on physical states.

II. REVIEW OF SOME FORMALISM

The free-field Fourier expansion for the quark field operator $\psi(x)$ and other fields in the theory must be modified in the presence of their interactions, since the fields no longer have the space-time coordinate dependence given by solutions of the free-field equations. In fact, we do not know in general what this coordinate dependence will be. Nevertheless, some definite statements can still be made without our specifying the interaction. Making a three-dimensional Fourier expansion of $\psi(x)$, we have (for fixed $x^+\equiv\tau$)

$$\psi(\tau,x^-,\vec{x}_\perp)=[2(2\pi)^3]^{-1/2}\int d^2p_\perp\int_0^\infty \frac{d\eta}{\eta}\sum_\lambda [b(p,\lambda;\tau)u(p,\lambda)\exp(-ip\cdot x)+d^\dagger(p,\lambda;\tau)u(-p,-\lambda)\exp(ip\cdot x)]\ . \tag{2.1}$$

This expansion holds for free fields; for interacting fields it only holds for the "good" components (defined below). Note that the spinors $u(p, \lambda)$ satisfy free-field equations of motion for a single spinor field of mass m, the renormalized quark mass. The scalar product $p \cdot x$ is given in the usual light-plane variables as

$$p \cdot x = p^- \tau + \eta x^- - \vec{p}_\perp \cdot \vec{x}_\perp , \tag{2.2}$$

where we have used the notation $p^+ \equiv \eta$ and $x^+ \equiv \tau$. The momentum p then satisfies the relation appropriate for a free quark of mass m:

$$p^- = (p_\perp{}^2 + m^2)/2\eta . \tag{2.3}$$

The development of $\psi(x)$ in τ is determined by the generator of τ displacements, P^-, which takes the place of the Hamiltonian in the conventional formalism. Explicitly, we have

$$i[P^-, \psi(x)] = \partial\psi(x)/\partial\tau . \tag{2.4}$$

With no interaction, it is straightforward to show that the above expansion holds for arbitrary τ, and that moreover

$$\begin{aligned} b_{\text{free}}(p, \lambda; \tau) &= b_{\text{free}}(p, \lambda; 0) , \\ d^\dagger_{\text{free}}(p, \lambda; \tau) &= d^\dagger_{\text{free}}(p, \lambda; 0) . \end{aligned} \tag{2.5}$$

In the presence of interaction these simple results no longer obtain; we rather have

$$\begin{aligned} b(p, \lambda; \tau) &= e^{iP^-\tau} b(p, \lambda; 0) e^{-iP^-\tau} , \\ d^\dagger(p, \lambda; \tau) &= e^{iP^-\tau} d^\dagger(p, \lambda; 0) e^{-iP^-\tau} , \end{aligned} \tag{2.6}$$

so that these operators are in general no longer equal to their values at $\tau = 0$. They cannot in general be simply interpreted as creation and destruction operators for single quanta of definite masses; nevertheless, they can be considered as creation and destruction operators in the present null-plane formulation. The states $|\eta, \vec{p}_\perp; \lambda\rangle$ of a particle with arbitrary momentum are defined by applying a particular Lorentz transformation to the rest states. We require two different choices for this Lorentz boost.

In the conventional formalism, one defines

$$|\eta, p_\perp; \lambda; \text{constituent}\rangle = e^{-\vec{\beta} \cdot \vec{K}} |m/\sqrt{2}\, , 0; \lambda\rangle , \tag{2.7}$$

where

$$\beta = \hat{p} \operatorname{arcsinh}(|\vec{p}|/m) , \tag{2.8}$$

and we have added the notation "constituent" to the state vector to emphasize the dependence of the construction on the choice of the boosting operation.

On the other hand, we might choose to define our states so that they transform simply under the Poincaré generators J_3, $\vec{E}_\perp$, and K_3 which leave the plane $\tau = 0$ invariant. These states have arisen naturally recently in discussions of field theories in the infinite-momentum frame. They are defined by

$$|\eta, \vec{p}_\perp; \lambda; \text{current}\rangle = e^{-i\vec{v}_\perp \cdot \vec{E}_\perp} e^{i\omega K_3} |m/\sqrt{2}\, , \vec{0}; \lambda\rangle , \tag{2.9}$$

where

$$v_\perp = p_\perp/\eta \text{ and } e^\omega = \sqrt{2}\, \eta/m , \tag{2.10}$$

and we have added the notation "current" in analogy with the "constituent" notation above.

For the four-component quark spinors, the matrix $G = \frac{1}{2}(1 + \gamma_0\gamma_3)$ projects onto a two-dimensional invariant subspace spanned by spinors of the form

$$\psi_+(x) = G\psi(x) . \tag{2.11}$$

The projected spinor has two linearly independent components for each kind of quark. The spinor representation does not decompose, however; i.e., the subspace orthogonal to that spanned by ψ_+ is not invariant. In other words, if we define

$$\psi(x) = \psi_+(x) + \psi_-(x) , \tag{2.12}$$

then when $\psi_-(x)$ is Lorentz-transformed by operators leaving the $\tau = 0$ null-plane invariant, it cannot be expressed solely in terms of ψ_- but must involve ψ_+ as well.

Recall now the Fourier expansion of $\psi(x)$. Clearly, since the projection operator G is linear we may write an expansion for $\psi_+(x)$ where the only difference is that we must use projected spinors

$$u_+(p, \lambda) \equiv Gu(p, \lambda) . \tag{2.13}$$

Now we shall choose the normalization of the spinors so that

$$\sum_\lambda u^\dagger_{+\alpha}(p, \lambda) u_{+\beta}(p, \lambda) = \sqrt{2}\, \eta G_{\alpha\beta} . \tag{2.14}$$

III. CONSTRUCTION OF THE TRANSFORMATION

For the quark field, transforming as spin $\frac{1}{2}$, the action of an arbitrary Lorentz transformation is usually written

$$U[\Lambda]\psi_\alpha(x)U^{-1}[\Lambda] = \sum_\beta S^{-1}{}_{\alpha\beta}[\Lambda]\psi_\beta(\Lambda x) , \tag{3.1}$$

where S is a 4×4 matrix which operates on the four-component column vector $\psi(x)$. For spatial rotations, S is unitary, but this does not hold true for Lorentz boosts; in general,

$$S^{-1} = \gamma_0 S^\dagger \gamma_0 . \tag{3.2}$$

We have shown how the current-quark and constituent-quark bases may be related through a unitary transformation constructed from some knowledge of the matrix operator $S[\Lambda]$, and we have shown that for the free-quark model this transformation reduces precisely to one given by Melosh.[3]

The key is the realization that in the Fourier expansion of ψ the spinors $u(p,\lambda)$ satisfy free-field equations of motion. It follows that the free-field transformation matrix $S^{\text{free}}[\Lambda]$ may be used to express the spinors $u(p,\lambda)$ for arbitrary momentum in terms of spinors describing rest states ($\vec{p}=0$)

$$u_\beta(p,\lambda)=\sum_\alpha S^{\text{free}}_{\beta\alpha}u_\alpha(\text{rest},\lambda)\,. \qquad (3.3)$$

Actually, since we have defined two sets of spinors we must be more explicit; that is, we must say how the boost is to be effected. In other words, we write

$$u_\beta(p,\lambda;\text{constituent})=\sum_\alpha S^{\text{free}}_{\beta\alpha}[\Lambda_{\text{constituent}}]u_\alpha(\text{rest},\lambda)\,, \qquad (3.4)$$

$$u_\beta(p,\lambda;\text{current})=\sum_\alpha S^{\text{free}}_{\beta\alpha}[\Lambda_{\text{current}}]u_\alpha(\text{rest},\lambda)\,, \qquad (3.5)$$

where $\Lambda_{\text{constituent}}$ and Λ_{current} are the Lorentz boosts appropriate to the constituent and current bases described above. Now define new functions $\varphi(x;\text{constituent})$ and $\varphi(x;\text{current})$:

$$\varphi_\beta(x;\text{constituent})=(2\pi)^{-3/2}\sum_\lambda\int\frac{d^2p_\perp d\eta}{\eta}\,[\eta^{1/2}b(p,\lambda;\text{constituent})w_\beta(\lambda)e^{-ip\cdot x}+\eta^{1/2}d^\dagger(p,\lambda;\text{constituent})w_\beta(-\lambda)e^{-p\cdot x}]\,, \qquad (3.6)$$

where

$$w_\beta(\lambda)=2^{-1/4}m^{-1/2}u_\beta(\text{rest},\lambda)\,,$$

and analogous expressions for the current-quark basis. The procedure described earlier may be used to show that these functions are related to $\psi_+(x)$ by a unitary transformation; for the current-quark basis the transformation is trivial,

$$\psi_+(x)=\varphi_+(x;\text{current})\,, \qquad (3.7)$$

while for the constituent-quark basis

$$\varphi_+(x;\text{constituent})=V_{\text{free}}\psi_+(x)=\exp\!\left(i\arctan\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{m+|p_0+p_3|}\right)\psi_+(x)\,. \qquad (3.8)$$

IV. THE GENERATORS OF $SU(6)_{W,\text{strong}}$

The generators F_i^α of $SU(6)_{W,\text{currents}}$ can be defined in terms of bilinear products of ψ_+, in the free-quark model, and we will assume the same form for them in general. If further (good) fields are found to be necessarily included in F_i^α, then the discussion can be suitably extended. Therefore, we will define

$$F_i^\alpha=\frac{1}{\sqrt{2}}\int d^4x\,\delta(x^+)\psi_+^\dagger(x)\Gamma^\alpha\tfrac{1}{2}\lambda_i\psi_+(x), \qquad (4.1)$$

where λ_i is an SU(3) matrix and $\Gamma^\alpha=(2,\beta\sigma^1,\beta\sigma^2,\beta\sigma^3)$. The SU(6) generators are defined analogously, for free fields,

$$W_i^\alpha=\frac{1}{\sqrt{2}}\int d^4x\,\delta(x^+)\varphi_+^\dagger(x;\text{constituent})\Gamma^\alpha\tfrac{1}{2}\lambda_i\varphi_+(x;\text{constituent})\,. \qquad (4.2)$$

Next we make use of the Fourier expansion of $\varphi_+(x)$ as defined in the preceding section, Eq. (3.6), to obtain

$$W_i^\alpha(\tau)=\sum_{\lambda\lambda'}\int\frac{d^2p\,d\eta}{2\eta}\,[b^\dagger(p,\lambda;\tau;\text{constituent})\tfrac{1}{2}\lambda_i b(p,\lambda';\tau;\text{constituent})w_+^\dagger(\lambda)\Gamma^\alpha w_+(\lambda')+d^\dagger(p,\lambda;\tau;\text{constituent})\tfrac{1}{2}\lambda_i d(p,\lambda';\tau;\text{constituent})w_+^\dagger(-\lambda)\Gamma^\alpha w_+(-\lambda')]\,. \qquad (4.3)$$

The operators have a τ dependence which is fixed by the nature of the interaction. For the free case, they are τ dependent; so then the $W_i^\alpha(\tau)$ are τ independent, i.e., they generate a symmetry of the Hamiltonian. (Actually, this is true only if the quark masses are degenerate; if not, then $W_i^\alpha(\tau)$ will have a dependence on τ given by $\exp[i(p_f^- - p_i^-)\cdot\tau]$, where

$$p_{f,i}^-=\frac{p_\perp^2+m_{i,f}^2}{\eta} \qquad (4.4)$$

and the i (f) subscript refers to the quark annihilated (created) by W_i^α.

Note that up to now we have made no reference to the transformation properties of states under either the algebra $SU(6)_{W,\,\text{currents}}$ generated by the $F_i^\alpha(\tau)$ or $SU(6)_{W,\,\text{strong}}$ generated by the $W_i^\alpha(\tau)$. Physical "in" and "out" states are created[4] by operators such as $b^\dagger_{\text{in}}(p,\lambda;\text{constituent})$ etc.

Heisenberg "in" operators are defined by

$$Q_{\text{in}}(x)=\Omega^{(+)}(\tau)Q_H(x)\Omega^{(+)}(\tau)^{-1}\,, \quad x^+=\tau \tag{4.5}$$

for an arbitrary Heisenberg operator $Q_H(x)$, where

$$\Omega^{(+)}(\tau)=e^{iP^-\tau}\Omega^{(+)}e^{-iP^-\tau} \tag{4.6}$$

and $\Omega^{(+)}$ is the Møller wave matrix, having the property that it transforms a positive-energy eigenstate of $P_0^-=P_0^-(\tau=0)$ into an ingoing eigenstate of P^- of the same energy. We have split up the "Hamiltonian" P^- into free and interaction pieces

$$P^-=P_0^-+P_I^-. \tag{4.7}$$

Since P^- governs the τ evolution of $Q_H(x)$,

$$Q_{\text{in}}(x)=e^{iP^-\tau}Q_{\text{in}}(0)e^{-iP^-\tau}\,, \tag{4.8}$$

as is appropriate for a Heisenberg operator. For the special case of $Q_H(x)=P^-$ we find

$$P^-(\tau)=P^-(0)=P^-=P^-_{0,\,\text{in}}(\tau)=P^-_{0,\,\text{in}}(0)\,, \tag{4.9}$$

while for the quark field $\psi(x)$

$$\begin{aligned}\psi_{\text{in}}(x)&=\Omega^{(+)}(\tau)\psi(x)\Omega^{(+)}(\tau)^{-1}\\&=[2(2\pi)^3]^{-1/2}\int d^2p_\perp\int_0^\infty\frac{d\eta}{\eta}\sum_\lambda\,[b_{\text{in}}(p,\lambda;\tau)u(p,\lambda)\exp(-ip\cdot x)+d^\dagger_{\text{in}}(p,\lambda;\tau)u(-p,-\lambda)\exp(ip\cdot x)]\,.\end{aligned} \tag{4.10}$$

Now the τ evolution of $\psi_{\text{in}}(x)$ is governed by $P^-_{0,\,\text{in}}$, that is, by the free "Hamiltonian" expressed in terms of "in" fields. Therefore, $b_{\text{in}}, d_{\text{in}}$ are τ independent.

The Lorentz transformation properties of $\psi_{\text{in}}(x)$ are those of a free field,

$$U_{\text{in}}[\Lambda]\psi_{\text{in},\alpha}(x)U_{\text{in}}[\Lambda]^{-1}=S^{\text{free}}[\Lambda]^{-1}{}_{\alpha\beta}\psi_{\text{in},\beta}(\Lambda x)\,, \tag{4.11}$$

even though the matrix $S[\Lambda]$ associated with $\psi(x)$ is not $S^{\text{free}}[\Lambda]$. To underline this point, we consider

$$\begin{aligned}U[\Lambda]\psi(x)U[\Lambda]^{-1}&=\Omega^{(+)}(\tau)^{-1}(\tau)U_{\text{in}}[\Lambda]\psi_{\text{in}}(x)U_{\text{in}}[\Lambda]^{-1}\Omega^{(+)}(\tau)\\&=S^{\text{free}}[\Lambda]\Omega^{(+)-1}(\tau)\psi_{\text{in}}(\Lambda x)\Omega^{(+)}(\tau).\end{aligned} \tag{4.12}$$

But

$$\Omega^{(+)\,-1}(\tau)\psi_{\text{in}}(\Lambda x)\Omega^{(+)}(\tau)\neq\psi(\Lambda x) \tag{4.13}$$

unless $(\Lambda x)^+=\tau$, which is not true unless Λ leaves the light-plane τ constant invariant. Therefore $S[\Lambda]\neq S^{\text{free}}[\Lambda]$ unless Λ leaves the light plane invariant. The operators $b_{\text{in}}, d_{\text{in}}$ transform simply under $U_{\text{in}}[\Lambda]$; for example,

$$\begin{aligned}\exp(-i\vec\beta\cdot\vec K_{\text{in}})b_{\text{in}}(p,\lambda;\text{constituent})\exp(i\vec\beta\cdot\vec K_{\text{in}})\\=b_{\text{in}}(\Lambda p,\lambda;\text{constituent})\,,\end{aligned} \tag{4.14}$$

where Λp is obtained from p by K-boosting an amount $\vec\beta$.

Since we are interested in constructing an SU(6) algebra of operators which have simple Lorentz transformation properties and which may be used to classify physical states in a simple fashion, (see Appendix A for a discussion), we are led to a consideration of the operators

$$W^\alpha_{i,\,\text{in}}=\Omega^{(+)}(\tau)W_i^\alpha(\tau)\Omega^{(+)}(\tau)^{-1}\,, \tag{4.15}$$

which are appropriate for classifying Heisenberg "in" states, and the operators

$$W^\alpha_{i,\,\text{out}}=\Omega^{(-)}(\tau)W_i^\alpha(\tau)\Omega^{(-)}(\tau)^{-1}\,, \tag{4.16}$$

which are appropriate for classifying Heisenberg "out" states. We have used the notation

$$\Omega^{(\pm)}(\tau)=e^{iP^-\tau}\Omega^{(\pm)}e^{-iP^-\tau}. \tag{4.17}$$

The $\Omega^{(\mp)}$ operators have the well-known expansions

$$\Omega^{(+)}=\mathcal{T}\exp\left[-i\int_{-\infty}^0 d\tau\,P^-_{I,D}(\tau)\right]\,, \tag{4.18}$$

$$\Omega^{(-)}=\mathcal{T}\exp\left[+i\int_0^{\infty} d\tau\,P^-_{I,D}(\tau)\right]\,, \tag{4.19}$$

where $P^-_{I,D}$ is the interaction part of P^- in the Dirac picture and $\mathcal{T}$ denotes τ ordering of the exponential (analogous to time ordering in the usual spacelike formalism). The $W^\alpha_{i,\,\text{in}}$ nad $W^\alpha_{i,\,\text{out}}$ generate SU(6) algebras which are connected by the scattering matrix S:

$$W^\alpha_{i,\,\text{out}}=S^{-1}W^\alpha_{i,\,\text{in}}S\,, \tag{4.20}$$

$$S=\Omega^{(+)}\Omega^{(-)*}. \tag{4.21}$$

These algebras are distinct unless there is no scattering. The Fock-space expansion

$$W^\alpha_{i,\,\mathrm{in}}=\sum_{\lambda\lambda'}\int\frac{d^2p\,d\eta}{2\eta}\,[b^\dagger_{\mathrm{in}}(p,\lambda;\mathrm{constituent})\tfrac{1}{2}\lambda^i b_{\mathrm{in}}(p,\lambda';\mathrm{constituent})w^\dagger_+(\lambda)P^\alpha w_+(\lambda')$$
$$+d^\dagger_{\mathrm{in}}(p,\lambda;\mathrm{constituent})\tfrac{1}{2}\lambda^i d_{\mathrm{in}}(p,\lambda';\mathrm{constituent})w^\dagger_+(-\lambda)P^\alpha w_+(-\lambda')] \tag{4.22}$$

shows explicitly that the $W^\alpha_{i,\,\mathrm{in}}$ have the correct Lorentz-group transformation properties that we desire for an $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ classification group for "in" states; similarly, the Fock-space expansion for $W^\alpha_{i,\,\mathrm{out}}$ does likewise for "out" states.

V. THE TRANSFORMATION AND WHAT IT MEANS

The eigenstates of $P^-_{0,\,\mathrm{in}}$ are constructed in the same way as in the free-particle case, so the n-incoming quark state

$$\frac{1}{\sqrt{n!}}b^\dagger_{\mathrm{in}}(p_1,\lambda_1;\mathrm{constituent})\times\cdots\times b^\dagger_{\mathrm{in}}(p_n,\lambda_n;\mathrm{constituent})|0\rangle \tag{5.1}$$

is an eigenstate of $P^-_{0,\,\mathrm{in}}$ with eigenvalue $\sum_{i=1}^n P^-_i$ and an eigenfunction of $\vec{P}_\perp$ (P^+) with eigenvalue $\sum_{i=1}^n \vec{P}_{i\perp}$ ($\sum_{i=1}^n \eta_i$). However, it is important to realize that since $P^-_{0,\,\mathrm{in}}=P^-$, this n-incoming quark state is also an eigenstate of the total "Hamiltonian." In particular, it is a satisfactory state describing the collision and subsequent development of outgoing waves of collision products. The incoming state consists of dressed (physical) particles.

Since "in" and "out" states are related via the Møller wave operators to Dirac-picture eigenstates of P^-_0

$$|a\rangle_{\mathrm{in}}=\Omega^{(+)}|a_D\rangle\,, \tag{5.2}$$

$$|b\rangle_{\mathrm{out}}=\Omega^{(-)}|b_D\rangle\,, \tag{5.3}$$

a matrix element of an arbitrary Heisenberg operator $Q_H(x)$ may be expressed as follows:

$${}_{\mathrm{out}}\langle b|Q_H(x)|a\rangle_{\mathrm{in}}=\langle b_D|\mathcal{T}(Q_D(x)S)|a_D\rangle\,, \tag{5.4}$$

which is the familiar starting point for the development of perturbation theory.

Since $F^\alpha_i(\tau)$ do not commute with P^-_0, then $F^\alpha_{i,\,\mathrm{in}}(\tau)$ do not commute with $P^-_{0,\,\mathrm{in}}$. This implies that "in" states that are built out of a fixed number of constituent quarks, via $b^\dagger_{\mathrm{in}}(p,\lambda;\mathrm{constituent})$, will not have a fixed number of current quarks in general.

Consider a state $|a\rangle_{\mathrm{in}}$ that transforms in a definite way under the $W^\alpha_{i,\,\mathrm{in}}$. Then

$$\begin{aligned}W^\alpha_{i,\,\mathrm{in}}|a\rangle_{\mathrm{in}}&=\Omega^{(+)}W^\alpha_i(0)\Omega^{(+)-1}|a\rangle_{\mathrm{in}}\\&=V_{\mathrm{free}}(0)\,\Omega^{(+)}F^\alpha_i(0)\Omega^{(+)-1}V_{\mathrm{free}}^{-1}(0)|a\rangle_{\mathrm{in}}\\&=V_{\mathrm{in}}F^\alpha_i(0)\,|a,\mathrm{currents}\rangle\,,\end{aligned} \tag{5.5}$$

where we have defined[5]

$$|a,\mathrm{currents}\rangle\equiv V_{\mathrm{in}}^{-1}|a\rangle_{\mathrm{in}}\,, \tag{5.6}$$

$$V_{\mathrm{in}}\equiv V_{\mathrm{free}}(0)\,\Omega^{(+)}\,. \tag{5.7}$$

Therefore, $|a,\mathrm{currents}\rangle$ is a Dirac-picture state that transforms under F^α_i in the same way as $|a\rangle_{\mathrm{in}}$ does under $W^\alpha_{i,\,\mathrm{in}}$, and

$$W^\alpha_{i,\,\mathrm{in}}=V_{\mathrm{in}}F^\alpha_i V_{\mathrm{in}}^{-1}\,. \tag{5.8}$$

The algebraic structure of matrix elements of an arbitrary Heisenberg operator Q with respect to $\mathrm{SU}(6)_{W,\,\mathrm{strong},\,\mathrm{in}}$ is given by the corresponding properties of $V_{\mathrm{in}}^{-1}QV_{\mathrm{in}}$ with respect to $\mathrm{SU}(6)_{W,\,\mathrm{currents}}$, since

$${}_{\mathrm{in}}\langle b|Q|a\rangle_{\mathrm{in}}=\langle b,\mathrm{currents}|V_{\mathrm{in}}^{-1}QV_{\mathrm{in}}|a,\mathrm{currents}\rangle\,, \tag{5.9}$$

just as discussed first by Melosh.

Similarly,

$$|a,\mathrm{currents}\rangle=V_{\mathrm{out}}^{-1}|a\rangle_{\mathrm{out}}\,, \tag{5.10}$$

$$V_{\mathrm{out}}\equiv\Omega^{(-)}V_{\mathrm{free}}(0)\,, \tag{5.11}$$

so that $|a,\mathrm{currents}\rangle$ transforms in the same way under F^α_i as $|a\rangle_{\mathrm{out}}$ does under $W^\alpha_{i,\,\mathrm{out}}$, and

$$W^\alpha_{i,\,\mathrm{out}}=V_{\mathrm{out}}F^\alpha_i V_{\mathrm{in}}^{-1}\,. \tag{5.12}$$

Therefore, the matrix element ${}_{\mathrm{out}}\langle b|Q|a\rangle_{\mathrm{in}}$, which commonly occurs in any treatment of transition amplitudes, satisfies

$${}_{\mathrm{out}}\langle b|Q|a\rangle_{\mathrm{in}}=\langle b,\mathrm{currents}|V_{\mathrm{out}}^{-1}QV_{\mathrm{in}}|a,\mathrm{currents}\rangle\,. \tag{5.13}$$

To convert this expression into a more convenient form for purposes of phenomenology, we write

$$\begin{aligned}V_{\mathrm{out}}^{-1}QV_{\mathrm{in}}&=V_{\mathrm{free}}^{-1}\Omega^{(-)*}Q\Omega^{(+)}V_{\mathrm{free}}\\&=(V_{\mathrm{free}}^{-1}\Omega^{(-)*}F_{\mathrm{free}})(V_{\mathrm{free}}^{-1}QV_{\mathrm{free}})\\&\quad\times(V_{\mathrm{free}}^{-1}\Omega^{(+)}V_{\mathrm{free}})\\&\equiv\tilde{\Omega}^{(-)*}\tilde{Q}\tilde{\Omega}^{(+)}\,,\end{aligned} \tag{5.14}$$

so

$$\begin{aligned}{}_{\mathrm{out}}\langle b|Q|a\rangle_{\mathrm{in}}&=\langle b,\mathrm{currents}|\tilde{\Omega}^{(-)*}\cdot\tilde{Q}\cdot\tilde{\Omega}^{(+)}|a,\mathrm{currents}\rangle\\&=\langle b,\mathrm{currents}|\mathcal{T}(\tilde{S}\tilde{Q})|a,\mathrm{currents}\rangle\,.\end{aligned} \tag{5.15}$$

Recall that $\Omega^{(\pm)}$ may be expressed as a τ-ordered exponential in $P^-_{I,D}(\tau)$. But since Dirac-picture operators have their τ development determined by the free field P_0^-,

$$P_{I,D}^-(\tau) = \exp(iP_0^-\tau)P_{I,D}^-(0)\exp(-iP_0^-\tau)\,, \tag{5.16}$$

and since P_0^- commutes with V_{free} (after all, V_{free} is just a change of spin basis for free quarks) we have the result

$$\tilde{P}_{I,D}^-(\tau) = V_{\text{free}}^{-1}P_{I,D}^-(\tau)V_{\text{free}} = \exp(iP_0^-\tau)\tilde{P}_I^-(0)\exp(-iP_0^-\tau)\,, \tag{5.17}$$

so that $\tilde{\Omega}^{(\pm)}$ may be explicitly computed in any given model by simply transforming $P_I^-(0)$ with V_{free}

$$\tilde{\Omega}^{(+)} = \mathcal{T}\exp\left[-i\int_{-\infty}^{0}\tilde{P}_I^-(\tau)\,d\tau\right]\,. \tag{5.18}$$

There have been two completely disjoint approximations used heretofore in making phenomenological use of this relation. The approach of Melosh[3] and his followers has been to ignore (or argue away) the effect of interaction, which menas to set $\Omega^{(\pm)}=1$, while including the effects of the transformation V_{free}. The approach of Yan[6] and others has been to ignore the distinction between current- and constituent-quark bases by setting $V_{\text{free}}=1$, while including the effects of interaction which are generated by the dressing operators $\Omega^{(\pm)}$; this approach is reasonable in the context of scalar quarks.

Within the context of the formalism developed here, we may hope to do better than these extreme approximations have been able to do. Since we know V_{free} explicitly for the spin-$\frac{1}{2}$ quark model, we may include its effects completely in an interacting-quark model while making some hopefully realistic approximation to $\tilde{\Omega}^{(\pm)}$ so as to account for the effects of interaction to some degree.

We have, of course, assumed throughout that the existence of bound states of quarks does not modify our discussion in a violent manner; of the problem of quark confinement we have nothing to say.

APPENDIX A

The problem is to define operators $\vec{L},\vec{S}$ so that

(a) $\vec{J}=\vec{L}+\vec{S}$,

(b) each set of operators $\{J_\beta\}$, $\{L_\beta\}$, $\{S_\beta\}$ satisfy the algebra of SU(2) within itself,

(c) $\vec{L}$ and $\vec{S}$ are vectors under space rotation, and

(d) $[L_\beta, W_j^\alpha]=0$.

For the free-quark model, Carlitz and Tung[7] have shown that the Melosh transformation V_{free} gives the solution in the sense that $W_{i,\text{free}}^\alpha = V_{\text{free}}F_i^\alpha V_{\text{free}}^{-1}$ and $\vec{L}_{\text{free}},\vec{S}_{\text{free}}$ satisfy (a)–(d) above. In fact,

$$S_{\beta,\text{free}} = W_{0,\text{free}}^\beta\,, \quad \beta=1,2,3\,. \tag{A1}$$

We have shown[1] that *the same* transformation V_{free} results if we require the states $|p,\lambda;\text{constituent}\rangle$ to be defined via K boosts from corresponding rest states. (See Marinescu and Kugler.[8]) That is, by this requirement not only does one obtain states whose transformation properties under $SU(6)_{W,\text{strong}}$ are momentum independent, but one also obtains a separation of total angular momentum into quark orbital and quark spin angular momenta which satisfy the conditions (a)–(d).

The "in" and "out" fields satisfy free-field equations, and the corresponding states transform simply according to the $SU(6)_W$ algebra generated by $W_{i,\text{in}}^\alpha$ and $W_{i,\text{out}}^\alpha$, respectively. Therefore, this is the proper solution for "in" and "out" fields. Hence

$$[L_{\beta,\text{in}}, W_{j,\text{in}}^\alpha]=0\,. \tag{A2}$$

But

$$W_i^\alpha = \Omega^{(+)-1}W_{i,\text{in}}^\alpha\Omega^{(+)}\,, \tag{A3}$$

so if we define

$$L_\beta \equiv \Omega^{(+)-1}L_{\beta,\text{in}}\Omega^{(+)}\,, \tag{A4}$$

$$S_\beta \equiv \Omega^{(+)-1}S_{\beta,\text{in}}\Omega^{(+)} = W_0^i\,, \tag{A5}$$

then

$$[L_\beta, W_j^\alpha]=0\,, \quad \alpha,\beta=1,2,3 \text{ for all } j \tag{A6}$$

as desired.

Suppose we now restrict our attention to single-particle "in" states in the constituent-quark basis. Then

$$b_{\text{in}}^\dagger(q,\lambda) = U([q])b_{\text{in}}^\dagger(\hat{q},\lambda) \tag{A7}$$

when $[q]$ denotes[9] the element of $SL(2,C)$ corresponding to the K boosts which takes $\hat{q}=(m,\vec{0})$ to q. Since U is unitary,

$$b_{\text{in}}(q,\lambda) = b_{\text{in}}(\hat{q},\lambda)U^{-1}([q])\,. \tag{A8}$$

The transformation which takes a state $|q,\lambda\rangle$ into the state $|q',\lambda\rangle$ is just $U([q'][q]^{-1})$. It is easy to check that such transformations U commute with all $W_{i,\text{in}}^\alpha$. Note that this does not mean that $\vec{K}$ commutes with W_i^α.

Furthermore, multiparticle states can be treated similarly. Consider the two-particle state $|q_1,\lambda_1;q_2,\lambda_2\rangle$ for example. Let

$$q_1'=Aq_1\,, \quad q_2'=Aq_2 \tag{A9}$$

for some Lorentz transformation A. Then let $U([q_i'][q_i]^{-1})$ act on the ith particle, so that

$$U_1([q_1'][q_1]^{-1})\otimes U_2([q_2'][q_2]^{-1})|q_1,\lambda_1;q_2,\lambda_2\rangle = |q_1',\lambda_1;q_2',\lambda_2\rangle. \quad (A10)$$

The two-particle state of momentum $Q=q_1+q_2$ is thus transformed into one of momentum $Q'=AQ$. Moreover, such a transformation commutes with all W_i^α, since it is a product of operators which so commute and W_i^α is a one-body operator. Thus we have constructed multiparticle states which are classified by $SU(6)_{W,\text{in}}\otimes O(3)$ independent of total momentum.

---

*This work was supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-1428.

[1]N. H. Fuchs, Phys. Rev. D 11, 1569 (1975). References to earlier work on constituent- and current-quark bases and their interrelations are given in this paper.

[2]We will use the notation and conventions of Ref. 1 except where otherwise indicated.

[3]H. J. Melosh, Phys. Rev. D 9, 1095 (1974).

[4]S.-J. Chang and T.-M. Yan, Phys. Rev. D 7, 1147 (1973).

[5]For a related discussion, see H. J. Melosh, thesis, Caltech, 1973 (unpublished).

[6]T.-M. Yan, Phys. Rev. D 7, 1760 (1973).

[7]R. Carlitz and W.-K. Tung, Phys. Rev. D 13, 3446 (1976).

[8]N. Marinescu and M. Kugler, Phys. Rev. D 12, 3315 (1975).

[9]P. Moussa and R. Stora, in *Methods in Subnuclear Physics*, Lectures for the Herceg-Novi School, 1966, edited by M. Nikolić (Gordon and Breach, New York, 1968), Vol. II, p. 265.