Scattering at fixed t and fixed angle: A renormalization-group analysis*

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We use the renormalization group to study the extrapolation between the Regge region and fixed-angle region of high-energy scattering in ϕ^3 field theory in $6-\epsilon$ dimensions. We also show how the renormalization group can be used to improve weak-coupling expressions for Regge residues and trajectories by including specific self-energy and radiative corrections to all orders of perturbation theory.

I. INTRODUCTION

It has been asserted by Huang and Low that the renormalization group does not determine the form of scattering amplitudes in the Regge limit (s large, t fixed, on the mass shell).[1] In terms of the Callan-Symanzik equation, their observation follows from examination of the solution for the renormalized four-point function

$$T(\lambda p, g, m_I^2, \mu^2) = T(p, \hat{g}(\lambda^{-1}), \hat{m}_I^2(\lambda^{-1}), \mu^2) \times \exp\left[-\int_{-\ln\lambda^2}^{0} d(\ln\lambda'^2)\sigma(\lambda')\right]. \quad (1)$$

Now, for an asymptotically free field theory, $\hat{g}(\lambda^{-1})$ and $\hat{m}_I^2(\lambda^{-1})$ vanish at large λ. For a fixed-angle limit the falloff of $\hat{m}_I^2(\lambda^{-1})$ is harmless if the infrared singularities are sufficiently tame, because the momentum transfer t in T remains finite on the right-hand side of Eq. (1) and provides a cutoff. We can then evaluate T on the right-hand side in perturbation theory using asymptotic freedom, and predict the asymptotic amplitude. By contrast, in the Regge limit, t vanishes on the right-hand side of Eq. (1), and now T develops infrared singularities due to the falloff of $\hat{m}_I^2(\lambda^{-1})$. These singularities cancel the falloff of $\hat{g}(\lambda^{-1})$ and the scaling function T receives finite contributions from every order of perturbation theory.

Let us now consider a field theory which is not only asymptotically free (so that the foregoing discussion applies) but also has an approximately known Regge limit. In this case we have a controlled approximation to the scaling function on the right-hand side of Eq. (1), valid in the Regge limit and involving an infinite set of perturbation terms. We can now use the renormalization group to explore two questions: (1) How and where does the Regge limit contort itself into the fixed-angle limit as $|t|$ increases? (2) Is the approximate Regge amplitude consistent with the renormalization group, and can the renormalization group be used to improve it?

In this paper we explore these questions for ϕ^3 field theory in D space-time dimensions, where $4<D<6$. We cannot extend discussion to $D\leq 4$ because in this region the infrared singularities are not "tame" and the new improved renormalization-group equations we use do not apply.[2,3,4,5] At $D=6$ the Regge behavior of ϕ^3 field theory is extremely complicated[6,7] and we have chosen to avoid the elaboration required to discuss this case. For the same reason, we study the scattering of the elementary scalar particles of ϕ^3 field theory even though the scattering of bound states has features which more closely resemble the phenomenological quark model of wide-angle scattering.[5,8]

We must also take account of the fact that the Regge limit of ϕ^3 field theory is controlled by a leading fixed Regge pole at $j=0$. This Regge pole is of little interest, but we can remove it because it is due to the sum of diagrams which are one-particle reducible in the t channel. We therefore write the scattering amplitude as $T=T_0+T_1$, where T_0 is the sum of amplitudes which are single-particle reducible in the t channel, and T_1 is everything else. The Regge behavior of T_1 can be found by solving the t-channel Bethe-Salpeter equation; the leading approximation is given by the familiar ladder sum.[9]

In Sec. II we begin our analysis with a concise derivation of the Regge behavior of T_1 in D dimensions, $D<6$. Then, in Sec. III we use our Regge amplitude to evaluate the right-hand side of Eq. (1) in an approximation which is uniformly valid in the Regge and fixed-angle limits. We use this result to conjecture where the transition from Regge to fixed-angle behavior occurs in the data. In Sec. IV we study the consistency of our approximate Regge amplitude with the renormalization group. We find that the renormalization group can be used to modify Regge amplitudes in a manner which can be very simply characterized.

II. THE BETHE-SALPETER EQUATION IN $D<6$ DIMENSIONS

The Regge behavior of ϕ^3 field theory in D dimensions can be obtained by analysis of the Bethe-Salpeter equation written in the t-channel center-of-mass frame.[9] We imagine particles of four-momenta $(\vec{p}, \frac{1}{2}\sqrt{t} + \omega)$ and $(-\vec{p}, \frac{1}{2}\sqrt{t} - \omega)$ colliding to produce particles of primed momenta and energies. The Bethe-Salpeter equation reads

$$T_1(\vec{p}, \omega; \vec{p}', \omega'; t) = K(\vec{p}, \omega; \vec{p}', \omega'; t) + \int \frac{d^{D-1}p'' d\omega''}{(2\pi)^D} K(\vec{p}, \omega; \vec{p}'', \omega'', t)\Delta(\vec{p}'', \tfrac{1}{2}\sqrt{t} + \omega'')\Delta(-\vec{p}'', \tfrac{1}{2}\sqrt{t} - \omega'') T_1(\vec{p}'', \omega''; \vec{p}', \omega'; t). \tag{2}$$

K is the sum of all perturbation amplitudes which are one- and two-particle irreducible in the t channel, and Δ is the propagator. This equation is exact. In lowest-order perturbation theory

$$K = \frac{-i\lambda_0^2}{(\omega - \omega')^2 - (\vec{p} - \vec{p}')^2 - m_I^2 + i\epsilon}, \quad \Delta(\vec{p}'', \tfrac{1}{2}\sqrt{t} + \omega'') = \frac{i}{(\frac{1}{2}\sqrt{t} + \omega'')^2 - \vec{p}''^2 - m_I^2 + i\epsilon}, \tag{3}$$

and T_1 becomes the sum of ladder diagrams in the t channel. This approximation leads to the Regge amplitude in the limit of weak coupling, which is the only limit amenable to study.

Equation (2) is diagonalized by expanding the dependence on the scattering angle $z = \hat{p} \cdot \hat{p}'$ in $(D-1)$-dimensional spherical harmonics.[6] For T_1 the expression is

$$T_1(\vec{p}, \omega; \vec{p}', \omega'; t) = \sum_{l=0}^{\infty} \left(\frac{2l + D - 3}{D - 3}\right) C_l^{(D-3)/2}(z) T_l(p, \omega; p', \omega'; t) \tag{4}$$

and similarly for K; $C_l^{(D-3)/2}$ is a Gegenbauer polynomial.[10] Diagonalization results through use of the equation[11]

$$\int d\Omega_n C_l^{(D-3)/2}(\hat{p} \cdot \hat{n}) C_{l'}^{(D-3)/2}(\hat{n} \cdot \hat{p}') = \frac{2\pi^{(D-1)/2} \delta_{ll'} C_l^{(D-3)/2}(\hat{p} \cdot \hat{p}')}{[l + \frac{1}{2}(D-3)]\Gamma(\frac{1}{2}(D-3))} \tag{5}$$

in evaluating the angular integral in Eq. (2). This result is an equation for T_l:

$$T_l(p, \omega; p', \omega'; t) = K_l(p, \omega; p', \omega'; t) - \frac{4}{(4\pi)^{(D+1)/2}\Gamma(\frac{1}{2}(D-1))} \times \int_{-\infty}^{\infty} d\omega'' \int_0^{\infty} \frac{p''^{D-2} dp'' K_l(p, \omega; p'', \omega''; t) T_l(p'', \omega''; p', \omega'; t)}{[(\frac{1}{2}\sqrt{t} + \omega'')^2 - p''^2 - m_I^2 + i\epsilon][(\frac{1}{2}\sqrt{t} - \omega'')^2 - p''^2 - m_I^2 + i\epsilon]}. \tag{6}$$

The projection K_l is[12]

$$K_l = \frac{i\lambda_0^2 \Gamma(\frac{1}{2}(D-1)) 2^{(D-4)/2}}{\pi^{1/2} p p'} (u^2 - 1)^{(D-4)/4} e^{i\pi(D-4)/2} Q_{l+(D-4)/2}^{(4-D)/2}(u),$$
$$u = [m_I^2 - (\omega - \omega')^2 + p^2 + p'^2 - i\epsilon]/2pp'. \tag{7}$$

It can be shown that the kernel of Eq. (6) is L^2 for $D<6$ so that its solution can be constructed by Fredholm's method.[9,13] This gives T_l as a ratio of power series in λ_0^2. However, terms higher than λ_0^2 in these series cannot be trusted because we have only retained the single particle exchange graph of Fig. 1(a) in constructing K. We would also have to include vertex and self-energy corrections, Fig. 1(b), as well as Fig. 1(c), if we wish to accurately calculate the order-λ_0^4 terms. (Order-λ_0^2 self-energy corrections to Δ would have to be retained as well.) For these reasons, only the first Fredholm (trace) approximation has model-independent significance. In this approximation,

$$T_l = K_l / D_l,$$
$$D_l = 1 + \frac{8i\lambda_0^2 (m_I)^{(D-4)/2} e^{i\pi(D-4)/2}}{(4\pi)^{(D+2)/2}} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dp (m_I^2 + 4p^2)^{(D-4)/4} \times \frac{Q_{l+(D-4)/2}^{(4-D)/2}(1 + m_I^2/2p^2)}{[(\frac{1}{2}\sqrt{t} + \omega)^2 - p^2 - m_I^2 + i\epsilon][(\frac{1}{2}\sqrt{t} - \omega)^2 - p^2 - m_I^2 + i\epsilon]}, \tag{8}$$

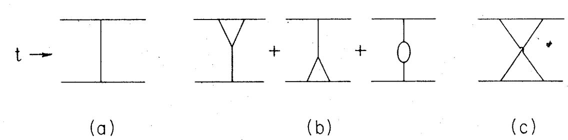

FIG. 1. (a) Graph used in Bethe-Salpeter kernel K, (b) corrections to K due to radiative corrections, and (c) remaining fourth-order contribution to K.

Regge poles occur for values of l at which D_l vanishes. For weak λ_0^2 this happens near $l=-1$ because Q is singular there. We therefore approximate Q by its singular term

$$e^{i\pi(D-4)/2}Q_{l+(D-4)/2}^{(4-D)/2}(u) \simeq \frac{2^{(4-D)/2}\pi^{1/2}(u^2-1)^{(4-D)/4}}{(l+1)\Gamma(\frac{1}{2}(D-3))}. \quad (9)$$

Applying this to Eq. (8) we obtain

$$D_l \simeq 1 - \frac{\Gamma(\epsilon/2)\lambda_0^2}{(l+1)(4\pi)^{D/2}}\int_0^1 dx[m_I^2 - tx(1-x)]^{-\epsilon/2}, \quad (10)$$

where $\epsilon = 6-D$. A similar approximation for K_l leads to the simple expression

$$T_l \simeq \frac{(i\lambda_0^2)(D-3)}{2pp'(l-\alpha)},$$
$$\alpha = -1 + \frac{\Gamma(\epsilon/2)\lambda_0^2}{(4\pi)^{D/2}}\int_0^1 dx[m_I^2 - tx(1-x)]^{-\epsilon/2}. \quad (11)$$

We are now set up to calculate the Regge behavior of T_1. Note first that in Eq. (4) we must replace $C_l^{(D-3)/2}(z)$ by $C_l^{(D-3)/2}(z)+C_l^{(D-3)/2}(-z)$; this adds the crossed and uncrossed t-channel ladders.[14] We recall that on the mass shell, $\omega=\omega'=0$, $p^2 = p'^2 = t/4 - m_E^2$, and

$$z = 1 - \frac{s}{2m_E^2 - t/2} = -1 + \frac{u}{2m_E^2 - t/2}. \quad (12)$$

The Regge limit therefore corresponds to the limit of Eq. (4) as $z \to -\infty$. This we calculate by converting the sum to a Sommerfeld-Watson integral, opening up the contour, and replacing the Gegenbauer polynomial by its large-angle approximation. The result is[15]

$$T_1 \underset{\substack{s\to\infty \\ t \text{ fixed}}}{\sim} -\frac{i\lambda_0^2}{s}\left(\frac{s}{m_E^2 - t/4}\right)^{1+\alpha} - \frac{i\lambda_0^2}{u}\left(\frac{u \pm i\epsilon}{m_E^2 - t/4}\right)^{1+\alpha}, \quad (13)$$

where α is given by Eq. (11). In the Regge limit, $u=-s$, but we distinguish between these variables for the purpose of extrapolating to wide angle.

Equation (13) can also be calculated by summing the leading Regge behavior of each uncrossed and crossed ladder diagram. The calculation by the Bethe-Salpeter equation has the advantage of showing that, when λ_0^2 is small, the Regge behavior of the ladder sum is actually the sum of the leading behaviors. We see from Eq. (11) that as $\epsilon \to 0$, λ_0^2 must be $\le O(\epsilon)$ for the weak-coupling approximation to have any meaning. This happens because at $\epsilon=0$ the Regge spectrum predicted by Eq. (6) abruptly changes from a leading pole to a leading cut in the angular momentum plane.[16,17,18] Cardy and Lovelace have further pointed out that Eq. (6) itself is inadequate at $D=6$.[6,7] We will further discuss these points in Sec. III.

III. JOINING THE REGGE AND FIXED-ANGLE ASYMPTOTIC AMPLITUDES

The amplitude T_1 must be renormalized when divergent diagrams [as in Fig. 1(b)] are included. Renormalized amplitudes are defined by normalization conditions on the renormalized inverse propagator and amputated vertex function

$$0 = i\Delta_R^{-1}(p=0, \lambda, m_I^2=0, \mu^2),$$
$$1 = \frac{\partial}{\partial p^2} i\Delta_R^{-1}(p^2, \lambda, m_I^2=0, \mu^2)\Big|_{p^2=-\mu^2},$$
$$-1 = \frac{\partial}{\partial m_I^2} i\Delta_R^{-1}(p^2=-\mu^2, \lambda, m_I^2, \mu^2)\Big|_{m_I^2=0}, \quad (14)$$
$$-i\lambda = \Gamma_R^{(3)}(p_i, \lambda, m_I^2=0, \mu^2)\big|_{p_i\cdot p_j=(1-3\delta_{ij})\mu^2/2}.$$

Normalization at zero internal mass can be done for $4<D\le 6$ without encountering infrared divergences.[3,4,5] We also introduce the dimensionless coupling g through $g=\lambda(\mu^2)^{-\epsilon/4}$. Equations (11) and (13) can be trivially renormalized through the replacement $\lambda_0^2 \to g^2(\mu^2)^{\epsilon/2}$. The complete amputated four-point function, one-particle irreducible in the t channel, satisfies the renormalization-group equation[2]

$$\left[\frac{\partial}{\partial\ln\lambda^2} + (1-K)\frac{\partial}{\partial\ln m_I^2} - \beta\frac{\partial}{\partial g} + 2\gamma + (1-\epsilon/2)\right] \times T_1^R(\lambda p, g, m_I^2, \mu^2) = 0. \quad (15)$$

β, γ, and K are functions of g only. In the zero-loop approximation, $\beta=-\epsilon g/4$, $\gamma=K=0$, and in the one-loop approximation

$$\beta = -\frac{\epsilon g}{4} + \frac{3}{4}\frac{\Gamma(4-D/2)[\Gamma(D/2-1)]^2}{(4\pi)^{D/2}\Gamma(D-2)}g^3 - \frac{\Gamma(4-D/2)}{(4\pi)^{D/2}}g^3\int\frac{dxdydz\,\delta(1-x-y-z)}{(xy+yz+zx)^{\epsilon/2}},$$
$$\gamma = \frac{\Gamma(4-D/2)[\Gamma(D/2-1)]^2}{2(4\pi)^{D/2}\Gamma(D-2)}g^2, \quad (16)$$
$$K = -\frac{(3D-8)\Gamma(4-D/2)[\Gamma(D/2-2)]^2}{8(4\pi)^{D/2}(D-3)\Gamma(D-4)}g^2.$$

The solution of Eq. (15) is given by Eq. (1), where the running coupling and internal mass are determined by

$$\hat{g}(1)=g,\quad \frac{d\hat{g}(\lambda)}{d\ln\lambda^2}=-\beta(\hat{g}(\lambda));$$
$$\hat{m}_I{}^2(1)=m_I{}^2,\quad \frac{d\ln\hat{m}_I{}^2(\lambda)}{d\ln\lambda^2}=1-K(\hat{g}(\lambda));\qquad (17)$$
$$\sigma(\lambda)=1-\epsilon/2+2\gamma(\hat{g}(\lambda)).$$

Equation (1) is an identity in λ. It can be turned into a relation involving on-mass-shell amplitudes for the s-channel process $p_1+p_2\to p_1'+p_2'$ by choosing

$$p_i=q_i+r_i/\lambda^2$$
$$q_1=\frac{\mu}{2}(1,0,0,+1),\quad r_1=\frac{m_E{}^2}{\mu}(1,0,1,0),$$
$$q_2=\frac{\mu}{2}(1,0,0,-1),\quad r_2=\frac{m_E{}^2}{\mu}(1,0,-1,0),$$
$$q_1'=\frac{\mu}{2}(1,\sin\theta,0,\cos\theta),$$
$$r_1'=\frac{m_E{}^2}{\mu}(1,-\cos\theta,0,\sin\theta),\qquad (18)$$
$$q_2'=\frac{\mu}{2}(1,-\sin\theta,0,-\cos\theta),$$
$$r_2'=\frac{m_E{}^2}{\mu}(1,\cos\theta,0,-\sin\theta),$$
$$\cos\theta=\frac{1}{s^2/2+2m_E{}^4}[s^2/2+st+2m_E{}^2(m_E{}^4-st-t^2)^{1/2}],$$
$$\sin\theta=\frac{1}{s^2/2+2m_E{}^4}[m_E{}^2s+2m_E{}^2t-s(m_E{}^4-st-t^2)^{1/2}],$$
$$\lambda^2=s/\mu^2.$$

We now write T_1 in terms of invariant variables and find Eq. (1) takes the form[19]

$$T_1^R(s,t,m_E{}^2,m_I{}^2,\mu^2,g)$$
$$=T_1^R\left(\mu^2,\frac{t\mu^2}{s},\frac{m_E{}^2\mu^2}{s},\hat{m}_I{}^2\left(\left(\frac{\mu^2}{s}\right)^{1/2}\right),\mu^2,\hat{g}\left(\left(\frac{\mu^2}{s}\right)^{1/2}\right)\right)$$
$$\times\left(\frac{s}{\mu^2}\right)^{2-D/2}\exp\left[2\int_{\hat{g}((\mu^2/s)^{1/2})}^{g}\frac{dg'\gamma(g')}{\beta(g')}\right].\qquad (19)$$

Let us first apply Eq. (19) to scattering at large s and fixed angle, t/s finite. For simplicity we work in the zero-loop approximation, where

$$\hat{m}_I{}^2\left(\left(\frac{\mu^2}{s}\right)^{1/2}\right)=\frac{m_I{}^2\mu^2}{s},\quad \hat{g}\left(\left(\frac{\mu^2}{s}\right)^{1/2}\right)=g\left(\frac{\mu^2}{s}\right)^{\epsilon/4}.\qquad (20)$$

This approximation is reasonable because asymptotic freedom makes $\hat{g}$ vanish as s becomes large. Now for $4<D\le 6$, every perturbation diagram is finite in this fixed-angle region when $m_E{}^2=m_I{}^2=0$.[3,4,5] Since $\hat{g}$ vanishes as s becomes large, we can use perturbation theory to evaluate T_1^R on the right-hand side of Eq. (19). The leading terms are given by s- and u-channel poles:

$$-i\hat{g}^2\left(\left(\frac{\mu^2}{s}\right)^{1/2}\right)(\mu^2)^{\epsilon/2}\left[\frac{1}{\mu^2}-\frac{1}{\mu^2(1+t/s)}\right].$$

We therefore find

$$T_1^R(s,t,m_E{}^2,m_I{}^2,\mu^2,g)\underset{\substack{s\to\infty\\ s/t\ \text{fixed}}}{\sim}-\frac{i\lambda^2}{s}-\frac{i\lambda^2}{u}.\qquad (21)$$

The result is trivial but proven: The Born approximation gives high-energy scattering at fixed angle.

In the Regge limit, the momentum transfer on the right-hand side of Eq. (19), $t\mu^2/s$, vanishes at large s. Now the scaling function T_1^R receives contributions from every order of perturbation theory because infrared singularities due to the vanishing of $\hat{m}_I{}^2((\mu^2/s)^{1/2})$ exactly compensate the falloff of $\hat{g}((\mu^2/s)^{1/2})$. We see this compensation explicitly in our expression for α, Eq. (11) provided we renormalize by setting $\lambda_0{}^2=g^2(\mu^2)^{\epsilon/2}$. However, we know the correct scaling function to all orders in perturbation theory; it is provided by evaluating Eq. (13) with the scaled variable and running coupling constants specified on the right-hand side of Eq. (19). The proof that this is the correct scaling function, in the zero-loop approximation, follows from evaluating Eq. (19); the result is the Regge amplitude of Eq. (13).

From this analysis it follows that Eq. (13) is valid in *both* the Regge and fixed-angle limits. As t increases beyond the Regge region, all terms higher than $O(\hat{g}^2)$ on the right-hand side of Eq. (19) are negligible because of asymptotic freedom; and Eq. (13) passes over to Eq. (21). The transition occurs when

$$1\simeq\frac{\lambda^2\Gamma(\epsilon/2)}{(4\pi)^{D/2}}\ln\left(\frac{s}{m_E{}^2-t/4}\right)\times\int_0^1 dx[m_I{}^2-tx(1-x)]^{-\epsilon/2}.\qquad (22)$$

If we ignore $m_I{}^2$ the transition momentum transfer is

$$\left\{\frac{\lambda^2\Gamma(\epsilon/2)[\Gamma(1-\epsilon/2)]^2}{(4\pi)^{D/2}}\ln(s/m_E{}^2)\right\}^{2/\epsilon}\simeq -t.\qquad (23)$$

The reference mass is provided primarily by the dimensional coupling constant, not by hadronic masses.

IV. CONSTRAINTS ON REGGE AMPLITUDES

The transition from Regge to fixed-angle behavior that we have found is about as simple as can be imagined. Ten years ago the idea of extrapolating Regge amplitudes to $|t|\sim s$ was used to fit data, but without much justification.[20] But the simplicity we have found is slightly deceptive, because if we had used the one-loop renormalization-group functions in Eq. (19), we would have found that the Regge amplitude of Eq. (13) is inconsistent with Eq. (19) when it is substituted on both sides. By demanding consistency we can improve the Regge amplitude in a manner we will specify.

Once this is done, the fixed-angle limit can again be recovered by extrapolation, and we return to the simple situation of Sec. III.

First let us write the renormalization-group equation in terms of dimensionless variables. It is easy to show that Eq. (19) is the general solution of the equation

$$\left(-\frac{\partial}{\partial \ln x}-K\frac{\partial}{\partial \ln y}-\beta\frac{\partial}{\partial g}+2\gamma\right)\phi(\sigma,\tau,x,y,g)=0, \tag{24}$$

where

$$\sigma=s/m_E^2,\quad \tau=t/m_E^2,\quad x=\mu^2/m_E^2,\quad y=m_I^2/m_E^2,$$
$$T_1^R=s^{\epsilon/2-1}\phi(\sigma,\tau,x,y,g). \tag{25}$$

The representation for T_1 follows from counting dimensions. Our program is to apply Eq. (24) in the Regge limit but to correct Eq. (13) by adding higher powers in g^2 to the Regge residue and exponent in Eq. (13). For ϕ we thus make the ansatz

$$\phi=A(\tau,x,y,g)\left(\frac{x}{\sigma}\right)^{\epsilon/2}\left(\frac{\sigma}{1-\tau/4}\right)^{B(\tau,x,y,g)},$$

$$A=-ig^2+\cdots, \tag{26}$$

$$B=\frac{\Gamma(\epsilon/2)g^2(x)^{\epsilon/2}}{(4\pi)^{D/2}}\int_0^1 d\xi[y-\tau\xi(1-\xi)]^{-\epsilon/2}+\cdots .$$

For this to be a solution of Eq. (24) we require

$$\begin{aligned}&-\frac{\partial B}{\partial \ln x}-K\frac{\partial B}{\partial \ln y}-\beta\frac{\partial B}{\partial g}=0,\\&-\frac{\partial A}{\partial \ln z}-K\frac{\partial A}{\partial \ln y}-\beta\frac{\partial A}{\partial g}+2\gamma A-\frac{\epsilon A}{2}=0 .\end{aligned} \tag{27}$$

At this point we can verify that in the zero-loop approximation the $O(g^2)$ expressions for A and B satisfy Eq. (27). We analyze the situation in the one-loop approximation for B only; the equations for A are similar. Now we have from Eq. (16)

$$\begin{aligned}&\beta=-\tfrac{1}{4}\epsilon g-\beta_1 g^3,\quad \gamma=\gamma_1 g^2,\quad K=-K_1 g^2,\\&B=\sum_{n=1}^{\infty}g^{2n}B_n .\end{aligned} \tag{28}$$

The various orders are linked by Eq. (27) in the fashion

$$\begin{aligned}&-\frac{\partial B_1}{\partial \ln x}+\frac{\epsilon}{2}B_1=0,\\&-\frac{\partial B_2}{\partial \ln x}+\epsilon B_2=-K_1\frac{\partial B_1}{\partial \ln y}-2\beta_1 B_1,\\&-\frac{\partial B_3}{\partial \ln x}+\frac{3\epsilon}{2}B_3=-K_1\frac{\partial B_2}{\partial \ln y}-4\beta_1 B_2\\&\qquad+\text{terms involving higher-order approximation of }\beta,\gamma,K,\\&\ldots .\end{aligned} \tag{29}$$

When these equations are solved, we encounter integration constants we do not know: B_{20}, B_{30}, $\ldots$,

$$\begin{aligned}B_1&=B_{10}(\tau,y)(x)^{\epsilon/2},\\B_2&=B_{20}(\tau,y)(x)^{\epsilon}+\frac{2}{\epsilon}\left(-K_1\frac{\partial B_{10}}{\partial \ln y}-2\beta_1 B_{10}\right)(x)^{\epsilon/2},\\B_3&=B_{30}(\tau,y)(x)^{3\epsilon/2}+\frac{2}{\epsilon}\left(-K_1\frac{\partial B_{20}}{\partial \ln y}-4\beta_1 B_{20}\right)(x)^{\epsilon}\\&\quad+\frac{2}{\epsilon^2}\left[K_1\left(K_1\frac{\partial^2 B_{10}}{\partial^2 \ln y}+2\beta_1\frac{\partial B_{10}}{\partial \ln y}\right)\right.\\&\qquad\left.+4\beta_1\left(K_1\frac{\partial B_{10}}{\partial \ln y}+2\beta_1 B_{10}\right)\right](x)^{\epsilon/2},\\&\ldots .\end{aligned} \tag{30}$$

We see the development of the infrared singularities in x which require us to go beyond perturbation theory in evaluating the Regge-limit scaling function. The undetermined integration constants provide the freedom to improve the Bethe-Salpeter kernel by adding terms like that in Fig. 1(c). These terms we do not know, so we set them to be zero. However, the coefficient of $(x)^{\epsilon/2}$ is nonzero and is determined by B_1. The *minimal* required modification of B is therefore determined by setting

$$\begin{aligned}&B(\tau,x,y,g)=(x)^{\epsilon/2}\bar{B}(\tau,y,g),\\&A(\tau,x,y,g)=\bar{A}(\tau,y,g) .\end{aligned} \tag{31}$$

The x independence of A is determined by an analysis similar to that we have given for B. We further see from Eq. (30) that, to every order in g, $\bar{B}$ will depend on τ and y in the same manner as $\bar{B}_1$, and $\bar{A}$ will be independent of τ and y, as $\bar{A}_1$ is. Therefore, in the minimal required modification,

$$\begin{aligned}&B(\tau,x,y,g)=g^2(x/y)^{\epsilon/2}\tilde{B}(\tau/y,g),\\&A(\tau,x,y,g)=g^2\tilde{A}(g) .\end{aligned} \tag{32}$$

The functions $\tilde{A}$ and $\tilde{B}$ are most efficiently determined by substituting Eq. (32) into Eq. (27):

$$\begin{aligned}&\left[-K\frac{\partial}{\partial \ln y}-\beta\frac{\partial}{\partial g}-\frac{2\beta}{g}-\frac{\epsilon}{2}(1-K)\right]\tilde{B}\left(\frac{\tau}{g},g\right)=0,\\&\left(-\beta\frac{d}{dg}+2\gamma-2\frac{\beta}{g}-\frac{\epsilon}{2}\right)\tilde{A}(g)=0 .\end{aligned} \tag{33}$$

The solutions of these equations are

$$B(\tau,x,y,g)=B_1(\tau,x,y(1+4\beta_1 g^2/\epsilon)^{-K_1/2\beta_1},g)/(1+4\beta_1 g^2/\epsilon)$$
$$=\frac{g^2\Gamma(\epsilon/2)(x)^{\epsilon/2}}{(4\pi)^{D/2}(1+4\beta_1 g^2/\epsilon)}\int_0^1 d\xi[y(1+4\beta_1 g^2/\epsilon)^{-K_1/2\beta_1}-\tau\xi(1-\xi)]^{-\epsilon/2}\,,$$
$$A(g)=A_1(1+4\beta_1 g^2/\epsilon)^{-1-\gamma_1/\beta_1}=-ig^2(1+4\beta_1 g^2/\epsilon)^{-1-\gamma_1/\beta_1}\,. \tag{34}$$

We see that the magnitude of the trajectory and residue are slightly changed, and in addition the scale of the trajectory's t dependence is modified.

As we have emphasized, the modifications we have made do not include all higher-order contributions to A and B. We would expect them to include all single-loop vertex and self-energy insertions on the ladders, including insertions into insertions. For example, it is readily verified that we have included all order-g^4 contributions of the form $(1/s)(\ln s)^p$ except that of Fig. 1(c).[21] The simplicity of the answer obscures the diagrammatic complexity of the modification. It should be noted that asymptotic freedom has not been invoked in Sec. IV, so our results apply to any superrenormalizable theory.

The divergence of Eqs. (11) and (34) at $\epsilon=0$ can be traced to an ultraviolet divergence of the Fredholm trace in the renormalizable dimension, $D=6$. Cardy and Lovelace have pointed out that this divergence is an illusion since the Bethe-Salpeter kernel of Fig. 1(a) falls off more rapidly than in perturbation theory when asymptotic freedom is invoked at the two vertices.[6,7] In order to exhibit this modification of the Regge spectrum, it is necessary to apply the renormalization group to the dynamics. Our calculation applies when vertex and self-energy insertions change the residue and trajectory, leaving the character of the Regge singularities unchanged.

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