

High-energy scattering of quarks in gauge theories

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High-energy quark-quark scattering is calculated in quantum chromodynamics (QCD) using the infrared equation of Cornwall and Tiktopoulos. In contrast to ordinary quantum electrodynamics, the high-energy behavior is determined by the infrared structure. The infrared divergence is regulated by introduction of a small gluon mass; the amplitudes obtained agree with previous perturbation calculations (for t fixed, $s \rightarrow \infty$) in QCD modified by the addition of a Higgs meson in the limit of small gluon mass. A proposal to calculate the infrared-finite behavior at fixed angle using asymptotic freedom is made, based on the hypothesis that the infrared divergence can be factored from an infrared-finite and renormalization-point-dependent part. The latter is calculable for the colorless exchange amplitude but not for the color exchange amplitude.

I. INTRODUCTION

In this paper we study high-energy scattering of quarks according to quantum chromodynamics (QCD). QCD is a Yang-Mills theory¹ of quarks and massless gluons having SU_3 as the local gauge symmetry. We shall be interested in two high-energy regimes, namely fixed momentum transfer and fixed angle, though we shall concentrate on rather different aspects of the two. At fixed momentum transfer we shall study the high-energy behavior in leading-log approximation, which turns out to be equivalent to studying the infrared behavior^{2,3} [something which, incidentally, is *not* true in quantum electrodynamics (QED)]. At fixed angle, we shall study the asymptotic behavior of the infrared-*finite* part of the amplitude—that part which remains after the infrared-singular (and dominant at high energy in leading-log approximation) part of the amplitude is factored out. In this regime, since QCD is asymptotically free, the high-energy behavior of the infrared-finite amplitude is correctly given by the leading-log sum and may be found through use of the renormalization group.⁴ In the forward direction, in contrast, we effectively study the high-energy behavior by looking at the infrared (IR) behavior, and since QCD is decidedly *not* asymptotically free in the IR region, one may well be suspicious of any results obtained through a leading-log sum. Nevertheless, we feel results in this approximation can be of value, both in that they reproduce with miniscule effort results previously obtained only through laborious calculations in perturbation theory,⁵⁻⁹ and that the techniques

employed may be rather easily modified to include non-leading-log effects (and perhaps even *all* non-leading-log effects).

Our work is an extension of previous efforts in the following ways. At fixed momentum transfer, as mentioned above, the only thing which has been done before is leading-log approximations through the eighth order⁵ in perturbation theory. We are able to sum all orders of perturbation theory. At fixed angles the infrared-singular part of the amplitude has been studied, again in leading log, in all orders of perturbation theory.¹⁰ We study the remaining infrared-finite part, and in a regime where the leading-log approximation is justified through the asymptotic freedom of the theory.

A major defect of QCD as presently developed is the lack of a convincing demonstration of the confinement of the quarks and colored gluons. It is hoped that a deeper understanding of the infrared behavior will shed light on confinement. And since the high-energy behavior which concerns us is intimately intertwined with the infrared behavior we should in principle be concerned with this problem.

One basic question is whether or not soft gluon emission cancels the infrared singularities of exclusive amplitudes, so that inclusive amplitudes are finite in analogy to the situation in QED. There are, at present, conflicting views on this question. On the one hand, several authors¹¹⁻¹³ have found order-by-order cancellation of internal and external infrared singularities in close analogy with QED, concluding thereby that there is no suppression of gluon emission in perturbation expansions of mass-shell quark amplitudes. On the other

hand, Cornwall and Tiktopoulos¹⁰ summed certain infinite classes of graphs and found suppression of gluon emission and, moreover, of non-color-singlet amplitudes in leading-log approximation. (The same set of graphs has been summed by a different technique by Crewther.¹⁴)

It seems likely to us that the confinement issue cannot be settled within the IR leading-log approximation, and an understanding of other IR effects, particularly those associated with the behavior of the coupling constant $g(k^2)$ (renormalized at gluon mass k^2) for small k^2 is essential. Insofar as we deal with leading logs only, the question of where the coupling constant is renormalized is unimportant since the choice of renormalization point affects only nonleading terms) and we are therefore able to finesse this interesting but difficult issue, both in the fixed-momentum regime (since there we deal with leading log only) and at fixed angles (since there we deal with the IR-finite amplitude only).

We are to deal with infrared singularities. The question of how to regularize these therefore comes up. Two basic choices exist. One is to deal only with off-shell amplitudes, which are all IR finite, and to study their behavior near the mass shell. The other is to insert a cutoff. This can be done in several ways; one may insert an artificial gluon mass λ and study the limit as $\lambda \rightarrow 0$; one may insert a real gluon mass through the use of Higgs scalars and study the zero-mass limit; one may dimensionally regularize. The last two of these manifestly preserve the gauge invariance of the theory, and are therefore "safe." The first does not (and is "unsafe") but is technically easier. Our experience is that essentially the same results are obtained with all three methods; therefore we opt for the artificial gluon-mass cutoff because of its technical simplicity.

We have remarked above that the high-energy limit of a given Feynman graph is determined by the infrared contributions of virtual gluons, as has been noted by various authors.^{2,3} However, the leading contribution of a given graph may be cancelled by other graphs in the same order of perturbation theory so that the true asymptotic behavior actually depends on surviving subasymptotic terms. In QCD the superficially leading graphs involving gluon-gluon interactions cancel extensively. In QED, extensive cancellations also occur in electron scattering as explained by Cheng and Ma.¹⁵ However, in QCD the internal-symmetry factors lead to constructive interference of the leading behavior of the QED analog graphs so that the longest-range (infrared singular) forces do determine the high-energy behavior of the quark-quark scattering amplitude. With this in

mind, the very different qualitative results of massive QED found by Cheng and Wu¹⁶ and of QCD^{5,6,17} do not seem astonishing. (In contrast, the vertex functions of QED and QCD are very similar in the limit $q^2 \rightarrow \infty$.)

In order to sum the (virtual) infrared singularities we use an equation proposed¹⁰ by Cornwall and Tiktopoulos (CT) (a similar approach to the QED vertex was given by Korthals-Altes and de Rafael).¹⁸ This equation is supposed to include the leading¹⁹ infrared singularities of the amplitude as $s \rightarrow \infty$ (at t fixed or at s/t fixed) and may include *all* (important) infrared singularities with appropriate choice of the renormalization point of the coupling constant (see Sec. III). In this limit the structure of the QCD equation is topologically similar to the QED case. The infrared limit is approached by inserting (by hand) a gluon mass λ and examining the theory in the limit $\lambda \rightarrow 0$. For t fixed, $s \rightarrow \infty$ it is easy to solve the CT equation in coordinate space.¹⁷ When expanded in power series the solution agrees with the small-gluon-mass limit of the explicit perturbation calculations of McCoy and Wu, Tyburski, and others to eighth order for both the color-singlet and isovector (they use SU_2 rather than SU_3) exchange amplitudes. This lends support to the CT equation and further suggests that the limit of the McCoy *et al.* variant of QCD joins smoothly with our way of approaching the infrared limit.

The decisive calculational advantage of a closed-form solution, in contrast to arduous perturbation calculations involving amazing and tricky cancellations among individual graphs indicates the value of further work to understand better the foundation and range of validity of the CT equation.

Simple consequences of our solution are the explicit asymptotic forms

$$T_0 = \frac{3i\pi}{4} T_1^B \frac{\exp[(-g^2/4\pi^2) \ln(-s/m^2) \ln(-t/\lambda^2)] - 1}{\ln(-s/m^2)}, \quad (1.1)$$

$$T_1 = T_1^B \exp[(-g^2/4\pi^2) \ln(-s/m^2) \ln(-t/\lambda^2)] \quad (1.2)$$

for the color-singlet exchange amplitude T_0 and the isovector exchange amplitude T_1 . [SU_3 results are given in the text; Eqs. (1.1) and (1.2) are for SU_2 in order to compare with most other work in this field.] T_1^B is the Born term. That T_1 is a Regge pole was known previously to eighth order. The simple structure found for T_0 had not been surmised previously from the complicated form of the individual terms in perturbation theory.

Whether the amplitudes (1.1) and (1.2) have any-

thing to do with the true high-energy scattering of hadrons is a moot point because of the singular behavior as $\lambda \rightarrow 0$. The scattering of positronium off positronium is a useful analogy. The infrared singularity in electron-electron or electron-positron scattering is associated with the long-range Coulomb force. But since positronium is electrically neutral, the long-range Coulomb force cancels in positronium scattering; the residual Van der Waals force is of finite range and is infrared finite. The inference is very strong that the infrared-singular part of quark-quark scattering has nothing to do with hadron-hadron scattering. Indeed, it is easy to see that the leading IR singularities cancel for color-singlet cluster scattering.^{10,17} The only part of (1.1) that is infrared finite is the $-1/\ln s$ term. Only this could conceivably be relevant to physically interesting processes.

The arrangement of this paper is as follows. In Sec. II we analyze the infrared singularities of the fourth-order scattering amplitude in detail to motivate the CT equation. The K - G photon decomposition of Grammer and Yennie²⁰ is shown to be a useful formal device to simplify the spin structure. In Sec. III the CT equation is analyzed, solved in suitable limits, and compared with perturbation expansions. Finally in Sec. IV we study the infrared-finite part of the s dependence for fixed-angle scattering using Weinberg's variant²¹ of the renormalization group. This application is based on the (thus far unproved) assumption that the full amplitude can be factored into an infrared-singular part and a renormalization-point-dependent part in analogy in QED. The use of the renormalization group for on-shell amplitudes to obtain asymptotic behaviors is possible only for amplitudes having no mass singularities. We find that T_0 , at fixed angle, has no singularities as the quark mass $m \rightarrow 0$ in fourth and sixth order while T_1 is singular. Hence it should be permissible to use the asymptotic freedom of QCD to obtain the fixed-angle asymptotic behavior of the infrared-finite part of T_0 , but not that of T_1 .

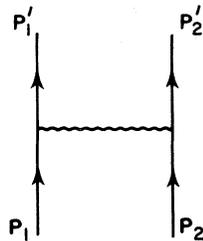


FIG. 1. The Born term for quark scattering is shown.

II. ANALYSIS OF FOURTH ORDER

The quark-gluon coupling has the form $\bar{q}\gamma^\mu t_i q A_\mu^i$, where the t_i are matrix representations of the gauge group in the fundamental representation and the gauge fields A_μ^i belong to the adjoint representation. For SU_2 and SU_3 $\vec{t} = \frac{1}{2}\vec{\tau}$ and $\frac{1}{2}\vec{\lambda}$, respectively, where $\vec{\tau}$ are SU_2 spin matrices and $\vec{\lambda}_i$ are the SU_3 matrices. In these cases we write the internal color symmetry structure as

$$T = T_0 + \vec{\tau}_1 \cdot \vec{\tau}_2 T_1 \quad (SU_2), \quad (2.1)$$

$$T = T_0 + \vec{\lambda}_1 \cdot \vec{\lambda}_2 T_1 \quad (SU_3), \quad (2.2)$$

respectively. T_0 has the significance of the color-singlet exchange while T_1 corresponds to pure color-vector or color-octet exchange. Generalizations to SU_n are immediate.

The Born term (Fig. 1) is pure T_1 :

$$T_B = \frac{g^2 \vec{t}_1 \cdot \vec{t}_2 C}{t - \lambda^2} \quad (2.3)$$

where²² the factor C is

$$C = \bar{u}(p'_1) \gamma_\mu u(p_1) \bar{u}(p'_2) \gamma_\mu u(p_2). \quad (2.4)$$

The factor C simplifies at large energies:

$$C \rightarrow s \delta_{\lambda'_1 \lambda_1} \delta_{\lambda'_2 \lambda_2} \times \begin{cases} 2, & \lambda_1 = \lambda_2 \\ 1 + \cos \theta, & \lambda_1 = -\lambda_2 \end{cases} \quad (2.5)$$

where λ_1, λ_2 (λ'_1, λ'_2) are the initial (final) helicities of quarks 1 and 2, and $s = (p_1 + p_2)^2$ is the squared c.m. energy and θ the c.m. scattering angle. We shall use the variables

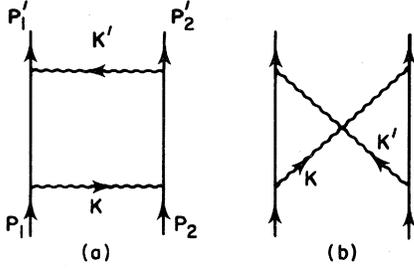
$$\begin{aligned} P &= p_1 + p_2, & s &= P^2, \\ Q &= p'_1 - p_1, & t &= Q^2, \\ R &= p'_1 - p_2, & u &= R^2. \end{aligned} \quad (2.6)$$

where $s \rightarrow \infty$ with t fixed the factor (2.5) is simply 2 for helicity-nonflip scattering. The necessity of the vanishing of T_B for $\theta = \pi$, $\lambda_1 = -\lambda_2$ is a consequence of angular momentum conservation. We shall frequently suppress the factor C from our equations, understanding that it is to be reinstated in the final results.

It is instructive to analyze the fourth-order box diagram (Fig. 2). The invariant amplitude is (in the Feynman gauge)

$$\begin{aligned} T_4^{(s)} &= -ig^4 (\vec{t}_1 \cdot \vec{t}_2)^2 \\ &\times \int \frac{d^4 k}{(2\pi)^4} \frac{g^{\mu\alpha}}{k'^2 - \lambda^2} \frac{g^{\nu\beta}}{k^2 - \lambda^2} \\ &\times \frac{N_{\mu\alpha, \nu\beta}}{(k^2 - 2k \cdot p)(k^2 + 2k \cdot p)}, \end{aligned} \quad (2.7)$$

where the numerator spin structure factor is

FIG. 2. The s - and u -channel box diagrams are shown.

$$N_{\mu\alpha,\nu\beta} = \bar{u}(p_1')\gamma_\mu(\not{p}_1 - \not{k} + m)\gamma_\nu u(p_1) \times \bar{u}(p_2')\gamma_\alpha(\not{p}_2 + \not{k} + m)\gamma_\beta u(p_2). \quad (2.8)$$

It is useful to note that $k' = k + Q$ and that $Q \cdot p_1 = -Q \cdot p_2 = -\frac{1}{2}t$. Equation (2.7) has infrared singularities for $k \approx 0$ and for $k' \approx 0$. (To expose the symmetry note that $k^2 - 2k \cdot p_1 = k'^2 - 2k' \cdot p_1$ and $k^2 + 2k \cdot p_2 = k'^2 + 2k' \cdot p_2$.) For these dominant regions the detailed spin structure of N is irrelevant and, in addition, the numerator k factors do not contribute. Consideration of $p_1, p_2 \rightarrow \infty$ leads to the same conclusion.

A systematic way to collect the infrared-domi-

$$I_s(p_1, p_2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \lambda^2)(k'^2 - \lambda^2)(k^2 - 2k \cdot p_1)(k^2 + 2k \cdot p_2)}. \quad (2.13)$$

The same result is obtained if we replace the k' line by its K component [Fig. 3(b)]. The IR-dominant contributions are not sensitive to the detailed spin structure of the coupled current, and the "K photon" method is just a convenient technical way of isolating the dominant zero-spin components of that current. The occurrence of the spin term C as a factor persists in higher orders. The crossed-box diagram [Fig. 4(a)] gives, with one K photon substitution,

$$T_4^u(K) = -ig^4 \vec{t}_1 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_2 C 4p_1 \cdot p_2' I_u(p_1, p_2'), \quad (2.14)$$

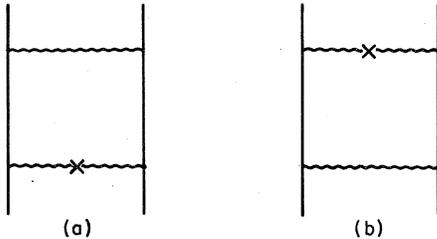


FIG. 3. The x on a gluon line means that the $g_{\mu\nu}$ in the propagator has been replaced by $K_{\mu\nu}$ of Eq. (2.9). The contribution of Fig. 3(a) [equal to that of Fig. 3(b)] contains the entire infrared singularity.

nant terms is the K, G photon method of Grammer and Yennie.²⁰ These authors treated the vertex in detail, and a slight generalization is needed for scattering amplitudes.²³ The method is to write in the photon (or gluon) propagator

$$g_{\mu\nu} = K_{\mu\nu} + G_{\mu\nu}, \quad (2.9)$$

$$K_{\mu\nu} = b(p, p') k_\mu k_\nu,$$

where p, p' are the momenta of the fermion lines and the factor b is chosen so that $G_{\mu\nu}$ gives $O(k)$ for small k , thereby suppressing the IR divergence. We can isolate the IR divergence of Fig. 2 by replacing the line k by a K photon having²⁴

$$b(p_1, p_2) = \frac{4p_1 \cdot p_2}{(2p_1 \cdot k - k^2)(2p_2 \cdot k + k^2)}. \quad (2.10)$$

With (2.10) the K -photon contribution simplifies the numerator factor as follows:

$$b(p_1, p_2) k^\nu k^\beta N_{\mu\alpha,\nu\beta} = 4p_1 \cdot p_2 \bar{u}(p_1')\gamma_\mu u(p_1) \bar{u}(p_2')\gamma_\alpha u(p_2). \quad (2.11)$$

The IR-singular part [denoted by Fig. 3(a)] is therefore

$$T_4^s(K) = -ig^4 (t_1 \cdot t_2)^2 C 4p_1 \cdot p_2 I_s(p_1, p_2), \quad (2.12)$$

$$I_u(p_1, p_2') = I_s(p_1, -p_2'). \quad (2.15)$$

A K substitution on the other line [Fig. 4(b)] gives the same result, noting that

$$\vec{t}_1 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_2 = \vec{t}_2 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_1.$$

Similarly the IR contribution due to the vertex correction [Fig. 5(g)] is

$$T_4^t(K) = -ig^4 \vec{t}_1 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_1 C \frac{4p_1 \cdot p_1'}{t - \lambda^2} I_t(p_1', p_1), \quad (2.16)$$

where

$$I_t(p_1', p_1) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \lambda^2)(k^2 - 2k \cdot p_1)(k^2 - 2k \cdot p_1')}. \quad (2.17)$$

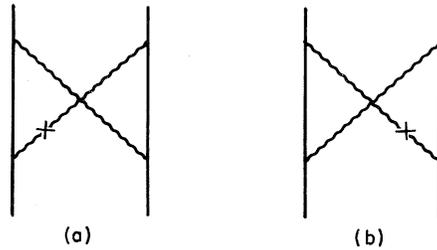


FIG. 4. The u -channel analog of Fig. 3 is shown.

The mirror image of 5(g) gives an identical contribution to (2.16). The corrections of the vertex due to gluon self-coupling [Fig. 5(i)] are IR finite and nonleading for $-t \gg \lambda^2$. We shall not consider them further.

The asymptotic behavior of the functions I_s , I_u , and I_t can be obtained by standard techniques. For $s \gg m^2$, $-t \gg \lambda^2$, $m^2 \gg \lambda^2$ we have

$$I_s \sim \frac{i}{8\pi^2 s t} \ln(-s/m^2) \ln(-t/\lambda^2), \tag{2.18}$$

$$I_u \sim \frac{i}{8\pi^2 u t} \ln(-u/m^2) \ln(-t/\lambda^2),$$

$$I_t \sim \frac{i}{16\pi^2 t} \ln(-t/m^2) \ln(t/\lambda^2). \tag{2.19}$$

(For massless quarks replace m by λ .)

For fixed t and $s \rightarrow \infty$, the leading terms come from the s - and u -channel boxes

$$T_4^s + T_4^u \sim \frac{C}{t} \frac{g^4}{4\pi^2} \ln\left(-\frac{t}{\lambda^2}\right) \times [(\vec{t}_1 \cdot \vec{t}_2)^2 \ln(-s/m^2) - \vec{t}_1 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_2 \ln(s/m^2)] \tag{2.20}$$

In order to find the corresponding T_0 and T_1 we use the identities (for SU_2)

$$\begin{aligned} (\vec{\tau}_1 \cdot \vec{\tau}_2)^2 &= 3 - 2\vec{\tau}_1 \cdot \vec{\tau}_2, \\ \vec{\tau}_1 \cdot (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\tau}_2 &= \vec{\tau}_2 \cdot (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\tau}_1 = 3 + 2\vec{\tau}_1 \cdot \vec{\tau}_2, \\ \vec{\tau}_1 \cdot (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\tau}_1 &= \vec{\tau}_2 \cdot (\vec{\tau}_1 \cdot \vec{\tau}_2) \vec{\tau}_2 = -\vec{\tau}_1 \cdot \vec{\tau}_2, \end{aligned} \tag{2.21}$$

or, for SU_3 ,

$$\begin{aligned} (\vec{\lambda}_1 \cdot \vec{\lambda}_2)^2 &= \frac{32}{9} + \frac{14}{3} \vec{\lambda}_1 \cdot \vec{\lambda}_2, \\ \vec{\lambda}_1 (\vec{\lambda}_1 \cdot \vec{\lambda}_2) \vec{\lambda}_2 &= \vec{\lambda}_2 \cdot (\vec{\lambda}_1 \cdot \vec{\lambda}_2) \vec{\lambda}_1 = \frac{32}{9} - \frac{4}{3} \vec{\lambda}_1 \cdot \vec{\lambda}_2, \\ \vec{\lambda}_1 \cdot (\vec{\lambda}_1 \cdot \vec{\lambda}_2) \vec{\lambda}_1 &= \vec{\lambda}_2 \cdot (\vec{\lambda}_1 \cdot \vec{\lambda}_2) \vec{\lambda}_2 = -\frac{8}{3} \vec{\lambda}_1 \cdot \vec{\lambda}_2. \end{aligned}$$

For brevity we give explicit results for SU_2 only. In T_0 the leading behavior cancels; the term in the square brackets in Eq. (2.20) becomes $3[\ln(-s) - \ln(s)] = -3i\pi$ giving

$$T_0^{(4)} \sim -\frac{3i\pi C}{16t} \frac{g^4}{4\pi^2} \ln(-t/\lambda^2). \tag{2.22}$$

There is no competition from other (nonleading) graphs since they are pure isospin-one exchange. For T_1 we find the behavior

$$T_1^{(4)} \sim -\frac{C}{4t} \frac{g^4}{4\pi^2} \ln(-t/\lambda^2) \ln(s/m^2). \tag{2.23}$$

These results are in agreement with previous calculations. Note that $T_1^B = Cg^2/4t$ occurs as a factor in (2.22) and (2.23).

For fixed angle the contribution of each channel is of the same order of magnitude, giving (see Ref. 17 for a more detailed analysis)

$$T_4^s + T_4^u + T_4^t \sim \frac{-g^2}{16\pi^2} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{-t}{\lambda^2}\right) \left(\sum_i C_i\right) T_B, \tag{2.24}$$

where the sum runs over the Casimir operators $C_i = t_i^2$ of all the external particles. In differential form Eq. (2.24) reads

$$\lambda \frac{\partial}{\partial \lambda} T_4 = \frac{g^2}{8\pi^2} \ln\left(\frac{s}{m^2}\right) \left(\sum_i C_i\right) T_2, \tag{2.25}$$

which is the prototype of the equation holding for all orders.

The operation $\lambda \partial / \partial \lambda$ removes IR-finite parts of the amplitude and in addition has a simple graphical interpretation. From the identity

$$\lambda \frac{\partial}{\partial \lambda} \frac{1}{k^2 - \lambda^2} = \frac{1}{k^2 - \lambda^2} 2\lambda^2 \frac{1}{k^2 - \lambda^2} \tag{2.26}$$

we see that $\lambda \partial / \partial \lambda$ acts as a mass insertion in the gluon propagator. Thus $\lambda \partial / \partial \lambda$ applied to a given Feynman graph produces a sum of graphs in which each gluon propagation receives an insertion in turn. For example, to fourth-order the relevant graphs are shown in Fig. 5. One makes (single) insertions in all possible ways and adds up the result. We notice that graphs 5(g)–5(k) can be combined with the Born term 5(a). Henceforth we omit all terms obtained by differentiating the (one-par-

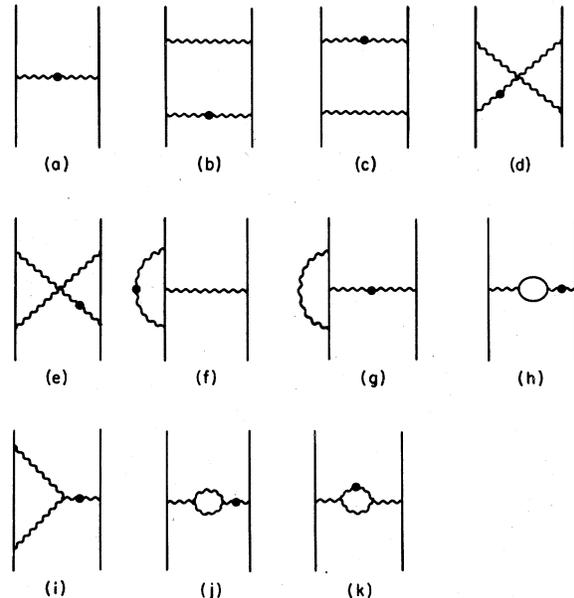


FIG. 5. The dot denotes that a mass insertion has been made on the gluon line [see Eq. (2.26)]. Contributions due to various mass insertions are shown to fourth order. Graphs (g)–(j) combine with the Born term to make $g(t)$ the effective coupling constant. Graph (k) also contributes to $g(t)$. The soft-gluon infrared corrections have the structure of Eq. (3.1).

ticle-reducible) gluon exchange line. As before, the spin structure is much simplified by replacing inserted gluons by inserted K gluons. After some relabeling, we find the contributions of graphs 5(b)–5(g) to $\lambda \partial T_4 / \partial \lambda$ to be given by the right-hand side of Eq. (3.1) below, with T replaced by T_B . The basic infrared function is

$$K(p, p', k; \lambda) = \frac{-8ig^2 \lambda^2 p \cdot p'}{(k^2 - \lambda^2)^2 (k^2 - 2k \cdot p)(k^2 + 2k \cdot p')} \quad (2.27)$$

This function has the symmetry properties

$$\begin{aligned} K(p, p', k; \lambda) &= K(p', p, -k; \lambda) \\ &= K(-p, -p', -k; \lambda). \end{aligned} \quad (2.28)$$

Equation (2.23) simply says that the IR-singular part of T_4 is obtained by making (inserted) infra-

red corrections to the Born term in all possible ways. One simply connects external quark lines in all possible ways.

III. THE CORNWALL-TIKTOPOULOS EQUATION AND ITS SOLUTION FOR QUARK-QUARK SCATTERING

The results of Sec. II suggest that the (at least the leading) IR singularities might have a simpler structure than is evident on casual inspection of the full array of extra graphs occurring in QCD. Indeed, Cornwall and Tiktopoulos have suggested,¹⁰ on the basis of a detailed examination of several examples, that the extensive cancellation of QCD graphs leads to the following simple rule for the leading IR singularities: attach inserted soft K gluons among all pairs of external lines. The corresponding infrared equation is just a generalization of the fourth-order calculation

$$\begin{aligned} \lambda \frac{\partial T}{\partial \lambda} &= \lambda \frac{\partial T_B}{\partial \lambda} + \int \frac{d^4 K}{(2\pi)^4} [K(-p_1, -p_2, k; \lambda) T(P, Q - k, R + k, \lambda) \tilde{t}_1 \cdot \tilde{t}_2 + K(p'_1, p'_2, k; \lambda) \tilde{t}_1 \cdot \tilde{t}_2 T(P, Q - k, R - k, \lambda) \\ &\quad - K(p_1, -p'_2, k; \lambda) \tilde{t}_1 \cdot T(P - k, Q + K, R, \lambda) \tilde{t}_2 + K(p'_1, -p_2, k; \lambda) \tilde{t}_2 \cdot T(P - k, Q - k, R, \lambda) \tilde{t}_1 \\ &\quad - K(p_1, -p'_1, k; \lambda) \tilde{t}_1 \cdot T(P - k, Q, R - k, \lambda) \tilde{t}_1 - K(p_2, -p'_2, k; \lambda) \tilde{t}_2 \cdot T(P - k, Q, R + k, \lambda) \tilde{t}_2]. \end{aligned} \quad (3.1)$$

We can cancel the common spin factor C [of Eq. (2.4)] from Eq. (3.1) for simplicity of writing. This can be seen by recalling Eq. (2.11) ff. and considering the iterated version of (3.1).

Equation (3.1) is advertised as an equation valid in leading-log approximation only. In this approximation it is not necessary to specify where the coupling constants occurring in T_B and in K are renormalized. But if, as is the case in QED, the equation is to have validity beyond leading log, it can only do so for a particular choice of g . In QED, the correct coupling constant is the physical one, renormalized at the physical electron mass and at zero photon mass, as is evident from the fact that the near-mass-shell virtual particles control the infrared singularities. In QCD the same is true, as is evident from the corrections of Figs. 5(h)–5(k) discussed in the preceding section, with only the modification that the photon mass at which the renormalization is made should be t in T_B and k^2 in K due to the singularity expected in $g(t)$ and $g(k^2)$ as t and k^2 approach zero. Thus one would conjecture that (3.1) is exactly true, to all orders (not just leading log) in the infrared singularity, if one uses $g(t)$ in T_B and $g(k^2)$ in K . If this is true, and if $g(k^2)$ really is

singular as $k^2 \rightarrow 0$, the leading-log results we shall obtain here could be drastically altered.

Equation (3.1) is of interest in two limits:

(a) fixed angle

$$s \rightarrow \infty, \quad s/t, s/u \text{ fixed}; \quad (3.2)$$

(b) fixed momentum transfer

$$s \rightarrow \infty, \quad t \text{ fixed}. \quad (3.3)$$

Our basic assumption is that the dominant contributions to the integrand of (3.1) come from the infrared region. Hence for fixed angle we can ignore the k dependence of the off-shell amplitude T

$$\begin{aligned} P \pm k &\approx P, \quad Q \pm k \approx Q, \\ R \pm k &\approx R \quad (\text{fixed } \theta), \end{aligned} \quad (3.4)$$

while for fixed $t = Q^2$ we must keep k relative to Q but can write²⁵

$$P \pm k \approx P, \quad R \pm k \approx R' \quad (\text{fixed } t). \quad (3.5)$$

Equation (3.1) is a pair of coupled equations for the amplitudes T_0 and T_1 . Using the identities (2.21) we get the coupled equations for fixed t (in the case of SU_2 gauge group²⁶)

$$\lambda \frac{\partial T_0}{\partial \lambda} = -\frac{3}{4} \int \frac{d^4 k}{(2\pi)^4} K_t(k, \lambda) T_0(P, Q, R, \lambda) + \frac{3}{4} \int \frac{d^4 k}{(2\pi)^4} [K_s(k, \lambda) - K_u(k, \lambda)] T_1(P, Q - k, R, \lambda), \quad (3.6)$$

$$\begin{aligned} \lambda \frac{\partial T_1}{\partial \lambda} = & \lambda \frac{\partial T_1^B}{\partial \lambda} + \frac{1}{4} \int \frac{d^4 k}{(2\pi)^4} [K_s(k, \lambda) - K_u(k, \lambda)] T_0(P, Q - k, R, \lambda) - \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} [K_s(k, \lambda) + K_u(k, \lambda)] T_1(P, Q - k, R, \lambda) \\ & + \frac{1}{4} \left(\int \frac{d^4 k}{(2\pi)^4} K_t(k, \lambda) \right) T_1(P, Q, R, \lambda), \end{aligned} \quad (3.7)$$

where the functions $K_{s,t,u}$ are defined by

$$\begin{aligned} K_s(k, \lambda) &= K(-p_1, -p_2, k; \lambda) + K(p'_1, p'_2, k; \lambda), \\ K_u(k, \lambda) &= K(-p_1, p'_2, k; \lambda) + K(p'_1, -p_2, k; \lambda), \\ K_t(k, \lambda) &= K(p_1, -p'_1, k; \lambda) + K(p_2, -p'_2, k; \lambda). \end{aligned} \quad (3.8)$$

It should be remembered that these functions depend on P , Q , and R . For fixed angle we can use (3.4) to simplify (3.6) and (3.7) as follows:

$$\lambda \frac{\partial T_0}{\partial \lambda} = -\frac{3}{2} B_t T_0 + \frac{3}{2} (B_s - B_u) T_1, \quad (3.9)$$

$$\lambda \frac{\partial T_1}{\partial \lambda} = \lambda \frac{\partial T_1^B}{\partial \lambda} + \frac{1}{2} (B_s - B_u) T_0 - (B_s + B_u) T_1 + B_t T_1. \quad (3.10)$$

The functions $B_{s,t,u}$, defined by

$$\begin{aligned} B_s &= \int \frac{d^4 k}{(2\pi)^4} K(p_1, p_2, k; \lambda), \\ B_u &= \int \frac{d^4 k}{(2\pi)^4} K(p_1, -p_2, k; \lambda), \\ B_t &= \int \frac{d^4 k}{(2\pi)^4} K(p_1, -p'_1, k; \lambda), \end{aligned} \quad (3.11)$$

have several equivalent forms due to the symmetry relations (2.25). They are the usual infrared factor due to attaching an external line (see Ref. 27) as shown by writing B_s in the form

$$\begin{aligned} B_s &= -ig^2 \lambda \frac{\partial}{\partial \lambda} \\ &\times \left(\int \frac{d^4 k}{(2\pi)^4} \frac{4p'_1 p_2}{(k^2 - \lambda^2)(k^2 - 2k \cdot p_1)(k^2 + 2k \cdot p_2)} \right). \end{aligned} \quad (3.12)$$

As $s \rightarrow \infty$, and for fixed angle, the asymptotic forms are

$$\begin{aligned} B_s &\rightarrow -(g^2/4\pi^2) \ln(-s/m^2), \\ B_u &\rightarrow -(g^2/4\pi^2) [\ln(s/m^2) + \ln \alpha_u], \\ B_t &\rightarrow -(g^2/4\pi^2) [\ln(s/m^2) + \ln \alpha_t], \end{aligned} \quad (3.13)$$

where $\alpha_u, \alpha_t = \frac{1}{2}(1 \pm \cos \theta)$. [When $m=0$, the scale in (3.13) is λ .]

The fixed-angle case was treated in detail in Ref. 17. We will not repeat this calculation but will

point out several features in common with the fixed- t case, which we now treat. As suggested by the perturbation calculations of Sec. II T_0 is smaller than T_1 by $1/\ln s$ in a given order of perturbation theory. Just as $(B_s + B_u)T_1$ is $O(\ln^2 s)$ larger than $(B_s - B_u)T_0$ in (3.10) we expect that the term $(K_s - K_u)T_0$ is negligible compared with $(K_s + K_u)T_1$ in Eq. (3.7). For fixed t the term $B_t T_1$ is $O(1/\ln s)$ times the $(K_s + K_u)T_1$ term (and in any event can be dealt with by an integrating factor.²⁸) A similar inspection of (3.6) shows that the first term $-\frac{3}{2}B_t T_0$ is expected to be $O(1/\ln s)$ times the term involving T_1 . Hence we adopt as an approximate leading-log version of Eqs. (3.6) and (3.7) the following:

$$\lambda \frac{\partial T_0}{\partial \lambda} = \frac{3}{4} \int \frac{d^4 k}{(2\pi)^4} (K_s - K_u) T_1(P, Q - k, R), \quad (3.14)$$

$$\lambda \frac{\partial T_1}{\partial \lambda} = \lambda \frac{\partial T_1^B}{\partial \lambda} - \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} (K_s + K_u) T_1(P, Q - k, R). \quad (3.15)$$

The structure of the analogous (single) equation for QED is not so simple because all terms are of the same order of magnitude due to cancellation rather than enhancement of the leading (logs) terms associated with the s - and u -channel terms. We have not been able to solve that equation²⁹; indeed, the equation is probably not even justified, due to this cancellation of the leading behavior, for the purpose of finding the dominant s dependence of the amplitude. Hence the QCD scattering amplitude behaves more simply than its counterpart in QED. This simplicity is due to the disposition of signs associated with the non-Abelian gauge group.

Since Eqs. (3.14) and (3.15) have convolution structure³⁰ we go over to coordinate space, obtaining (suppressing the fixed- P and fixed- R dependence)

$$\lambda \frac{\partial}{\partial \lambda} T_0(x, \lambda) = \frac{3}{4} K_-(x, \lambda) T_1(x, \lambda), \quad (3.16)$$

$$\lambda \frac{\partial}{\partial \lambda} T_1(x, \lambda) = \lambda \frac{\partial}{\partial \lambda} T_1^B - \frac{1}{2} K_+(x, \lambda) T_1(x, \lambda), \quad (3.17)$$

where $K_{\pm} \equiv K_s \pm K_u$.

Equation (3.17) has the form (t is $\ln\lambda$)

$$\frac{\partial f}{\partial t} = g(t) + k(t)f(t), \tag{3.18}$$

so that the solution is (apart from the solution to the homogeneous equation, $f_0 = \text{const} \times \exp[\int_{t_0}^t k(t')dt']$)

$$f(t) = \int_{t_0}^t dt' g(t') \exp\left(\int_{t'}^t k(t'')dt''\right). \tag{3.19}$$

In our case g is a derivative $\partial B/\partial t$ and (3.19) can be written

$$f(t) = B(t) + \int_{t_0}^t dt' k(t')G(t, t'), \tag{3.20}$$

where $G(t, t')$ is defined to be $\exp[\int_{t'}^t k(t'')dt'']$.

Applying the result (3.20) to Eq. (3.17) gives

$$T_1(P, Q, R, \lambda) = T_1^B(P, Q, R, \lambda) - \frac{1}{2} \int d^4x e^{iQ \cdot x} \times \int_{\lambda_0}^{\lambda} \frac{d\lambda'}{\lambda'} T_1^B(P, x, R, \lambda') K_+(x, \lambda') \times \exp\left[-\frac{1}{2} \int_{\lambda'}^{\lambda} \frac{d\lambda''}{\lambda''} K_+(x, \lambda'')\right]. \tag{3.21}$$

The choice of lower limit λ_0 is arbitrary since changing it only changes the definition of the infrared-finite part of the amplitude. Although the original equation was only valid when $\lambda^2 \ll m^2, -t, s$ we can choose $\lambda_0 = \infty$ for convenience. This has the effect of suppressing completely the contribution from the lower limit. For example, the first non-trivial term in the expansion of (3.21) gives the exact fourth-order s and u K -photon box diagrams (2.13) and (2.15) with this choice of λ .

T_0 is found by direct integration of Eq. (3.16):

$$T_0(P, Q, R; \lambda) = \frac{3}{4} \int d^4x e^{iQ \cdot x} \int_{\infty}^{\lambda} \frac{d\lambda'}{\lambda'} K_-(x, \lambda') \times T_1(P, x, R, \lambda'). \tag{3.22}$$

The explicit solutions (3.21) and (3.22) have a simple form in coordinate space, which probably merits further study. Here we show how to make contact with ordinary leading-log perturbation theory by expanding the exponential and working out the asymptotic behavior term by term. In (3.21) we express $T_B(x)$ and $K_+(x)$ in terms of Fourier coefficients and perform the x integration to get

$$T_1 = T_1^B + \frac{g^2}{4} \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n(n-1)!} \int_{\lambda_0}^{\lambda} \frac{d\lambda_1}{\lambda_1} \int_{\lambda_1}^{\lambda} \frac{d\lambda_2}{\lambda_2} \dots \int_{\lambda_{n-1}}^{\lambda} \frac{d\lambda_n}{\lambda_n} \int \frac{d^4k_1}{(2\pi)^4} \dots \int \frac{d^4k_n}{(2\pi)^4} \frac{K_+(k_1, \lambda_1) \dots K_+(k_n, \lambda_n)}{(Q - k_1 - k_2 - \dots - k_n)^2 - \lambda_1^2}, \tag{3.23}$$

$$\frac{4}{g^2} T_1 = \frac{1}{t - \lambda^2} - \frac{1}{2} \int_{\lambda_0}^{\lambda} \frac{d\lambda_1}{\lambda_1} \int \frac{d^4k_1}{(2\pi)^4} \frac{K_+(k_1, \lambda_1)}{(Q - k_1)^2 - \lambda_1^2} + \left(-\frac{1}{2}\right) \int_{\lambda_0}^{\lambda} \frac{d\lambda_1}{\lambda_1} \int_{\lambda_1}^{\lambda} \frac{d\lambda_2}{\lambda_2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{K_+(k_1, \lambda_1) K_+(k_2, \lambda_2)}{(Q - k_1 - k_2)^2 - \lambda_1^2} + \dots \tag{3.24}$$

The second term of (3.24) is easily integrated explicitly. It yields the exact fourth-order s - and u -channel box graphs with one K photon inserted (cf. Sec. II), as of course has to be true according to our previous discussion. Its asymptotic value is

$$\frac{4}{g^2} T_1^{(4)} = \frac{-g^2}{4\pi^2 t} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{t}{\lambda^2}\right). \tag{3.25}$$

The asymptotic form of

$$\tau(P, Q, R, \lambda) \equiv \int \frac{d^4k}{(2\pi)^4} \int_{\infty}^{\lambda} \frac{d\lambda'}{\lambda'} \frac{K_+(k, \lambda')}{(Q - k)^2 - \lambda'^2} \tag{3.26}$$

is therefore $(g^2/2\pi^2 t) \ln(s/m^2) \ln(-t/\lambda^2)$; cf. Eq. (2.22).

If we interchange λ integrations in the last term of (3.24) according to the rule

$$\int_{\infty}^{\lambda} d\lambda_1 \int_{\lambda_1}^{\lambda} d\lambda_2 = \int_{\infty}^{\lambda} d\lambda_2 \int_{\infty}^{\lambda_2} d\lambda_1, \tag{3.27}$$

we find that this (sixth-order) term has the form

$$\frac{4}{g^2} T_1^{(6)} = \left(-\frac{1}{2}\right) \int_{\infty}^{\lambda} \frac{d\lambda'}{\lambda'} \int \frac{d^4k}{(2\pi)^4} K_+(k, \lambda') \times \tau(P, Q - k, R; \lambda'). \tag{3.28}$$

Except near $k \approx Q$ and large λ' (which is unphysical since we expect τ to cut off around $\lambda^2 \approx -t$) we can use the asymptotic form of τ . Hence we write (3.28) as

$$\frac{4}{g^2} T_1^{(6)} = \left[-\frac{g^2}{4\pi^2} \ln\left(\frac{s}{m^2}\right)\right] \times \left(-\frac{1}{2}\right) \int_{\infty}^{\lambda} \frac{d\lambda'}{\lambda'} \ln\left(\frac{-t}{\lambda'^2}\right) \int \frac{d^4k}{(2\pi)^4} \frac{K_+(k, \lambda')}{(Q - k)^2}. \tag{3.29}$$

To investigate the nature of the singularity, note that for large s the t dependence of τ is actually of the form

$$\int_0^1 \frac{dx}{\lambda^2 - tx(1-x)} = \frac{-2}{(t^2 - 4t\lambda^2)^{1/2}} \ln\left(\frac{(t^2 - 4\lambda^2 t)^{1/2} + t}{(t^2 - 4\lambda^2 t)^{1/2} - t}\right). \tag{3.30}$$

For $-t \gg \lambda^2$ this becomes $-2 \ln(-t/\lambda^2)/t$ as before. For small $-t \ll \lambda^2$ the t pole goes away, giving $2/\lambda^2$. So we can fix up the pole by writing $(Q-k)^2 - (Q-k)^2 - p\lambda'^2$ with $p=O(1)$ in (3.29) as an interpolation formula. The numerator is smooth so we have simply dropped the k dependence in $\ln[-(Q-k)^2/\lambda^2]$. It is now straightforward to express (3.29) in terms of the fourth-order box contribution. Comparing with (3.26), we have for the integrals in (3.29)

$$\int^\lambda d\lambda' \frac{\partial \tau_p}{\partial \lambda'} \ln\left(\frac{-t}{\lambda'^2}\right) = \tau_p \ln\left(\frac{-t}{\lambda^2}\right) - \int^\lambda \tau_p d \ln\left(\frac{-t}{\lambda'^2}\right) \approx \frac{g^2}{4\pi^2 t} \ln\left(\frac{s}{m^2}\right) \ln^2\left(\frac{-t}{\lambda^2}\right). \quad (3.31)$$

Note that τ_p differs from the usual box in that one of the propagators has its pole at $p\lambda^2$ instead of λ^2 . Evaluation shows that the $\ln(-t/\lambda^2)$ is replaced by $\ln(-t/\lambda\mu)$ if the two poles are at λ^2 and μ^2 respectively. Hence the asymptotic behavior is unchanged; we continue to use λ as the length scale. Combining with the factors in (3.29) gives

$$\frac{4}{g^2} T_1^{(6)} = \frac{1}{2t} \left[-\frac{g^2}{4\pi^2} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{-t}{\lambda^2}\right) \right]^2. \quad (3.32)$$

The first three terms now suggest exponentiation of the series

$$T_1 = T_1^B \left\{ 1 - \frac{g^2}{4\pi^2} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{-t}{\lambda^2}\right) + \frac{1}{2!} \left[-\frac{g^2}{4\pi^2} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{-t}{\lambda^2}\right) \right]^2 + \dots \right\}. \quad (3.33)$$

We now extend (3.33) to all orders by using T_6 to evaluate T_8 , etc. in the manner just described.

Repeating the calculation of Eqs. (3.28)–(3.32) for $T_1^{(8)}$ using the identity

$$\int_\infty^\lambda d\lambda_1 \int_{\lambda_1}^\lambda d\lambda_2 \int_{\lambda_2}^\lambda d\lambda_3 = \int_\infty^\lambda d\lambda_3 \int_\infty^{\lambda_3} d\lambda_2 \int_\infty^{\lambda_2} d\lambda_1 \quad (3.34)$$

gives the expected next term in (3.33) and shows

$$T_0 = -\frac{3ig^2}{16\pi} T_1^B \ln\left(\frac{-t}{\lambda^2}\right) \sum_{n=0}^{\infty} \left(\frac{-g^2}{4\pi}\right)^n \ln^n\left(\frac{s}{m^2}\right) \ln^n\left(\frac{-t}{\lambda^2}\right) / (n+1)! \\ = \frac{3i\pi}{4} T_1^B \frac{\exp\left[(-g^2/4\pi^2) \ln(s/m^2) \ln(-t/\lambda^2)\right] - 1}{\ln(s/m^2)}. \quad (3.41)$$

Again, this formula agrees with perturbation evaluations to eighth order³¹ in the limit of small gluon mass.

Note that T_0 has a fixed cut at $j-1$ in addition to a moving Regge cut located at the same position as the Regge pole in T_1 .

If we take the limit $\lambda \rightarrow 0$ in (3.41) there is a con-

stant surviving term; T_0 then has the form

$$T_1 = T_1^B \exp\left[\frac{-g^2}{4\pi^2} \ln\left(\frac{s}{m^2}\right) \ln\left(\frac{-t}{\lambda^2}\right)\right]. \quad (3.35)$$

This has a Regge form as confirmed in low orders by other authors.^{5,6}

T_0 is evaluated using Eq. (3.22) rewritten in the form

$$T_0(P, Q, R; \lambda) = \frac{3}{4} \int_\infty^\lambda \frac{d\lambda'}{\lambda'} \int \frac{d^4 k}{(2\pi)^4} K_-(k, \lambda') \times T_1(P, Q-k, R, \lambda'). \quad (3.36)$$

Introducing the expansion of (3.35) and treating the $k=Q$ singularity as before, we obtain

$$\frac{4}{g^2} T_0(P, Q, R; \lambda) = \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{-g^2}{4\pi} \ln\left(\frac{s}{m^2}\right) \right]^n \times \int^\lambda \frac{d\lambda'}{\lambda'} \left[\ln\left(\frac{-t}{\lambda'^2}\right) \right]^n \times \int \frac{d^4 k}{(2\pi)^4} \frac{K_-(k, \lambda')}{(Q-k)^2 - p\lambda'^2}. \quad (3.37)$$

Defining the function

$$M(Q, \lambda) = \int_\infty^\lambda \frac{d\lambda'}{\lambda'} \int \frac{d^4 k}{(2\pi)^4} \frac{K_-(k, \lambda')}{(Q-k)^2 - p\lambda'^2}, \quad (3.38)$$

the integral in (3.37) can be written as

$$\ln^n\left(\frac{-t}{\lambda^2}\right) M(Q, \lambda) + 2n \int^\lambda \frac{d\lambda'}{\lambda'} \ln^{n-1}\left(\frac{-t}{\lambda'^2}\right) M(Q, \lambda'). \quad (3.39)$$

The function $M(Q, \lambda)$ is the difference of the s - and u -channel box contributions and so involves $\ln(-s) - \ln s = -i\pi$:

$$M(Q, \lambda') = \frac{-i\pi g^2}{t 4\pi^2} \ln\left(\frac{-t}{\lambda'^2}\right). \quad (3.40)$$

Evaluating the leading part of (3.39) now allows (3.37) to be written as

stant surviving term; T_0 then has the form

$$T_0 = \text{const} \times i \frac{s}{t \ln s} \quad (3.42)$$

on reinstating the spin factor of the Born term. Thus we obtain a nonzero amplitude for the scattering of colored particles. This does not quite

contradict the statements of Cornwall and Tik-topoulous since T_0 is nonleading relative to T_1 . Moreover, the vanishing exponential is really the sum of a series whose terms are divergent as $\lambda \rightarrow 0$ and one can wonder if it makes sense to keep the -1 . We expect, however, to find the same structure for the finite off-shell amplitude so that the infrared-finite factor is to be taken seriously.

Although (3.41) clearly is useful as an explicit amplitude for the small, but finite, λ case its meaning in the truly massless gluon case depends on whether external gluon emission occurs. If not, then the $1/\ln s$ term corresponds to "physics." If so, then the divergent radiation amplitude will presumably cancel the vanishing exponential term in (3.41). This interpretational difficulty should disappear in a correct off-shell treatment of the quarks in a bound-state problem. This approach is currently under investigation.

It is interesting that the (nonleading) amplitude T_0 is given correctly (i.e., in accordance with perturbation theory) by the infrared equation. The differential equation approach is so much simpler than brute-force summation of Feynman graphs that this approach deserves further work to determine more clearly the range of validity of the CT equation. In fact, comparison with the leading diagrams even in sixth-order (cf. especially Ref. 6) which all involve gluon-gluon couplings make the agreement appear somewhat miraculous.

IV. THE INFRARED-FINITE PART

The discussion in the preceding sections suggests (though it does not prove) that the quark-quark scattering amplitude can be factored, as in QED, into the product of an infrared-finite part and an infrared-singular part:

$$T = T_{\text{IR}} \bar{T}, \quad (4.1)$$

where T_{IR} contains all dependence on the gluon mass λ and \bar{T} is independent of λ . Equation (4.1) is presumed written for the unrenormalized amplitude; thus T depends on the bare coupling g_0 and an ultraviolet cutoff Λ . If T is the color-flip amplitude T_1 , the lowest-order contribution \bar{T}_1 is the Born term T_B ; if T is the nonflip amplitude T_0 , the lowest contribution to \bar{T}_0 is of order g_0^4 . In fact, if the infrared-singular factor T_{IR} is defined³² through the K, G gluon decomposition outlined in Sec. II, the lowest-order contribution to \bar{T}_0 is

precisely the sum of the box graph and the crossed-box graph calculated with a G gluon for one (not both) of the exchanged gluon lines, while the lowest-order contribution to the infrared-singular factor consists of the same boxes with a K gluon inserted for one (not both) of the exchanged gluon lines. Further, with our explicit choice of the K -gluon propagator, T_{IR} is ultraviolet finite so it is independent of the ultraviolet cutoff Λ . Conversely, \bar{T} is independent of the infrared cutoff λ ,³³ if the IR divergence of the coupling-constant renormalization³⁴ is included in T_{IR} .

Upon renormalization \bar{T} becomes replaced by a renormalized amplitude depending on a renormalization point M and a coupling constant g_M and is no longer dependent on Λ . Furthermore, \bar{T} (since it is independent of λ) will satisfy a (new improved) renormalization-group equation²¹:

$$\left(M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + \gamma(g) + \hat{\gamma}(g) \bar{m} \frac{\partial}{\partial \bar{m}} \right) \bar{T} = 0 \quad (4.2)$$

(\bar{m} here is related to, but not identical to, the quark mass). As a result, we can obtain the asymptotic behavior, at fixed angle, of \bar{T} from asymptotic freedom provided that there are no mass singularities [i.e., factors like $\ln(s/m^2)$] in the perturbation expansion of \bar{T} .³⁵

In fourth order, direct evaluation of the box diagram with one G gluon shows it to contain mass singularities of the form $\ln(t/m^2)$, but not $\ln(s/m^2)$. [The box with one K gluon, in contrast, does contain terms like $\ln(s/m^2)$ as we have already seen; but this contributes to T_{IR} , not \bar{T} .] In the amplitude T_0 , the crossed box with one G gluon exactly cancels this mass singularity so that the entire \bar{T}_0 amplitude, in order g^4 , contains no mass singularities. This does *not* hold for T_1 ; \bar{T}_1 *does* have mass singularities in order g^4 .

Let us elaborate a bit on these remarks. First, look at the box graph with one G gluon. This is proportional to (for SU_2 as the gauge group)

$$g^4 (\vec{\tau}_1 \cdot \vec{\tau}_2)^2 \int d^4 k \frac{1}{k^2} \frac{1}{(Q+k)^2} \frac{1}{(p_1-k)^2 - m^2} \frac{1}{(p^2+k)^2 - m^2} \\ \times \{ [\gamma_\mu (\not{p}_1 - \not{k} + m) \gamma_\nu]_1 [\gamma_\mu (\not{p}_2 + \not{k} + m) \gamma_\nu]_2 - 4p_1 \cdot p_2 \}. \quad (4.3)$$

Upon inserting Feynman parameters α_i and shifting the integration variable k this becomes (dropping an obviously finite term as $m \rightarrow 0$)

$$g^4 (\vec{\tau}_1 \cdot \vec{\tau}_2)^2 \int_0^1 d\alpha_2 d\alpha_3 d\alpha_4 \theta(1 - \alpha_2 - \alpha_3 - \alpha_4) \frac{1}{[s\alpha_3\alpha_4 + t\alpha_2(1 - \alpha_2 - \alpha_3 - \alpha_4) - m^2(\alpha_3 + \alpha_4)^2]^2} \\ \times \{ 4p_1 \cdot p_2 [-\alpha_2 - \alpha_3 + \alpha_2\alpha_3 - 2\alpha_4(1 - \alpha_2 - \alpha_3 - \alpha_4)] (\gamma_\mu)_1 (\gamma_\mu)_2 \\ + \alpha_3\alpha_4 (\gamma_\mu \not{p}_2 \gamma_\nu)_1 (\gamma_\mu \not{p}_1 \gamma_\nu)_2 + 8\alpha_4(1 - \alpha_2 - \alpha_3 - \alpha_4) (\not{p}_2)_1 (\not{p}_1)_2 \}. \quad (4.4)$$

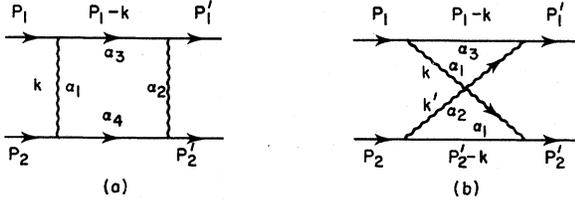


FIG. 6. The fourth-order box graphs are labeled with momenta and Feynman parameters to facilitate discussion of the mass singularities.

Inspection of the various numerator terms in (4.4) shows us that the only term which can be singular as $m \rightarrow 0$ is that proportional to $(\alpha_1 + \alpha_2)$. Evaluating this term gives the result, for the $m \rightarrow 0$ singularity,

$$g^4 (\vec{t}_1 \cdot \vec{t}_2)^2 \frac{1}{t} (\gamma_\mu)_1 (\gamma_\nu)_2 \{ [\ln(t/m^2)]^2 + [\ln(s/t)]^2 \}. \quad (4.5)$$

The corresponding crossed graph gives

$$g^4 \vec{t}_1 \cdot (\vec{t}_1 \cdot \vec{t}_2) \vec{t}_2 \frac{1}{t} (\gamma_\mu)_1 (\gamma_\nu)_2 \{ [\ln(t/m^2)]^2 + [\ln(u/t)]^2 \} \quad (4.6)$$

The isospin factors [of Eq. (2.20)] are such that these two subtract in \bar{T}_0 and add in \bar{T}_1 . Thus the m singularities in the box and crossed box cancel in \bar{T}_0 , but do not in \bar{T}_1 . The surviving factors in T_0 involve $\ln(s/t)$ and $\ln(u/t)$; these are, of course, finite in the fixed-angle limit. The same structure persists in SU_n .

The other fourth-order graphs involve vertex and propagator corrections. These are all individually finite as $m \rightarrow 0$. Thus, in general, to order g^4 , \bar{T}_0 has no mass singularities but \bar{T}_1 does.

The cancellation of the leading $m^2 \rightarrow 0$ singularity for the color-singlet exchange amplitude is in fact systematic and easy to see in higher orders. The mechanism can be seen in the box and crossed-box diagrams, Eqs. (4.5) and (4.6). These expressions differ only by the exchange $p_2 \rightarrow p'_2$ (i.e., $s \rightarrow u$), an overall sign, and the color factor. The singlet pieces of these color factors are however, identical. If one can show in general that the m^2 dependence is governed by s or u , and the λ^2 dependence governed by t , then the overall sign difference between the two graphs will give cancellation of the leading behavior in any order of perturbation theory. To see that the t dependence is scaled by λ^2 and the s (or u) dependence is scaled by m^2 , we write the complete box [Fig. 6(a)] in Feynman parametrized form. When the k -loop integral is performed, the denominator in the integral over Feynman parameter space is then the second power of

$$s\alpha_3\alpha_4 + t\alpha_1\alpha_2 - m^2(\alpha_3 + \alpha_4)^2 - \lambda^2(\alpha_1 + \alpha_2), \quad (4.7)$$

as already noted in Eq. (4.4). The fact that t is scaled by λ^2 is a simple consequence of their common Feynman parameters in Eq. (4.7), and we find a similar situation for s and m^2 . This fact would continue to be true in the IR-finite behavior of the two graphs under discussion, because one requires a modification (by, e.g., the K, G photon method discussed above) of the α_1 and α_2 propagators. The color-singlet exchange, IR-finite piece, of double-gluon exchange is therefore singularity-free.

While we cannot show the complete cancellation of singularities in higher order for the color-singlet exchange amplitude, we can show that the leading logarithmic singularities cancel, in pairs of graphs. We demonstrate the technique for sixth order; extension is straightforward and perhaps provable by induction. Figure 7 shows the sixth-order graphs considered. (Pure form factor or t -channel pole terms do not contribute to the color-singlet exchange amplitude, and in any case are nonsingular for the finite- t limit. Other graphs are similarly uninteresting.)

Graphs 7(a) and 7(b) (or rather the color-singlet pieces of them) again differ by an overall sign and the exchange $s \rightarrow u$. For the scale of the logarithmic factors, we need consider only a ϕ^3 analog, and perform the k_1, k_2 loop integrations exactly after applying the Feynman parametric identity. With the parametric labeling as in Fig. 7(a) we need to notice only that the coefficient of m^2 in the remaining denominator vanishes only with $(\alpha_5, \alpha_6, \alpha_7)$, and at the same rate as the coefficient of s (namely, quadratically with any pair, as for the fourth-order case). Similarly, t and λ^2 are controlled by the set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. The cancellation of leading logs between Figs. 7(a) and 7(b) then follows as for the fourth-order case.

Also in Fig. 7, (a)=(b), (c)=(d), and (c)=(f) with $s \rightarrow u$ with no color factors. For these graphs, the color algebra gives us the required minus sign in the above equalities in the color-singlet piece only. In this way, we again need only check the controlling kinematic variable in the $m^2 \rightarrow 0$ limit. The same calculation outlined above shows the required relationship: The coefficients of both s and m^2 vanish quadratically with pairs of $(\alpha_4, \alpha_5, \alpha_6, \alpha_7)$ (see Fig. 7) while the t and λ^2 dependence is governed by $(\alpha_1, \alpha_2, \alpha_3)$.

We finish by noting that it is really the large- s and not the $m^2 \rightarrow 0$ limit which is of interest, and that this difference can be nontrivial. The pair of tower graphs in Fig. 8 bear the same relationship to each other as the graphs of Fig. 6. When parametrized as in Fig. 8, the s coefficient is governed by $\alpha_7\alpha_{10}$. While α_7 and α_{10} do come into the m^2 coefficient, clearly so do $(\alpha_5, \alpha_6, \alpha_8, \alpha_9)$

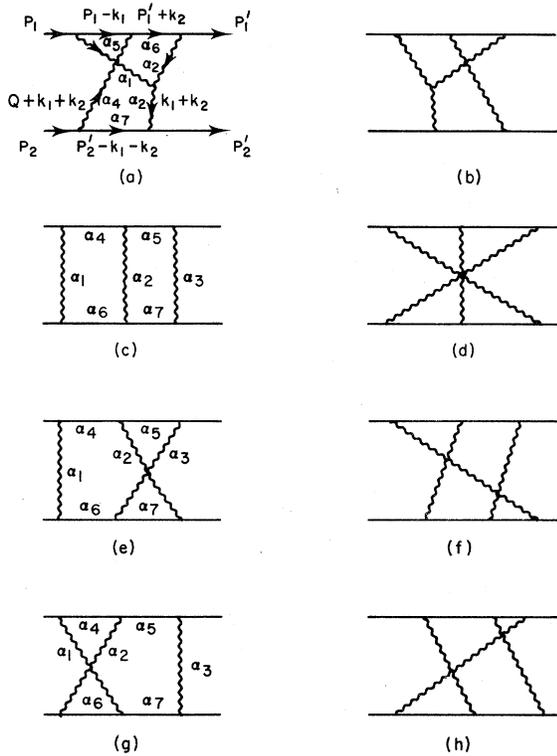


FIG. 7. Some sixth-order graphs pertinent to the mass-singularity problem are shown.

from the internal loop. However, this is enough to ensure that the leading logarithmic s dependence will be scaled by m^2 , not λ^2 .

We are therefore led to conjecture that \bar{T}_0 has no mass singularities in any order. Hence Eq. (4.2) permits us to obtain the fixed-angle asymptotic form of \bar{T}_0 :

$$\bar{T}_0 \rightarrow [-b \ln(s/\mu^2)]^{-2-c/2b} \bar{T}_0^{(4)}(\theta), \quad (4.8)$$

where μ^2 is the renormalization-group invariant mass, defined by

$$\mu^2 = M^2 e^{1/bg_M^2} \quad (4.9)$$

so that the running coupling constant is $g_s = 1/[-b \ln(s/\mu^2)]^{1/2}$. Here $\beta = bg^3 + \dots$ and $\gamma = cg^2 + \dots$ define b and c , and finally, $\bar{T}_0^{(4)}(\theta)$ is the fixed-angle limit of the fourth-order color-nonflip amplitude with one exchanged G gluon which is a function only of the scattering angle θ .

Several authors^{4, 34, 36-38} have used the renormalization group to investigate mass-shell scattering amplitudes of theories having tame infrared behavior. The present argument purports to strip away the troublesome infrared singularities in a manner allowing similar arguments for the infrared-finite part of the amplitude.

V. CONCLUSIONS

It must be stressed that our results do not necessarily have a lot to do with the properties of the Pomeron, which describes the near forward scattering of physical hadrons. To apply our results to this problem would require proof that the overall process is describable by the contributions of individual qq (and also $q\bar{q}$) scattering amplitudes. However, consideration of the scattering of quark clusters immediately reveals the existence of a host of potentially important graphs not included in the "additive" quark picture. For fixed t we could calculate the infrared dominant part of qq or $q\bar{q}$ scattering but could not estimate the subdominant terms. Since the infrared singularities cancel^{8, 10, 17} for color-singlet scattering, as is physically obvious whether confinement occurs or not, the amplitudes obtained here do not bear a close relation to the physical ones for near forward scattering. For fixed angle the situation is more hopeful, both because we could obtain part of the noninfrared determined energy dependence and because the kinematical situation is simpler.³⁹

Thus the gauge theory description of the Pomeron remains in a primitive state. Besides Ref. 8, we refer to the papers of Low⁴⁰ and Nussinov⁴¹ for further ideas. We are presently investigating colorless current-current scattering in the hope of obtaining information about the Pomeron without confronting the infrared singularities directly.

Our results do, however, confirm a more detailed agreement of the infrared equation of Cornwall and Tiktopoulos with perturbation calculations to rather high order. Since the remarkable cancellations among individual graphs show that the perturbation expansion is at best exceedingly clumsy, we are encouraged to study further non-perturbative approaches to the infrared problem. These can be of several varieties. For example, we can exploit the conjecture that the equation of Cornwall and Tiktopoulos is exact with the use of coupling constants $g(t)$ and $g(k^2)$ renormalized at

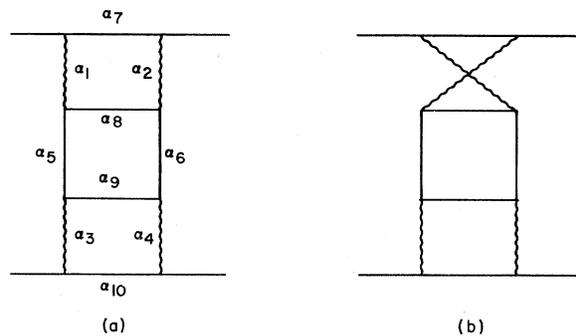


FIG. 8. These tower graphs are discussed in the text.

gluon masses t and k^2 as discussed in Sec. III. We can (in the absence of any theoretical guidance as to how these behave) guess some singular behavior such as $g^2(k^2) \propto k^{-2}$, and see what effect this will have on high-energy fixed-angle quark-quark scattering. It will certainly modify our leading-log results in a nontrivial way. Or, we can study

other ways of regularizing the infrared singularity, in particular by dealing with off-shell quark-quark scattering, since this is obviously what is relevant in any physically interesting process.

Either of these (and doubtless other) nonperturbative studies will be of interest, and we hope to explore them in the future.

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¹C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).

²E. Poggio and H. Quinn, Phys. Rev. D **12**, 3279 (1975).

³J. G. Halliday, J. Huskins, and C. T. Sachrajda, Nucl. Phys. **B83**, 189 (1974).

⁴C. G. Callan, Jr., and D. Gross, Phys. Rev. D **11**, 2905 (1975).

⁵B. M. McCoy and T. T. Wu, Phys. Rev. D **12**, 3257 (1975).

⁶L. Tyburski, Phys. Rev. Lett. **37**, 319 (1976); Phys. Rev. D **13**, 1107 (1976).

⁷J. Carazzone, E. Poggio, and H. Quinn, Phys. Rev. D **11**, 2286 (1975).

⁸C. D. Stockham, Phys. Rev. D **15**, 1736 (1977).

⁹H. T. Nieh and Y.-P. Yao, Phys. Rev. D **13**, 1082 (1976).

¹⁰J. M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. **35**, 338 (1975); Phys. Rev. D **13**, 3370 (1976); in *New Pathways in High Energy Physics*, edited by A. Perlmutter (Plenum, New York, 1976), Vol. II, p. 213.

¹¹T. Appelquist, J. Carazzone, H. Kluberg-Stern, and M. Roth, Phys. Rev. Lett. **36**, 768 (1976); **36**, 1161(E) (1976).

¹²E. Poggio and H. Quinn, Phys. Rev. D **14**, 578 (1976).

¹³Y.-P. Yao, Phys. Rev. Lett. **36**, 653 (1976).

¹⁴D. Crewther, Caltech Report No. CALT 68-556 (unpublished).

¹⁵S. J. Chang and S. K. Ma, Phys. Rev. **188**, 2385 (1969).

¹⁶H. Cheng and T. T. Wu, Phys. Rev. Lett. **24**, 1456 (1970), and references therein.

¹⁷P. Carruthers and F. Zachariassen, Phys. Lett. **62B**, 338 (1976); in *New Pathways in High Energy Physics* (Ref. 10), Vol. II, p. 265.

¹⁸C. Korthals-Altes and E. de Rafael, Nucl. Phys. **B106**, 237 (1976).

¹⁹T. Kinoshita and A. Ukawa, Phys. Rev. D **13**, 1573 (1976); Cornell report (unpublished). These authors show that the leading infrared singular diagrams are different in the asymptotic and nonasymptotic cases.

²⁰G. Grammer and D. R. Yennie, Phys. Rev. D **8**, 4332 (1973).

²¹S. Weinberg, Phys. Rev. D **8**, 3497 (1973).

²²Our conventions are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1969), Vol. I, except that our spinors obey $\bar{u}u = 2m$.

²³Grammer and Yennie (Ref. 32) show how to isolate the infrared singularities in fourth-order scattering, but

their approximation discards some of the s, t dependence in which we are interested, as is shown by a more detailed analysis of the fourth-order box graphs.

²⁴The k dependence of this factor was chosen differently in Refs. 17 and 20. Our choice is motivated by a desire to cleanly separate the IR and UV divergence in a given Feynman graph.

²⁵Neglecting the k dependence here would give the correct λ dependence but loses the correct (s, t) dependence as mentioned in footnote 26.

²⁶For SU_3 the coefficients $(-\frac{3}{4}, \frac{3}{4})$ in Eq. (3.6) become $(-\frac{4}{3}, \frac{3}{3})$. In Eq. (3.7) the first equation has the same coefficient for the $K_s - K_u$ term, while $-\frac{1}{2}(K_s + K_u)$ is replaced by $-\frac{1}{12}(2K_s + 7K_u)$ and $\frac{1}{4}K_t$ becomes $\frac{1}{12}K_t$. Note that the order-of-magnitude arguments for the various terms in Eqs. (3.6) and (3.7) are still valid when we change the gauge group from SU_2 to SU_3 .

²⁷D. R. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961).

²⁸By writing $T_0 = f_0 \exp(-\frac{3}{2} B_t \ln \lambda)$ we eliminate the T_0 term from the equation for $\lambda \partial f_0 / \partial \lambda$; which then can be solved by direct integration once T_1 has been found. But $T_0 = \exp(-\frac{3}{2} B_t \ln \lambda) \cong 1 - (3g^2/8\pi^2) \ln(-t/m^2) \ln \lambda$ is not significant as $s \rightarrow \infty$, t fixed and will be ignored here.

²⁹The QED equation for fixed t , $s \rightarrow \infty$ is

$$\lambda \frac{\partial T}{\partial \lambda} = \lambda \frac{\partial T_B}{\partial \lambda} + \int \frac{d^4 k}{(2\pi)^4} (K_s - K_u) T(P, Q - k, R) - \left(\int \frac{d^4 k}{(2\pi)^4} K_t \right) T(P, Q, R).$$

³⁰To be sure, K_s, K_u have a slight Q dependence but this is negligible in the present limit. Equations (3.16) and (3.17) remain valid in the general case but with $K_{\pm}(x, \lambda)$ replaced by

$$\tilde{K}_{\pm}(x, \lambda) = \int d^4 x' \int \frac{d^4 Q}{(2\pi)^4} e^{iQ \cdot (x' - x)} K_{\pm}(Q, x', \lambda).$$

³¹We are indebted to Professor B. McCoy and Professor T. T. Wu for conversations and correspondence on this point. See also C. Y. Lo and H. Cheng, Phys. Rev. D **13**, 1131 (1976).

³²The generalization of the K, G trick has apparently not been made when gluons couple gluons rather than quarks. However, Eq. (3.1) (the result of extensive cancellation of infrared singularities of graphs including this type) says that the surviving infrared singular contributions involve gluons attached to external quark lines only.

³³While, as we have already remarked, it seems evident that T can have little or no direct physical significance because it contains infrared singularities, there is some hope that \bar{T} might. If the incorporation of T (with virtual quarks) into a hadronic (or indeed any physical) scattering amplitude removes the IR-sin-

gular factor (the IR cutoff λ is replaced by factors like $p^2 - m^2$ for off-shell quarks of momentum p) it is conceivable that the finite factor \bar{T} will be exposed. Thus the asymptotic behavior of \bar{T} may be of real interest.

³⁴G. Tiktopoulos, Phys. Rev. D 11, 2252 (1975).

³⁵For off-shell external quarks, the expression (1) becomes

$$s\alpha_3\alpha_4 + t\alpha_1\alpha_2 + m_1^2\alpha_1\alpha_3 + m_2^2\alpha_2\alpha_3 + m_2^2\alpha_1\alpha_4 \\ + m_2^2\alpha_2\alpha_4 - m^2(\alpha_3 + \alpha_4) - \lambda^2(\alpha_1 + \alpha_2).$$

The linear rather than quadratic $(\alpha_3 + \alpha_4)$ coefficient

of m^2 indicates a less singular m^2 dependence for off-shell as opposed to on-shell amplitudes. See the penultimate paragraph in Sec. III.

³⁶N. Khuri, Phys. Rev. D 14, 2665 (1976).

³⁷H. B. Thacker and W. I. Weisberger, Phys. Rev. D 14, 2658 (1976).

³⁸K. Nishijima, T.-N. Pharn, and Z. F. Ezawa (unpublished).

³⁹S. J. Brodsky and G. R. Farrar, Phys. Rev. D 11, 1309 (1975).

⁴⁰F. E. Low, Phys. Rev. D 12, 163 (1975).

⁴¹S. Nussinov, Phys. Rev. D 14, 246 (1976).