Radially separated monopole solutions in non-Abelian gauge models*

Louis Michel and Lochlainn O'Raifeartaigh†
Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette 91440, France

Kameshwar C. Wali
Physics Department, Syracuse University, Syracuse, New York 13210
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We consider a Yang-Mills-Higgs Lagrangian invariant under local gauge transformations belonging to an arbitrary compact group G. The Higgs fields are assumed to belong to a real representation of G. We analyze in detail the conditions imposed on the fields due to the requirement that the static Hamiltonian or the total energy of the system be finite. We then seek static finite-energy solutions for which the radial dependence of the fields is factorized. We show that for the coupled system of nonlinear equations emerging from the Lagrangian the equations for angular functions which carry the internal-symmetry labels and the equations for the radial functions separate into two separate systems of coupled nonlinear equations. We solve the equations for angular functions completely and show that the gauge fields vanish outside a fixed SO(3) subgroup of G and that inside the SO(3) group they reduce to the 't Hooft–Polyakov solution *with unit magnetic charge in appropriate units*. The Higgs fields may belong to any integer representation of this SO(3) group. The static Hamiltonian and consequently the total energy or mass of the monopole depend on the representation of the Higgs field. Thus we obtain in principle a mass formula for the monopoles, the one with the lowest mass corresponding to the 't Hooft–Polyakov case.

I. INTRODUCTION

Non-Abelian gauge theories of the Yang-Mills type have created a great deal of interest recently as possible candidates for a unified theory of strong, weak, and electromagnetic interactions.[1] While the full quantum field aspects of such theories are still far from being thoroughly understood, classical solutions to the field equations have proved to be quite interesting from several points of view.[2] Thus 't Hooft[3] and Polyakov[4] independently found a static, finite-energy solution in a non-Abelian model consisting of a triplet of isovector gauge mesons and a triplet of isovector Higgs mesons with interactions invariant under local gauge group SO(3). The solution has the very interesting feature that it can be thought of as corresponding to a magnetic monopole with magnetic charge of strength $4\pi/e$, which satisfies the famous Dirac quantization condition.[5] Subsequently several authors have attempted to generalize the model in various ways.[6]

In this paper we consider the 't Hooft-Polyakov model generalized to an arbitrary compact gauge group G. In the next section we describe the notation, set up the Lagrangian, and derive the covariant field equations. We consider the most general Higgs potential allowed by renormalization requirements. The conditions for the minima of such a potential and the ensuing mass spectrum of the Higgs mesons are discussed at some length in the Appendix. In the third section we examine in detail the consequences of the assumption that the total energy be finite. This leads to several interesting and strong restrictions on the Higgs and the gauge fields on a sphere at infinity. Of special importance is the integrability condition for the well-known equation[2] which relates the value of the Higgs field at one point on the sphere to that at another point by a gauge transformation. In Sec. IV, we derive the necessary condition for the existence of a solution for this equation and explain its physical significance.

To proceed further, we assume in the remaining sections that the fields can be expressed as products of functions involving separately the radial coordinate r and the polar angles $\omega=(\theta,\psi)$. The internal-symmetry indices are associated with the angular functions. This hypothesis is a natural generalization of the 't Hooft–Polyakov model and it enables us to accomplish a complete separation of the nonlinear coupled system of equations into nonlinear coupled equations for the radial functions and the angular functions. We solve the angular equations and show that the gauge-field angular part is unique; it vanishes outside a fixed-SO(3) subalgebra of the original algebra. Inside the subalgebra, it is the 't Hooft–Polyakov solution. The Higgs fields must also belong to a representation of the fixed SO(3) but, in contrast to the 't Hooft–Polyakov case, they may belong to any integer-valued representation. However, the strength of the magnetic charge is still a unit multiple of $4\pi/e$ as in the 't Hooft–Polyakov model before. The static Hamiltonian and consequently the mass or total energy of the monopole depend on the representation of the Higgs field. We discuss the existence of solutions to the ensuing radial equations and show that the 't Hooft–Polyakov monopole is the one with the lowest mass.

II. THE LAGRANGIAN AND THE FIELD EQUATIONS

We consider a compact Lie group G; its Lie algebra, denoted by $\mathcal{G}$ is assumed to be k dimensional. Let the generators T^α, $\alpha=1,2,\ldots,k$ satisfy

$$[T^\alpha, T^\beta] = C^{\alpha\beta\gamma}T^\gamma\,, \tag{2.1}$$

where $C^{\alpha\beta\gamma}$ are the structure constants of G. If $\underline{a}$ and $\underline{b}\in\mathcal{G}$, then $(\underline{a},\underline{b})$ given by

$$(\underline{a},\underline{b}) = -\tfrac{1}{2}a^\alpha b^\beta\,\mathrm{tr}T^\alpha T^\beta\,,\quad \underline{a}=a^\alpha T^\alpha\,,\quad \underline{b}=b^\alpha T^\alpha\,, \tag{2.2}$$

defines the group-invariant Cartan inner product which is nondegenerate on the semisimple subalgebra of $\mathcal{G}$.

We are interested in a Lagrangian consisting of a set of gauge fields $\underline{A}_\mu(x)=A_\mu^\alpha(x)\tau^\alpha$ and a set of Higgs fields $\Phi(x)$ and invariant under local G transformations.

The gauge fields $\underline{A}_\mu(x)$ belong to the adjoint representation of G given by the antisymmetric matrices $\tau=(\tau^\alpha)$, where

$$(\tau^\beta)^{\alpha\gamma} = C^{\alpha\beta\gamma}. \tag{2.3}$$

The fields $\underline{A}_\mu(x)$ can be looked upon as vector fields valued in $\mathcal{G}$.

We shall assume that the Higgs fields $\Phi(x)$ belong to an arbitrary real representation (not necessarily irreducible) of G. Let the corresponding vector space be denoted by $\mathcal{E}$. There is actually no loss of generality in assuming that the Φ representation is real, since any complex representation can be combined with its conjugate representation to form a real representation and the resultant skew-symmetric invariant does not occur in the Lagrangian. We denote the matrices of the representation to which Φ belongs by $\underline{t}$,

$$(t^\alpha)^{AB} = -(t^\alpha)^{BA}, \tag{2.4}$$

$$\alpha=1,2,\ldots,k,\quad A,B=1,2,\ldots,K\,.$$

The covariant derivatives of the Φ and $\underline{A}_\mu$ fields are defined by

$$\underline{D}_\mu\Phi^A = \partial_\mu\Phi^A + e(t^\alpha)^{AB}A_\mu^\alpha\Phi^B\,, \tag{2.5}$$

$$\underline{D}_\mu A_\nu^\alpha = \partial_\mu A_\nu^\alpha + eC^{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma\,. \tag{2.6}$$

Using the above-described fields, their covariant derivatives, and the metric

$$-g_{00}=g_{11}=g_{22}=g_{33}=1\,, \tag{2.7}$$

we can write the Lagrangian $\mathcal{L}(x)$ invariant under local gauge transformations

$$A_\mu \to A'_\mu = e^{\tau\cdot\chi(x)}A_\mu\,,\quad \Phi\to\Phi' = e^{t\cdot\chi(x)}\Phi$$

as

$$\mathcal{L}(x) = -\tfrac{1}{4}(\underline{F}_{\mu\nu},\underline{F}_{\mu\nu}) - \tfrac{1}{2}(\underline{D}_\mu\Phi,\underline{D}_\mu\Phi) - V(\Phi)\,, \tag{2.8}$$

where

$$\underline{F}_{\mu\nu} = \frac{\partial}{\partial x^\mu}\underline{A}_\nu - \frac{\partial}{\partial x^\nu}\underline{A}_\mu + e[\underline{A}_\mu,\underline{A}_\nu]\,. \tag{2.9}$$

We note two other useful forms for $\underline{F}_{\mu\nu}$,

$$\underline{F}_{\mu\nu} = \underline{D}_\mu\underline{A}_\nu - \frac{\partial}{\partial x^\nu}\underline{A}_\mu = \frac{1}{e}[\underline{D}_\mu,\underline{D}_\nu]\,. \tag{2.10}$$

From the Lagrangian (2.8) we can derive the covariant field equations

$$\frac{\partial}{\partial x^\mu}\underline{D}^\mu\Phi^A - e(t^\alpha)^{BA}A_\mu^\alpha\underline{D}^\mu\Phi^B - \frac{\partial V}{\partial\Phi^A} = 0\,, \tag{2.11}$$

$$\frac{\partial}{\partial x^\mu}F^{\mu\nu,\alpha} - eC^{\gamma\beta\alpha}F^{\mu\nu,\gamma}A_\mu^\beta - e(\underline{D}^\nu\Phi^A)(t^\alpha)^{AB}\Phi^B = 0\,. \tag{2.12}$$

If we define

$$\underline{J}^\nu(\Phi) = (\Phi,\underline{t}\,\underline{D}^\nu\Phi)\quad[\text{i.e., }J^{\nu,\alpha} = \Phi^A(t^\alpha)^{AB}\underline{D}^\nu\Phi^B]\,, \tag{2.13}$$

the field equations (2.11) and (2.12) take the form

$$\underline{D}^\mu\underline{D}_\mu\Phi = \frac{\partial V}{\partial\Phi}\,, \tag{2.14}$$

$$\underline{D}_\mu\underline{F}^{\mu\nu} = -e\underline{J}^\nu(\Phi)\,. \tag{2.15}$$

Note from (2.10) that $\underline{F}^{\mu\nu}$ also satisfies the Jacobi identity,

$$\underline{D}_\mu\underline{\tilde{F}}^{\mu\nu} = 0\,, \tag{2.16}$$

where

$$\underline{\tilde{F}}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\sigma\rho}\underline{F}_{\sigma\rho}\,.$$

The potential $V(\Phi)$ is a function of Φ, which is bounded below and invariant under G. For classical solutions there are no further restrictions on the functional form of V but, if we require a renormalizable quantum field theory, $V(\Phi)$ is restricted to be a fourth-degree polynomial. In simple models with spontaneous symmetry breaking, one generally considers a $V(\Phi)$ of the form

$$V(\Phi) = \frac{\lambda}{4}(\Phi,\Phi)^2 - \frac{\mu^2}{2}(\Phi,\Phi)^2\,. \tag{2.17}$$

The symmetry-breaking direction is then given by an arbitrary vector v such that

$$(v,v)^{1/2} = \mu/\sqrt{\lambda}\,. \tag{2.18}$$

In general, however, a fourth-degree potential may contain trilinear and quadrilinear invariants which are not isotropic in the group, that is, they do not allow an arbitrary direction for spontaneous symmetry breaking. Full details for such potentials are given in the Appendix. The general form is

$$V(\Phi)=\frac{\alpha}{4}(t(\Phi),\Phi)+\frac{\lambda}{4}(\Phi,\Phi)^2+\frac{\beta\mu}{3}(b(\Phi),\Phi)-\frac{\mu^2}{2}(\Phi,\Phi)\,, \tag{2.19}$$

where $t(\Phi)$ and $b(\Phi)$ denote vectors in $\mathcal{E}$ which are trilinear and bilinear (and symmetric) in Φ. The condition for an extremum of $V(\Phi)$ is then

$$\frac{dV}{d\Phi}=\alpha t(\Phi)+\lambda(\Phi,\Phi)\Phi+\beta\mu b(\Phi)-\mu^2\Phi=0\,. \tag{2.20}$$

For any solution $\Phi=\Phi_0$ of (2.20), the mass-squared matrices for the Higgs and gauge fields are given in the usual way by

$$\mathfrak{M}_{AB}=\left(\frac{\partial^2 V}{\partial\Phi^A\partial\Phi^B}\right)_{\Phi=\Phi_0}, \tag{2.21}$$

$$e^2M_{\alpha\beta}=(t^\alpha\Phi_0,t^\beta\Phi_0)\,, \tag{2.22}$$

respectively.

III. FINITE-ENERGY CONDITIONS IN THE STATIC CASE

We are interested in the static solutions of the field equations (2.14) and (2.15). In addition we shall assume for the moment that $A_0=0$. When all the time derivatives and A_0 are equal to zero, the equations of motion reduce to

$$\underline{\vec{D}}^2\Phi=\frac{\partial V}{\partial\Phi}\,, \tag{3.1}$$

$$\underline{\vec{D}}\times\underline{\vec{F}}=-e\underline{\vec{J}}(\Phi)\,, \tag{3.2}$$

where

$$\underline{F}_i=\tfrac{1}{2}\epsilon_{ijk}\underline{F}_{jk}\,.$$

The Hamiltonian or the total energy of the system is

$$\mathcal{H}=\int\mathcal{H}(x)d^3x=\int[\tfrac{1}{2}(\underline{\vec{F}},\underline{\vec{F}})+\tfrac{1}{2}(\underline{\vec{D}}\Phi,\underline{\vec{D}}\Phi)+V(\Phi)]d^3x\,. \tag{3.3}$$

The boundary conditions for the solutions of (3.1) and (3.2) are imposed by the requirement that $\mathcal{H}$ be finite, i.e., $\mathcal{H}(x)$ be integrable,

$$r^3\mathcal{H}(x)\underset{r\to\infty}{\longrightarrow}0\,, \tag{3.4}$$

and by the spontaneous symmetry-breaking mechanism. The latter requires that all the group invariants in the potential $V(\Phi)$ tend to finite constants for large r. In particular

$$\lim_{r\to\infty}(\Phi(\vec{r}),\Phi(\vec{r}))=c^2<\infty,\quad c>0\,. \tag{3.5}$$

Hence we assume

$$\lim_{r\to\infty}\Phi(r,\omega)=c\phi(\omega),\quad(\phi(\omega),\phi(\omega))=1\,, \tag{3.6}$$

where $\phi(\omega)$ lies on a group orbit which is determined by the other invariants according to (2.20) with $\Phi_0=c\phi(\omega)$.

We next note that, since each term in the Hamiltonian (3.3) is positive, (3.4) imposes a condition separately on each term. For the pure gauge term we satisfy this condition by assuming in the standard way that

$$\lim_{r\to\infty}re\underline{\vec{A}}(r,\omega)=\underline{\vec{a}}(\omega)<\infty\,, \tag{3.7}$$

which implies that

$$\lim_{r\to\infty}r^2e\underline{\vec{F}}(r,\omega)=\underline{\vec{f}}(\omega)<\infty\,. \tag{3.8}$$

For the kinetic term for the Higgs field in (3.3), the finite-energy condition (3.4) implies

$$\lim_{r\to\infty}r(r\underline{\vec{D}}\Phi,r\underline{\vec{D}}\Phi)=0\,, \tag{3.9}$$

which takes the form

$$\underline{\vec{d}}\,\phi(\omega)=[\vec{\partial}+\vec{a}^{\alpha}(\omega)t^\alpha]\,\phi(\omega)=0\,, \tag{3.10}$$

where

$$\vec{\partial}=r\vec{\nabla}\,.$$

Equation (3.10) implies that on the infinite sphere the values of $\phi(\omega)$ for different ω are related by a group transformation, a result which is perhaps more familiar in the integrated form

$$\phi(\omega)=U(g(\omega,\omega_0))\,\phi(\omega_0)\,. \tag{3.11}$$

If we take the scalar and vector product of (3.10) by the unit vector $\hat{r}$, respectively, we obtain

$$\hat{r}\cdot\vec{a}^{\alpha}t^{\alpha}\phi(\omega)=0 \tag{3.12}$$

and

$$\underline{\vec{J}}\,\phi(\omega)=-(\hat{r}\times\underline{\vec{d}})\phi(\omega)=(\underline{\vec{L}}+\underline{\vec{b}})\phi(\omega)=0\,, \tag{3.13}$$

where

$$\underline{\vec{L}}=-\hat{r}\times\vec{\partial}\ \text{ and }\ \underline{\vec{b}}=-\hat{r}\times\vec{a}\,.$$

Note $\underline{\vec{L}}$ and $\underline{\vec{J}}$ are the ordinary and covariant angular momentum operators (except that they are anti-Hermitian since we have found it convenient to omit the usual factor i).

With (3.6), the mass-squared matrix (2.22) for the gauge field becomes

$$e^2M_{\alpha\beta}(\omega)=c^2(t^\alpha\phi(\omega),t^\beta\phi(\omega))\,. \tag{3.14}$$

Now, although this mass matrix is ω dependent, it is physical because the *covariant angular momentum* acting on it is zero on account of (3.10), that is,

$$J_i M_{\alpha\beta} = L_i M_{\alpha\beta} + a_i^\gamma (C^\gamma_{\alpha\alpha'} M_{\alpha'\beta} + C^\gamma_{\beta\beta'} M_{\alpha\beta'}) = 0 . \tag{3.15}$$

In particular the eigenvalues of $M_{\alpha\beta}$ are independent of ω.

Finally we note for later convenience that, taking the scalar product of (3.10) with $t^\beta \phi(\omega)$, we obtain

$$M_{\beta\alpha} \vec{a}^{\,\alpha} = (\phi, t^\beta \vec{\partial} \phi) , \tag{3.16}$$

which expresses the massive gauge fields in terms of the Higgs currents. From (3.16) we also have

$$a_i^\beta M_{\beta\alpha} a_j^\alpha = (\partial_i \phi, \partial_j \phi) . \tag{3.17}$$

IV. INTEGRABILITY CONDITIONS FOR $\phi(\omega)$

The equation (3.10) which connects the ω dependence of ϕ with its internal-symmetry structure is the most striking consequence of the condition that the energy should be finite. However, such a partial-differential equation has global solutions if and only if its integrability conditions are satisfied. Hence it is interesting and also turns out to be very useful to study these conditions. If we write (3.10) in the form

$$\vec{\partial} \phi(\omega) = -\vec{a}^{\,\alpha} t^\alpha \phi(\omega) , \tag{4.1}$$

we see that the integrability conditions for these equations are

$$\begin{aligned} \partial_i (a_j^\alpha t^\alpha \phi) - \partial_j (a_i^\alpha t^\alpha \phi) &= -(\partial_i \partial_j - \partial_j \partial_i)\phi \\ &= \epsilon_{ijk} L_k \phi \\ &= -\epsilon_{ijk} b_k \phi . \end{aligned} \tag{4.2}$$

Thus they can be written in the form

$$\underline{f}_{ij} \, \phi = 0 , \tag{4.3}$$

where

$$\underline{f}_{ij} = \partial_i \underline{a}_j - \partial_j \underline{a}_i + [\underline{a}_i , \underline{a}_j] + \epsilon_{ijk} \underline{b}_k . \tag{4.4}$$

The fields $\underline{f}_{ij}$ are the analogs, on the sphere, of the field $\underline{F}_{\mu\nu}$ in Minkowski space. If they vanish, the fields $\underline{a}_j$ can be gauged to zero locally. As a matter of fact, because $\pi_2(G) = 0$ for any compact Lie group, they can be gauged away globally. That is to say, if $\underline{f}_{ij} = 0$, there is a function $S(\omega)$ valued in $\mathcal{G}$ such that

$$\underline{a}_i(\omega) = S^{-1}(\omega) \partial_i S(\omega) , \tag{4.5}$$

and for ω in S_2 and G compact, $S(\omega)$ is homotopic to zero. We also note that

$$\underline{f}_{ij} = \lim_{r\to\infty} r^2 \underline{F}_{ij} . \tag{4.6}$$

Similarly, if we write (3.13) in the form

$$L_i \phi(\omega) = -b_i \phi(\omega) , \tag{4.7}$$

we can derive the integrability condition

$$\underline{h}_{ij} \, \phi = 0 , \tag{4.8}$$

where

$$\underline{h}_{ij} = [\underline{J}_i , \underline{J}_j] - \epsilon_{ijk} \underline{J}_k \tag{4.9}$$

$$= \underline{f}_{ij} - \epsilon_{ijk} [\underline{J}_k , \hat{r} \cdot \underline{\vec{a}}] . \tag{4.10}$$

In contrast to $\underline{f}_{ij}$ $\underline{h}_{ij}$ has the remarkable property that

$$\hat{r}_i \underline{h}_{ij} = 0 \tag{4.11}$$

and hence

$$\underline{h}_{ij} = \epsilon_{ijk} \hat{r}_k \underline{f} , \tag{4.12}$$

where $\underline{f}$ is the scalar

$$\underline{f} = \tfrac{1}{2} \epsilon_{ijk} \hat{r}_k \underline{h}_{ij} = \tfrac{1}{2} \epsilon_{ijk} \hat{r}_k \underline{f}_{ij} . \tag{4.13}$$

Thus $\underline{h}_{ij}$ has the same radial component $\underline{h}_r = \hat{r} \underline{f}$ as $\underline{f}_{ij}$ but, unlike $\underline{f}_{ij}$, it has no transverse components. The reason for this is that Eq. (3.13) is weaker than (3.10) since the former can be obtained by taking the cross product of (3.10) with $\hat{r}$, but not conversely. The result is that the vanishing of $\underline{h}_{ij}$ implies only that the transverse components of $\underline{\vec{a}}(\omega)$ can be gauged to zero.

Note in particular that

$$\underline{f} \phi = 0 . \tag{4.14}$$

Thus $\underline{f}$ is scalar in ordinary space, vector in the adjoint representation of G, and is in the little group of $\phi(\omega)$. From (3.14) we then have

$$M_{\alpha\beta} f^\beta = 0 . \tag{4.15}$$

Thus $f^\alpha(\omega)$ is massless and is therefore a suitable but generally not unique candidate for defining an electromagnetic direction in the Lie algebra.

In any case Eq. (4.14) will play an important role in the sequel and it is convenient to expand the function $\underline{f}$ into

$$\underline{f} = 2\underline{u} + \underline{v} , \tag{4.16}$$

where

$$2\underline{u} = -\vec{L} \cdot \underline{\vec{a}}, \quad \underline{v} = -\epsilon_{ijk} \hat{r}_i \underline{a}_j \underline{a}_k . \tag{4.17}$$

Finally we note that the integrability condition (4.3) shows that there are only three possibilities:

(i) $\underline{f}_{ij} = 0$. In this case, the potentials can be gauged away leading to a trivial solution.

(ii) $\underline{f}_{ij} \neq 0$, $\underline{f}_{ij} \phi = 0$. Then there exists a solution which is nontrivial.

(iii) $\underline{f}_{ij} \neq 0$, $\underline{f}_{ij} \phi \neq 0$. Then there are no solutions.

V. SEPARABILITY HYPOTHESIS: ANGULAR AND RADIAL EQUATIONS

In this section, we seek solutions to the field equations (3.1) and (3.2) in which $\Phi(\vec{r})$ and $\underline{\vec{A}}(\vec{r})$ have the form

$$\Phi^A(\vec{r}) = \phi(\omega)\, c\frac{S(r)}{r}, \quad e\underline{\vec{A}}(\vec{r}) = \underline{\vec{a}}(\omega)\frac{R(r)}{r} \tag{5.1}$$

with the boundary conditions, when $r\to\infty$,

$$\frac{S(r)}{r}\to 1,\ R(r)\to 1, \text{ and } S(0)=R(0)=0\,. \tag{5.2}$$

The above separation of variables requires for its definition the choice of a gauge which we take to be the Landau gauge specified by div $\underline{\vec{A}}=0$. Equation (5.1) constitutes a straightforward generalization of the Wu-Yang hypothesis[7] in the case of pure Yang-Mills fields and the later adaptation by 't Hooft[3] and Polyakov[4] to the case of Yang-Mills fields supplemented by Higgs fields. The internal-symmetry indices are carried by the angular functions $\phi(\omega)$ and $\underline{\vec{a}}(\omega)$. To begin with they are assumed to be arbitrary but for the usual smoothness properties. In particular, they are not assumed to be spherically symmetric.[8]

The radial separation (5.1) reduces the Landau gauge condition to two separate conditions

$$\hat{r}\cdot\vec{a}(\omega) = 0 = \vec{\partial}\cdot\vec{a}(\omega)\,, \tag{5.3}$$

implying $\vec{a}(\omega)$ has no radial component and it is divergence-free. Thus the Landau gauge is the most convenient and natural gauge to impose the separability hypothesis. Using (5.1) the static field equation (3.1) can be written as

$$\phi(\omega)r^2S''(r) = S(r)\left[-\underline{\vec{d}}_R{}^2\phi(\omega) + r^2\mu^2W\left(\frac{S(r)}{r}\right)\phi(\omega)\right], \tag{5.4}$$

where

$$\underline{\vec{d}}_R\phi = \vec{\partial}\phi + R(r)\underline{\vec{a}}\cdot\underline{t}\,\phi = \vec{d}_R\phi + [R(r)-1]\underline{\vec{a}}\cdot\underline{t}\phi \tag{5.5}$$

and

$$W(x) = (x-1)(\sigma^2x+1)\,, \tag{5.5$'$}$$

where $\sigma\geq 1$ is a dimensionless parameter which depends on the assumed form for the Higgs potential (see Appendix). Similarly, using (5.1) the static field equations (3.2) can be written as

$$\underline{\vec{a}}(\omega)r^2R''(r) = R(r)(\underline{\vec{j}}(R,\underline{\vec{a}})) - e^2c^2S^2(r)\underline{\vec{j}}(R,\phi)\,, \tag{5.6}$$

where

$$\underline{\vec{d}}_R\,\underline{\vec{a}}(\omega) = \vec{\partial}\,\underline{\vec{a}}(\omega) + R(r)\,\underline{\vec{a}}\cdot\underline{\tau}\,\underline{\vec{a}}(\omega)$$

and

$$\underline{\vec{j}}(R,\underline{\vec{a}}) = -\underline{\vec{d}}_R{}^2\,\underline{\vec{a}}(\omega) - R(r)(\underline{\vec{a}},\underline{\tau}\,\vec{\partial}\,\vec{a}),$$
$$\underline{\vec{j}}(R,\phi) = (\phi,\underline{t}\,\underline{\vec{d}}_R\phi)\,. \tag{5.7}$$

Equations (5.4) and (5.6) are coupled nonlinear equations. It is not obvious that the separability ansatz (5.1) will lead to separate equations for the radial functions $S(r)$ and $R(r)$ and the angular functions $\phi(\omega)$ and $\underline{\vec{a}}(\omega)$. We next prove that this indeed is true and derive the radial and angular equations.

A. Polynomials $q(R)$ and $Q(R)$

By taking the scalar product of (5.4) and (5.6) with $\phi(\omega)$ and $\underline{\vec{a}}(\omega)$, respectively, we obtain

$$r^2S''(r) = S(r)[q(R) + r^2\mu^2W(S/r)]\,, \tag{5.8}$$

$$\nu^2(\omega)r^2R''(r) = R(r)\nu^2(\omega)Q(R) + e^2c^2\frac{S^2(r)}{2}\frac{dq(R)}{dR}\,, \tag{5.9}$$

where we have used $(\phi(\omega),\phi(\omega))=1$ and defined

$$\nu^2(\omega) = (\vec{a}(\omega),\vec{a}(\omega))\,. \tag{5.10}$$

$q(R)$ and $\nu^2Q(R)$ are quadratic polynomials in $R(r)$,

$$\begin{aligned} q(R) &= -(\phi(\omega),\underline{\vec{d}}_R{}^2\phi(\omega)) \\ &= -[(\phi(\omega),\partial^2\phi(\omega)) + 2R(r)(\phi(\omega),\underline{t}\cdot\underline{\vec{a}}\cdot\vec{\partial}\phi(\omega)) \\ &\quad + R^2(r)(\phi(\omega),(\underline{t}\cdot\underline{\vec{a}})^2\phi(\omega))]\,, \end{aligned} \tag{5.11}$$

$$\begin{aligned} \nu^2(\omega)Q(R) &= (\underline{\vec{a}}(\omega),\underline{\vec{j}}(R,a)) \\ &= -[(\underline{\vec{a}}(\omega),\vec{\partial}^2\underline{\vec{a}}(\omega)) + 3R(r)(\underline{\vec{a}}(\omega),\underline{\tau}\cdot\underline{\vec{a}}\cdot\vec{\partial}\underline{\vec{a}}(\omega)) \\ &\quad + R^2(r)(\underline{\vec{a}}(\omega),(\underline{\tau}\cdot\underline{\vec{a}})^2\underline{\vec{a}}(\omega))]\,. \end{aligned} \tag{5.12}$$

Both the polynomials $q(R)$ and $Q(R)$ have apparent ω dependence in the coefficients of the different powers of R. However, consider (5.11); if we take two different values of ω, say ω and ω_0, in (5.8) and subtract one equation from the other, we obtain an equation of the form

$$0 = \sum_{k=1}^{3}[X_k(\omega) - X_k(\omega_0)]R^{k-1}(r)\,. \tag{5.13}$$

Now if $R(r)$ is a constant, $R(0)=0$ forces $R(r)$ to vanish everywhere leading to the trivial solution $\underline{\vec{A}}=0$ everywhere. If $R(r)$ is not a constant, since the different powers of R are linearly independent functions, it follows that

$$X_k(\omega) = X_k(\omega_0)\,, \tag{5.14}$$

i.e., the coefficient of different powers of $R(r)$ in $q(R)$ are ω-independent constants. Repeated arguments along similar lines prove that $\nu(\omega)$ is a constant independent of ω and so also are the coefficients in the polynomial $Q(R)$. Thus we obtain

$$(\phi(\omega), \partial^2\phi(\omega)) = -l(l+1)\,, \tag{5.15}$$

$$(\phi(\omega), \underline{t}\cdot\vec{\underline{a}}\cdot\vec{\partial}\phi(\omega)) = m\nu\,, \tag{5.16}$$

$$(\phi(\omega), (\underline{t}\cdot\vec{\underline{a}})^2\phi(\omega)) = -n\nu^2\,, \tag{5.17}$$

$$(\vec{\underline{a}}(\omega), \partial^2\vec{\underline{a}}(\omega)) = -L(L+1)\nu^2\,, \tag{5.18}$$

$$3(\vec{\underline{a}}(\omega), \underline{\tau}\cdot\vec{\underline{a}}\cdot\vec{\partial}\,\vec{\underline{a}}(\omega)) = 2M\nu^3\,, \tag{5.19}$$

$$(\vec{\underline{a}}(\omega), (\underline{\tau}\cdot\vec{\underline{a}})^2\vec{\underline{a}}(\omega)) = -\nu^4/N\,. \tag{5.20}$$

In the above equations l, m, n, L, M, N, and ν are constants. We have chosen to write them in the specified form for later convenience.

Using the finite-energy condition (3.10) and Eqs. (5.15)–(5.17), we see that the constants l, m, and n satisfy

$$l(l+1) = m\nu = n\nu^2 \tag{5.21}$$

and hence, from (5.11), that the polynomial $q(R)$ takes the form

$$q(R) = l(l+1)(1-R)^2\,, \tag{5.22}$$

where there is only one free parameter.

Derivation of an analogous expression for $Q(R)$ is more involved. The key to the derivation is to prove that the $r\to\infty$ limit $\vec{\underline{j}}(1,\vec{\underline{a}})$ of the current $\vec{\underline{j}}(R,\underline{a})$ defined in (5.7) is zero. For this purpose we observe that, from (5.7) and the finite-energy condition (3.10),

$$\vec{j}^{\alpha}(1,\vec{a})t^{\alpha}\phi = 0\,. \tag{5.23}$$

But then from the definition of $\vec{\underline{j}}(R,\phi)$ in (5.7),

$$(\vec{\underline{j}}(1,\vec{\underline{a}}), \vec{\underline{j}}(R,\phi)) = (\phi, \vec{\underline{j}}(1,\vec{a})\cdot\underline{t}\cdot\vec{\underline{d}}_R\phi)$$
$$= -(\vec{\underline{j}}(1,a)\cdot\underline{t}\,\phi, \vec{\underline{d}}_R\phi) = 0\,. \tag{5.24}$$

Note that Eq. (5.24) holds for all R in $\vec{\underline{j}}(R,\phi)$. If we now take the scalar product of (5.6) with $\vec{\underline{j}}(1,\vec{a})$ and use (5.24), the $\vec{\underline{j}}(R,\phi)$ term drops out and we obtain

$$(\vec{\underline{j}}(1,\vec{a}),\vec{a})r^2R'' = R(\vec{\underline{j}}(1,\vec{\underline{a}}), \vec{\underline{j}}(R,\vec{\underline{a}}))\,. \tag{5.25}$$

However, if $R(r)\to 1$ as $r\to\infty$, $r^2R''\to 0$ as $r\to\infty$. Consequently, taking the limit as $r\to\infty$, (5.25) yields

$$(\vec{\underline{j}}(1,\vec{\underline{a}}), \vec{\underline{j}}(1,\vec{\underline{a}})) = 0$$

and hence the desired result

$$\vec{\underline{j}}(1,\vec{\underline{a}}) = 0\,. \tag{5.26}$$

Then taking the limit $r\to\infty$ of (5.12), it follows that

$$Q(1) = 0\,. \tag{5.27}$$

Equation (5.27) has two consequences. First it shows that $Q(R)$ must have a factor $(1-R)$; also from (5.18)–(5.20) and (5.12),

$$2M\nu = L(L+1) + \nu^2/N$$

and hence

$$Q(R) = L(L+1)(1-R)\left[1 - \frac{\nu^2}{L(L+1)N}R\right] \tag{5.28}$$

This is the required analog of (5.22) for $Q(R)$. However, note that it contains two parameters, L and ν^2/N. Secondly, if we take the $r\to\infty$ limit of (5.9) and use (5.27), we obtain

$$\lim_{r\to\infty}\frac{S^2}{2}\frac{dq}{dR} = \lim_{r\to\infty} S^2(1-R) = 0\,. \tag{5.29}$$

Since $S\sim r$ as $r\to\infty$, the above equation shows that $(1-R)$ falls off faster than r^{-2} as $r\to\infty$. The asymptotic condition (5.29) will be crucial in what follows.

The physical meaning of the equations $q(1) = Q(1) = 0$ can be seen from the radial equations (5.8) and (5.9). As we shall see in Sec. VIII, because of $q(1) = Q(1) = 0$, the fields $(S-r)$ and $(R-1)$ fall off exponentially with increasing r and hence are massive. On the other hand if $q(1)$ or $Q(1)$ were not zero, then one can easily see that $(S-r)$ and $(R-1)$ would fall off like r^{-1} and r^{-2}, respectively, and hence they are long-range or massless fields.

B. Angular equations

We can extract more information than is contained in (5.15)–(5.20) from (5.4) and (5.5). For this purpose consider the Hilbert space V_ϵ of real functions defined on the sphere, valued in $\mathcal{E}$ with the scalar product (denoted by double bold parentheses)

$$((\psi(\omega), \chi(\omega))) = \frac{1}{4\pi}\int(\psi(\omega),\chi(\omega))d\omega\,. \tag{5.30}$$

Take the scalar product of (5.4) in V_ϵ by $\Psi(\omega)$ such that

$$((\Psi(\omega), \phi(\omega))) = \frac{1}{4\pi}\int(\Psi(\omega),\phi(\omega))d\omega = 0\,. \tag{5.31}$$

Then

$$((\Psi(\omega), \vec{\underline{d}}_R{}^2\phi(\omega))) = a + bR + cR^2 = 0\,, \tag{5.32}$$

where a, b, c are numerical constants. Since $R(r)$ is not a constant,

$$a = b = c = 0$$

for all $\Psi(\omega)$ satisfying (5.31). Hence

$$((\Psi(\omega), \partial^2\phi(\omega))) = ((\Psi(\omega), \underline{t}\cdot\vec{\underline{a}}\cdot\vec{\partial}\phi(\omega)))$$
$$= ((\Psi(\omega), (\underline{t}\cdot\vec{\underline{a}})^2\phi(\omega)))$$
$$= 0\,. \tag{5.33}$$

Equation (5.33) combined with (5.15), (5.16), (5.17) give

$$\partial^2\phi(\omega) = -l(l+1)\phi(\omega)\,, \tag{5.34}$$

$$\underline{t}\cdot\vec{\underline{a}}\cdot\vec{\partial}\phi(\omega) = m\nu\phi(\omega)\,, \tag{5.35}$$

$$(\underline{t}\cdot\vec{\underline{a}})^2\,\phi(\omega) = -n\nu^2\phi(\omega)\,. \tag{5.36}$$

Similarly, consider a Hilbert space V_g of real vector functions defined on the sphere and valued in the algebra $\mathcal{G}$. If $\vec{\underline{\chi}}(\omega)$ are vector functions such that

$$((\vec{\underline{\chi}}(\omega), \vec{\underline{a}}(\omega))) = 0\,, \tag{5.37}$$

taking the scalar product of (5.5) by $\vec{\underline{\chi}}(\omega)$, we obtain an equation of the form

$$a[1-R(r)] + b[1-R(r)]^2 + c[1-R(r)]^3 + S^2(r)\{g + h[1-R(r)]\} = 0\,, \tag{5.38}$$

where a, b, c, g, and h are numerical constants which can be easily determined from (5.5) and (5.18)–(5.20). Now, from the asymptotic conditions $R\to 1$ and $S^2(1-R)\to 0$, we see that (5.38) can be satisfied for nontrivial R only if

$$a = b = c = g = h = 0\,. \tag{5.39}$$

From identifying a, b, c appropriately, we obtain three equations

$$\partial^2\vec{\underline{a}}(\omega) = -L(L+1)\vec{\underline{a}}(\omega)\,, \tag{5.40}$$

$$2\vec{\underline{a}}\cdot\underline{\tau}\cdot\vec{\partial}\,\vec{\underline{a}}(\omega) + (\vec{\underline{a}}, \underline{\tau}\vec{\partial}\,\vec{\underline{a}}) = 2M\nu\vec{\underline{a}}(\omega)\,, \tag{5.41}$$

$$[\underline{\tau}\cdot\vec{\underline{a}}(\omega)]^2\vec{\underline{a}}(\omega) = -\frac{1}{N}\,\nu^2\vec{\underline{a}}(\omega)\,, \tag{5.42}$$

which are the counterparts of the corresponding equations (5.34)–(5.36) for $\phi(\omega)$. We also obtain two additional equations relating ϕ and $\vec{\underline{a}}$, viz.,

$$(\phi, \underline{t}\vec{\partial}\phi) = -(\phi, \underline{t}(\underline{t}\cdot\vec{\underline{a}})\phi) = n\vec{\underline{a}}\,, \tag{5.43}$$

where the constant n has been computed from (5.43) and (5.35).

C. Summary of all the equations

Utilizing the relations between the numerical constants to eliminate some of them, we summarize all the equations obtained above for both the radial functions $S(r)$ and $R(r)$ and the angular functions $\phi(\omega)$ and $\vec{\underline{a}}(\omega)$.

Equations for angular functions.

(a) $\vec{\underline{d}}\phi(\omega) = [\vec{\partial}1 + \underline{t}\cdot\vec{\underline{a}}(\omega)]\phi(\omega) = 0\,,\quad (\phi(\omega), \phi(\omega)) = 1\,,$ (5.44a)

(b) $\vec{\underline{d}}^2\vec{\underline{a}}(\omega) = -\vec{\underline{j}}(\vec{\underline{a}})\,,\quad (\vec{\underline{a}}(\omega), \vec{\underline{a}}(\omega)) = \nu^2\,,$ (5.44b)

(c) $\partial^2\phi(\omega) = -l(l+1)\phi(\omega)\,,\quad \partial^2\vec{\underline{a}}(\omega) = -L(L+1)\vec{\underline{a}}(\omega)\,,$ (5.44c)

(d) $(\vec{\underline{a}}\cdot\underline{\tau})^2\vec{\underline{a}} = [\underline{a}_i, [\underline{a}_i, \vec{\underline{a}}]] = -\dfrac{\nu^2}{N}\,\vec{\underline{a}}\,,$ (5.44d)

(e) $(\phi, \underline{t}\vec{\partial}\phi) = n\vec{\underline{a}}\,.$ (5.44e)

Equations for radial functions.

$$r^2S''(r) = \{l(l+1)[1-R(r)]^2 + r^2\mu^2W(S/r)\}S(r)\,, \tag{5.45a}$$

$$r^2R''(r) = [1-R(r)]\{L(L+1)R(r)[1-\epsilon R(r)] - e^2c^2nS^2(r)\}\,, \tag{5.45b}$$

where

$$\epsilon = \frac{l(l+1)}{L(L+1)Nn}\,.$$

VI. DISCUSSION OF THE CONSTANTS IN THE EQUATIONS

Before we discuss the solutions to Eqs. (5.44) and (5.45), it is instructive to analyze them briefly and see the nature and significance of the various constants appearing in them. First of all we note that the radial separation implies that all the equations valid in the $r\to\infty$ limit are valid for all r. This is the case of Eq. (5.44a), which is the basic finite-energy condition (3.10). The analogous equation for $\vec{\underline{a}}(\omega)$ contains a Higgs current term in the $r\to\infty$ limit, but the radial separation, as shown in the preceding section, leads to a stronger condition, namely, Eq. (5.44b) involving only $\vec{\underline{a}}$ fields. The latter equation can be written in the equivalent forms

$$(\underline{d}_i\underline{f}_{ij}) = 0\,,\quad (\underline{d}_i\underline{f}) = 0\,, \tag{6.1}$$

where $\underline{f}_{ij}$ and $\underline{f}$ are defined in (4.4) and (4.13). Equations (6.1) are pure Yang-Mills equations for the angular functions $\vec{\underline{a}}(\omega)$ or equivalently for $\underline{f}(\omega)$ on the unit sphere. However, note that (5.44e) relates $\vec{\underline{a}}(\omega)$ to $(\phi, \underline{t}\vec{\partial}\phi)$.

Equations (5.44c) show that $\phi(\omega)$ and $\vec{\underline{a}}(\omega)$ are spherical harmonics of definite order l and L, respectively. Thus this property true for a system of linear equations is also true in the case of the coupled nonlinear system of equations under study. Therefore, l and L are positive integers [if one of them were zero, we would have the trivial solution $\phi(\omega) = \vec{\underline{a}}(\omega) = \text{const}$]. If l is a positive integer, it then follows from (5.21) that $n>0$. Assuming $\nu>0$, we see from (5.21) that $m>0$. Finally, since $(\underline{a}\cdot\underline{\tau})^2$ is a sum of negative operators (being a sum of squares of antisymmetric operators), it follows from (5.44d) that $N>0$.

The same equation (5.44d) can be used to prove that, for each ω, the $\underline{a}_i(\omega)$ generate an SU(2) subalgebra which we shall denote $\text{SU}(2)_\omega$. For this purpose, consider at each ω a triad of unit vectors $\hat{n}^{(1)}(\omega)$, $\hat{n}^{(2)}(\omega)$, $\hat{r}$ such that

$$\hat{n}^{(1)}\cdot\hat{n}^{(2)} = 0\,,\quad \hat{r}(\omega) = \hat{n}^{(1)}\times\hat{n}^{(2)}\,. \tag{6.2}$$

Then

$$\hat{r}_k \hat{r}_l + \sum_{i=1}^{2} n_k^{(i)} n_l^{(i)} = \delta_{kl} . \tag{6.3}$$

Now, since $N > 0$, we can define

$$\underline{e}^{(i)}(\omega) = \frac{\sqrt{N}}{\nu}\, \hat{n}^{(i)} \cdot \underline{\vec{a}}(\omega), \quad i = 1, 2 , \tag{6.4}$$

and using (6.3) and $\hat{r} \cdot \underline{\vec{a}}(\omega) = 0$, we find

$$\underline{\vec{a}}(\omega) = \frac{\nu}{\sqrt{N}} \sum_{i=1}^{2} \hat{n}^{(i)} \underline{e}^{(i)} . \tag{6.5}$$

A straightforward computation using (6.2), (6.4), and (5.44d) yields

$$[\underline{e}^{(1)}, [\underline{e}^{(1)}, \underline{e}^{(2)}]] = -\underline{e}^{(2)}$$

and (6.6)

$$[\underline{e}^{(2)}, [\underline{e}^{(2)}, \underline{e}^{(1)}]] = -\underline{e}^{(1)} .$$

Hence, if we define

$$\underline{e}^{(3)} = [\underline{e}^{(1)}, \underline{e}^{(2)}] = -\frac{N}{\nu^2} \underline{v}(\omega) , \tag{6.7}$$

we see that the three $\underline{e}^{(i)}$ form the basis of an $SU(2)_\omega$ Lie subalgebra of $\mathcal{G}$ with the usual structure constants ϵ_{ijk}, i.e.,

$$[\underline{e}^{(i)}, \underline{e}^{(j)}] = \epsilon_{ijk} \underline{e}^{(k)} . \tag{6.8}$$

Now any compact Lie algebra splits canonically into the direct sum of an Abelian algebra and a semisimple algebra. From (5.44d), since we can write $\underline{\vec{a}}(\omega)$ as a commutator

$$\frac{\nu^2}{N}\, \underline{\vec{a}} = [\underline{v}, \underline{\vec{b}}] , \tag{6.9}$$

the $\underline{\vec{a}}(\omega)$ fields take their values inside the semisimple $\mathcal{G}_s$ of $\mathcal{G}$. There is only a finite set of SU(2) Lie subalgebras in any $\mathcal{G}_s$, up to a conjugation by the group G. Since we are interested only in the regular solution $\underline{\vec{a}}(\omega)$, we conclude that all $SU(2)_\omega$ for different ω are conjugated into each other by G.

The normalization constant ν^2 [Eq. (5.44b)] depends on the arbitrary scale which one can choose for the invariant orthogonal metric (i.e., the Cartan-Killing bilinear form) on $\mathcal{G}_s$. However, as can easily be verified, the scalar product of $\underline{e}$'s in G space is independent of ν^2, but depends only on N,

$$(\underline{e}^{(1)}, \underline{e}^{(1)}) = (\underline{e}^{(2)}, \underline{e}^{(2)}) = (\underline{e}^{(3)}, \underline{e}^{(3)}) = N/2 . \tag{6.10}$$

Since all the representations of the SU(2) Lie algebras are known, Eq. (6.10) yields

$$\begin{aligned} N &= -\tfrac{1}{2} \operatorname{tr} (\underline{e}^{(3)})^2 \\ &= \sum_{j=0}^{\infty} \tfrac{1}{3} c_j j (j+1)(2j+1) \\ &= \sum_{j=0}^{\infty} \tfrac{1}{2} c_j \binom{2j+2}{3} , \end{aligned} \tag{6.11}$$

where c_j is the multiplicity of the $(2j+1)$-dimensional irreducible representation in the reduction of the adjoint representation $\mathcal{G}$ restricted to a specific $SU(2)_\omega$ subalgebra. Of course $c_1 \geq 1$ and, since $\mathcal{G}$ is real, c_j is even when j is half-integer; then (6.11) shows that

$$N \text{ is an integer } \geq 2 . \tag{6.12}$$

To summarize, at each ω, $\underline{\vec{a}}(\omega)$ generate an $SU(2)_\omega$; for different ω the different $SU(2)_\omega$ are conjugated. N is a purely group-theoretic number which depends only on the embedding of these SU(2) in $\mathcal{G}$. For instance, the values of N for some low-rank groups are the following.

$\mathcal{G}$	SO(3)	SO(4)	SU(3)	
N	2	2, 2, 4	3, 12	(6.13)

The only constant that remains to be discussed is n. First of all we note that, from the definition (3.14) of $M_{\alpha\beta}$ and the Eqs. (5.44a) and (5.44e), we have

$$M_{\alpha\beta}(\omega) \vec{a}^{\beta} = c^2 n \vec{a}^{\alpha} , \tag{6.14}$$

which shows that the nonvanishing gauge fields are actually eigenfields of the mass-squared operator with the same mass squared $m^2 = c^2 n$. Next, Eq. (5.36) can be written as

$$-(\underline{e}^{(1)} \cdot \underline{t})^2 \phi - (\underline{e}^{(2)} \cdot \underline{t})^2 \phi = nN\phi \tag{6.15}$$

or also

$$\left[-\sum_{i=1}^{2} (\underline{e}^{(i)} \cdot \underline{t})^2 + (\underline{e}^{(3)} \cdot \underline{t})^2 \right] \phi = [t(t+1) - m_t^2] \phi = nN\phi , \tag{6.16}$$

where $2t$, $(t - m_t)$ are non-negative integers and $-t \leq m_t \leq t$. This implies that, if we call t the spin of the irreducible representation of $SU(2)_\omega$ which appears in the space of ϕ, $\phi(\omega)$ is an eigenvector of $(\underline{\vec{a}} \cdot \underline{t})^2$ and

$$2nN = 2[t(t+1) - m_t^2] \text{ is an integer } > 0 , \tag{6.17}$$

barring the trivial case $\phi(\omega) = \underline{\vec{a}}(\omega) = \text{const}$.

VII. SOLUTION OF THE ANGULAR EQUATIONS

In this section we obtain the complete solution of the angular equations (5.44). The first step is to use Eq. (5.44b) to show that the functions $\underline{u}$ and $\underline{v}$ which make up $\underline{f}$ in (4.16) are not independent but satisfy the relationship

$$\underline{f} = \underline{u} = -\underline{v} . \tag{7.1}$$

For this purpose we introduce the identity

$$L_i L_j + \partial_i \partial_j = (\delta_{ij} - \hat{r}_i \hat{r}_j) \vec{L}^2 \tag{7.2}$$

for functions on the sphere. Using this identity we have, from the definition (4.17) of $\underline{u}$, the relationship

$$\underline{\vec{a}}(\omega)=\frac{-2}{L(L+1)}\vec{L}\underline{u}(\omega), \tag{7.3}$$

which shows that $\underline{u}(\omega)$ acts as a potential for $\underline{\vec{a}}(\omega)$. We next note that, since for $\hat{r}\cdot\underline{\vec{a}}=0$

$$\underline{f}_{ij}=\epsilon_{ijk}\hat{r}_k\underline{f}, \tag{7.4}$$

Eq. (5.44b) yields

$$\underline{d}_i\underline{f}=0. \tag{7.5}$$

Using the definition $\underline{f}=2\underline{u}+\underline{v}$, we then have from (7.5)

$$\partial_i\underline{v}+2[\underline{a}_i,\underline{u}]=-2\partial_i\underline{u}-[\underline{a}_i,\underline{v}]. \tag{7.6}$$

The terms on the right-hand side of (7.6) can be evaluated as follows: From (7.3) we obtain by taking the cross product with $\hat{r}$

$$\partial_i\underline{u}=\frac{L(L+1)}{2}\underline{b}_i, \tag{7.7}$$

and from (5.44d) and the definition (4.17) of $\underline{v}$ we obtain

$$[\underline{a}_i,\underline{v}]=\frac{\nu^2}{N}\underline{b}_i. \tag{7.8}$$

Hence (7.6) becomes

$$\partial_i\underline{v}+2[\underline{a}_i,\underline{u}]=-[L(L+1)+\nu^2/N]\underline{b}_i. \tag{7.9}$$

If we now take the (noncovariant) divergence of (7.9), recall that $\underline{\vec{a}}$ is divergence-free, and use (7.7) twice, we obtain

$$\vec{\partial}^2\underline{v}+L(L+1)[\underline{a}_r,\underline{b}_r]=2[L(L+1)+\nu^2/N]\underline{u}, \tag{7.10}$$

which from the definition (4.17) of $\underline{v}$ reduces to

$$\vec{\partial}^2\underline{v}-2L(L+1)\underline{v}=2[L(L+1)+\nu^2/N]\underline{u}. \tag{7.11}$$

Now from the definition (4.17) of $\underline{u}$ and from (5.44c) we see that $\underline{u}$ is a spherical harmonic of order L. Hence operating on (7.11) with $\vec{\partial}^2+L(L+1)$, we obtain

$$[\vec{\partial}^2+L(L+1)][\vec{\partial}^2-2L(L+1)]\underline{v}=0. \tag{7.12}$$

But for nontrivial L ($L\neq0$) the operator $\vec{\partial}^2-2L(L+1)$ is negative definite and therefore nonsingular. Thus (7.12) implies that

$$[\vec{\partial}^2+L(L+1)]\underline{v}=0. \tag{7.13}$$

Thus $\underline{v}$ is also a spherical harmonic of order L. If we now insert (7.13) into (7.11) we obtain

$$\underline{v}(\omega)=\lambda\underline{u}(\omega), \tag{7.14}$$

where

$$\lambda=-2[L(L+1)+\nu^2/N]/3L(L+1).$$

Thus $\underline{v}$, and therefore $\underline{f}$, are proportional to $\underline{u}$; indeed (4.16) yields

$$\underline{f}=(2+\lambda)\underline{u}. \tag{7.15}$$

We assume that the integrability condition (4.14) is satisfied nontrivially, i.e., $\underline{f}\neq0$. Then Eq. (4.15) reads

$$M_{\alpha\beta}u^\beta=0=M_{\alpha\beta}v^\beta. \tag{7.16}$$

To determine λ, we consider the equation

$$\underline{\vec{J}}\cdot\underline{\vec{a}}=2(\underline{u}+\underline{v}) \tag{7.17}$$

and operate on it by the mass matrix M. Now

$$M_{\alpha\beta}(\underline{\vec{J}}\cdot\underline{\vec{a}})^\beta=\underline{\vec{J}}\cdot(M_{\alpha\beta}\vec{a}^\beta)=c^2n(\underline{\vec{J}}\cdot\vec{a}^\alpha), \tag{7.18}$$

where we have used (3.15) (vanishing of the covariant derivative of M) and (6.14). From (7.16), (7.17), and (7.18), since $c^2n\neq0$ for a nontrivial solution, we obtain

$$\underline{u}+\underline{v}=0 \text{ or } \lambda=-1. \tag{7.19}$$

Using $\lambda=-1$ in (7.14), we also pick up the relation

$$\frac{\nu^2}{N}=\frac{L(L+1)}{2} \tag{7.20}$$

for the constants. This completes the first step in the solution of the angular equations.

The next step is to construct a self-coupling equation for $\underline{f}$. This is not difficult since we have $\underline{f}=\underline{u}$. First from (7.1) and (7.13) we see that

$$\vec{\partial}^2\underline{f}=-L(L+1)\underline{f}, \tag{7.21}$$

so that $\underline{f}$ also is a spherical harmonic of order L, and from (7.3) we have

$$\underline{a}_i(\omega)=\frac{2}{L(L+1)}L_i\underline{f}, \tag{7.22}$$

but from the angular equation (6.1) we have

$$d_i\underline{f}=\partial_i\underline{f}+[\underline{a}_i,\underline{f}]=0. \tag{7.23}$$

Inserting (7.22) into this equation, we obtain

$$\partial_i\underline{f}=\frac{2}{L(L+1)}[\underline{f},L_i\underline{f}], \tag{7.24}$$

which is the required self-coupling equation. Furthermore (6.1), which is similar to (3.10), means that the values of $\underline{f}$ in $\mathcal{G}$ for different ω are obtained from each other by a transformation of G [similar to (3.11)]. Hence

$$(\underline{f}(\omega),\underline{f}(\omega))=\kappa^2, \tag{7.25}$$

where κ^2 is a constant. To evaluate κ^2 we use (7.22), which gives

$$\nu^2=(\underline{a}_i,\underline{a}_i)$$
$$=\left[\frac{2}{L(L+1)}\right]^2(L_i\underline{f},L_i\underline{f})$$
$$=-\left[\frac{2}{L(L+1)}\right]^2(\underline{f},L^2\underline{f}).$$

Hence from (7.21) we have

$$\nu^2=\frac{4}{L(L+1)}(\underline{f},\underline{f})=\frac{4\kappa^2}{L(L+1)} \tag{7.26}$$

and from (7.20)

$$\kappa^2=NL^2(L+1)^2/8. \tag{7.27}$$

Finally, we note from (7.21) that $\underline{f}$ has the expansion

$$\underline{f}^\alpha(\omega)=\sum_m t_m^\alpha Y_m^L(\omega),\quad \bar{t}_m^\alpha=(-1)^m t_{-m}^\alpha \tag{7.28}$$

in terms of one spherical harmonic Y_m^L. Since $\underline{f}$ is real and the Y_m^L are complex, the expansion coefficients t_m^α are complex, but they satisfy the relation displayed.

To exploit the self-coupling equation (7.24), using the form (7.28) for $f(\omega)$ we need the following inversion formula which can be proved using standard spherical-harmonic analysis.

Let $\chi(\omega)$ be a function such that

$$\chi(\omega)=\sum c_{rs}Y_r^L(\omega)Y_s^L(\omega),\quad c_{rs}=c_{sr}. \tag{7.29}$$

Then

$$c_{rs}=\sum_J\left(\begin{pmatrix}L & L & J\\ r & s & \mu\end{pmatrix}\right)\int Y_\mu^J(\omega)\chi(\omega)d\omega, \tag{7.30}$$

where

$$\left(\begin{pmatrix}L & L & J\\ r & s & \mu\end{pmatrix}\right)=[4\pi(2J+1)]^{1/2}\times\begin{pmatrix}L & L & J\\ r & s & \mu\end{pmatrix}\Big/\begin{pmatrix}L & L & J\\ 0 & 0 & 0\end{pmatrix}(2L+1), \tag{7.31}$$

and $\left(\begin{smallmatrix}L & L & J\\ r & s & \mu\end{smallmatrix}\right)$ are the conventional 3-j symbols.[9]

When written out explicitly in terms of the structure constants the third component of the self-coupling equation (7.24) reads

$$\partial_3 f^\alpha(\omega)=\frac{2}{L(L+1)}c^\alpha_{\beta\gamma}f^\beta(\omega)L_3f^\gamma(\omega), \tag{7.32}$$

and if we now use the expansion (7.28) we see that it becomes

$$\sum_m t_m^\alpha\partial_3Y_m^L(\omega)=\sum_{rs}\frac{3-r}{L(L+1)}c^\alpha_{\beta\gamma}t_r^\beta t_s^\gamma Y_r^L(\omega)Y_s^L(\omega), \tag{7.33}$$

where we have used the antisymmetry of the structure constants in β and γ. Equation (7.33) is then of the form (7.29). Hence if we use the inversion formula we obtain

$$\frac{s-r}{L(L+1)}c^\alpha_{\beta\gamma}t_r^\beta t_s^\gamma=t_m^\alpha\sum_J\left(\begin{pmatrix}L & L & J\\ r & s & \mu\end{pmatrix}\right)\int d\omega\, Y_\mu^J(\omega)\partial_3Y_m^L(\omega), \tag{7.34}$$

where m is not summed over because $m=-\mu=r+s$ on account of the integration over ω and the additivity of the magnetic quantum number in the 3-j symbols.

Similarly, if we insert the expansion (7.28) into (7.25) we obtain

$$\kappa^2=\sum_{rs}t_r^\alpha t_s^\alpha Y_r^L(\omega)Y_s^L(\omega), \tag{7.35}$$

which is also of the form (7.29). Hence, by the inversion formula, we have

$$t_r^\alpha t_s^\alpha=\kappa^2\left(\begin{pmatrix}L & L & 0\\ r & s & 0\end{pmatrix}\right)=(-1)^r\delta_{r+s,0}\frac{\sqrt{4\pi}}{2L+1}\kappa^2. \tag{7.36}$$

We now use (7.36) to analyze (7.34) as follows: Define the matrix

$$P_{\alpha\beta}=\rho\sum_r t_r^\alpha t_{-r}^\beta(-1)^r,\quad \rho=\frac{2L+1}{\sqrt{4\pi}\,\kappa^2}. \tag{7.37}$$

It is easy to see that the matrix $P_{\alpha\beta}$ is a projection operator, since from (7.28) it is real and symmetric and from (7.36) we have

$$P_{\alpha\beta}P_{\beta\gamma}=P_{\alpha\gamma}. \tag{7.38}$$

Furthermore from (7.36) we see that

$$P_{\alpha\beta}t_r^\beta=t_r^\alpha \text{ and } P_{\alpha\beta}f^\beta(\omega)=f^\alpha(\omega), \tag{7.39}$$

so that the t_r^α and $f^\alpha(\omega)$ are eigenvectors of $P_{\alpha\beta}$ with eigenvalue unity. Finally, since from (7.28) and (7.37), respectively, we have for any vector λ_α

$$f^\alpha(\omega)\lambda_\alpha=0\rightarrow t_r^\alpha\lambda_\alpha=0\rightarrow P_{\beta\alpha}\lambda_\alpha=0, \tag{7.40}$$

we see that $f^\alpha(\omega)$ spans the complete subspace $\mathfrak{F}$ onto which $P_{\alpha\beta}$ projects. But since from (7.36) the trace of $P_{\alpha\beta}$ is $2L+1$, this subspace is $2L+1$ dimensional.

If we now return to (7.34) and operate with the projection operator $(\delta_{\alpha\alpha'}-P_{\alpha\alpha'})$, the right-hand side drops out and we obtain from (7.39)

$$(\delta_{\alpha\alpha'}-P_{\alpha\alpha'})\frac{s-r}{L(L+1)}c^{\alpha'}_{\beta\gamma}t_r^\beta t_s^\gamma=0. \tag{7.41}$$

But since $c^{\alpha}_{\beta\gamma}t^{\beta}_{r}t^{\gamma}_{s}$ vanishes in any case for $r=s$, we then have

$$(\delta_{\alpha\alpha'}-P_{\alpha\alpha'})c^{\alpha'}_{\beta'\gamma'}P_{\beta'\beta}P_{\gamma'\gamma}=0\,. \tag{7.42}$$

Now let ${}^{(r)}\mu$, $r=1,2$, be any two elements of the Lie algebra lying in the $(2L+1)$-dimensional subspace $\mathfrak{F}$ projected out by $P_{\alpha\beta}$. Then by hypothesis

$${}^{(r)}\mu={}^{(r)}\mu_{\alpha}\tau_{\alpha}={}^{(r)}\mu_{\alpha'}P_{\alpha'\alpha}\tau_{\alpha}\,, \tag{7.43}$$

and hence

$$[{}^{(1)}\mu,{}^{(2)}\mu]={}^{(1)}\mu_{\alpha'}{}^{(2)}\mu_{\beta'}P_{\alpha'\alpha}P_{\beta'\beta}c^{\gamma}_{\alpha\beta}\tau_{\gamma}=W^{\gamma}\tau_{\gamma}\,. \tag{7.44}$$

But from (7.42) we have

$$P_{\gamma\gamma'}W^{\gamma'}=W^{\gamma}\,. \tag{7.45}$$

Thus the commutator also lies in the subspace $\mathfrak{F}$ projected out by $P_{\alpha\beta}$. It follows that the $2L+1$ subspace $\mathfrak{F}$ of the Lie algebra projected out by $P_{\alpha\beta}$ is actually a subalgebra of $\mathcal{G}$.

Every compact Lie algebra is the direct sum of an Abelian Lie algebra and semisimple Lie algebra, the latter being generated by the commutators of the algebra. The Lie algebra $\mathfrak{F}$, which is compact since it is a subalgebra of $\mathcal{G}$, is generated by the fields $\underline{f}(\omega)=-\underline{v}(\omega)$ which are commutators [see Eq. (4.17)]; hence $\mathfrak{F}$ is semisimple.

For the last step of the solution we return to (7.36) which shows that the spatial rotation group acts on $\mathfrak{F}$ by its $(2L+1)$-dimensional irreducible representation:

$$D^{(L)}(R)_{\alpha\beta}f^{\beta}(\omega)=f^{\alpha}(R\omega)\,. \tag{7.46}$$

By this action, SO(3) is a group of automorphisms of the Lie algebra $\mathfrak{F}$; indeed, with the use of (7.30) and (7.39), one obtains from (7.36) the explicit expression for the structure constants of $\mathfrak{F}$,

$$c^{\alpha}_{\beta\gamma}=-\frac{(2L+1)^2L(L+1)}{4\pi\kappa^4}\sum_{rs}\sum_{J\mu}\frac{1}{r-s}\bar{t}^{\beta}_{r}\bar{t}^{\beta}_{s}t^{\alpha}_{m}\begin{pmatrix}L & L & J\\ r & s & \mu\end{pmatrix}\int Y^{j}_{\mu}(\omega)\partial_3Y^{L}_{m}(\omega)d\omega\,, \tag{7.47}$$

and the integration in this equation shows that the structure constants are invariant by the rotation on ω defined in (7.46). For a semisimple Lie group every connected Lie group of automorphisms is a group of inner automorphisms. Hence $\mathfrak{F}$ contains an SO(3) subalgebra which acts irreducibly [by (7.46)] on the $(2L+1)$-dimensional algebra $\mathfrak{F}$; this is impossible unless this SO(3) is $\mathfrak{F}$ itself. This requires L to be unity.

We denote by $SO(3)_I$ the fixed-SO(3) subalgebra of $\mathcal{G}$ generated by the $\underline{f}(\omega)$'s. From (7.1) and (7.3), we see that $SO(3)_I$ is also generated by the gauge fields $\vec{\underline{a}}(\omega)$. If there are several nonconjugated SO(3) subalgebras of $\mathcal{G}$ the Higgs mechanism will in general choose among them. Equations (7.24), (7.1), and (7.22) yield, up to an arbitrary orthogonal choice of basis in $SO(3)_I$,

$$f^{\alpha}=-\kappa\hat{r}_{\alpha},\quad a^{\alpha}_{i}=\kappa\epsilon_{\alpha ij}\hat{r}_{j},\quad \kappa=(N/2)^{1/2}. \tag{7.48}$$

Since we had absorbed the charge e in the definition of $\underline{f}(\omega)$ and $\vec{\underline{a}}(\omega)$, Eq. (7.48) is exactly the 't Hooft–Polyakov solution for the charge $e_N=e(2/N)^{1/2}$. This charge, which varies with the embedding in G, is the natural charge for $SO(3)_I$, and the monopole strength has the minimum value of $4\pi/e_N$.

Equation (7.48) completely solves the angular Eqs. for $\underline{a}(\omega)$, and it remains only to consider the Higgs fields. From (5.44a) and (7.48)

$$\partial_i\phi^A=-(t^{\alpha})^{AB}a^{\alpha}_{i}\phi^B=-(t^{\alpha})^{AB}\kappa\epsilon_{\alpha ij}\hat{r}_j\phi^B, \tag{7.49}$$

which implies that

$$(L_i+\bar{t}_i)\phi(\omega)=0\,, \tag{7.50}$$

where the $\bar{t}_i$ are the generators of $SO(3)_I$, normalized in the conventional way. Equations (7.50) and (5.44c) show that the SO(3) representation of $\phi(\omega)$ if fixed to be $t(t+1)=l(l+1)$. For given $l=t$, it is easy to construct explicit solutions for $\phi(\omega)$ in terms of linear combinations of real and imaginary parts of appropriate spherical harmonics with real coefficients.

The above results show that the condition of separation of variables is so strong that it reduces the solution for general G to that for $G=SO(3)$, and that for SO(3) the solution differs from the original 't Hooft–Polyakov ansatz only in that the Higgs field can belong to an arbitrary integral representation of SO(3) [and then only if the spin and isospin compensate as in (7.50)]. Thus the 't Hooft–Polyakov solution for $G=SO(3)$ is essentially the only solution with separated variables for arbitrary G.

VIII. RADIAL EQUATIONS

Finally we come to the discussion of the radial equations. Using (7.50) to determine the constants and replacing $l(l+1)$ by its gauge-invariant counterpart $t(t+1)$, the radial equations (5.45a), (5.45b) are easily seen to reduce to

$$r^2H''=H[(t+1)K^2+\mu^2(S-r)(\sigma^2S+r)],\quad H=e_NcS\,,$$
$$r^2K''=K[(K^2-1)+\tfrac{1}{2}t(t+1)H^2],\quad K=1-R\,. \tag{8.1}$$

The boundary conditions for these equations are

$$K(0)=1,\quad H(0)=0$$

and (8.2)

$$K(r) \to 0, \quad S(r) \sim r \quad \text{as } r \to \infty.$$

Inserting the conditions at $r \to \infty$ in the radial equations we easily see that $K(r)$ and $S(r)$ take the asymptotic forms

$$K(r) \sim \text{const} \times \exp\{-e_N c[\tfrac{1}{2}t(t+1)]^{1/2} r\},$$
$$S(r) \sim r + \text{const} \times \exp[-\mu(1+\sigma^2)^{1/2} r]. \tag{8.3}$$

Thus $K(r)$ and $S(r) - r$ describe fields of mass $e_N c[t(t+1)/2]^{1/2}$ and $\mu(1+\sigma^2)^{1/2}$, respectively. This is exactly what we should expect since, as we have shown in the Appendix, $\mu(1+\sigma^2)^{1/2}$ is the mass acquired by the Higgs field after the spontaneous symmetry breaking and, from (2.22), $e_N c[t(t+1)/2]^{1/2}$ is the mass acquired by the gauge field through the Higgs mechanism.

It is worth noting that the radial equations are derivable directly from the Hamiltonian (3.3), which now takes the form

$$\mathcal{H} = \frac{4}{e_N^2} \int_0^\infty \frac{dr}{r^2} \left\{ (rK')^2 + \tfrac{1}{2}(rH' - H)^2 + \tfrac{1}{2}(1-K^2)^2 + \frac{t(t+1)}{2} H^2 K^2 + \frac{\mu^4}{4} H^2 [\sigma^2 S^2 + \tfrac{4}{3}(1-\sigma^2) r S - 2r^2] \right\}. \tag{8.4}$$

Now the existence of nontrivial solutions that minimize $\mathcal{H}$ [equivalently, solutions to the radial equations (8.1)] has been proved by Tyupkin *et al.*,[10] in the case when $t=1$. It is easy to verify that the proof can be extended to any $t>0$. Thus finite-energy solutions to the radial equations (8.1) certainly exist.

An interesting feature of the solutions for general t is that they provide a *mass formula* $m(t) = \mathcal{H}$ on account of the dependence on t in (8.4). Differentiating $\mathcal{H}$ with respect to t (which is isospin, not time), we obtain

$$\frac{d\mathcal{H}}{dt} = \int d^3x \frac{\delta \mathcal{H}}{\delta \psi_s} \frac{\partial \psi_s}{\partial t} + \frac{\partial \mathcal{H}}{\partial t}, \quad \psi_s = (H, K)_{\text{solution}} \tag{8.5}$$

and, since the first term on the right-hand side of (8.5) vanishes on account of the field equations, we have

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = \left(\frac{2t+1}{2}\right) \int_0^\infty \frac{dr}{r^2} H^2 K^2 > 0.$$

Thus the mass is a monotonically increasing function of the isospin and the solution with $t=1$ is the lowest-lying nontrivial solution.

In conclusion we note that the above formalism can be extended to include dyons by making the ansatz

$$eA_0^\alpha = \hat{r}^\alpha \left(\frac{J(r)}{r}\right). \tag{8.6}$$

Since A_0^α plays the role of a second Higgs field, which lies in the adjoint representation and which has no potential and no interaction[11] with $\phi(\omega)$, one sees by analogy with ϕ that the only effect of A_0^α on the field equations is to add to the Hamiltonian density a term of the form

$$\frac{1}{e^2 r^2} [\tfrac{1}{2}(rJ' - J)^2 + \tfrac{1}{2} J^2 K^2]. \tag{8.7}$$

The corresponding equations of motion are then the generalization of the Julia and Zee equations[12] for the special case $t=1$.

Note added in proof. We are grateful to John Rawnsley for pointing out that since the value of $m(t)$ is less than the value of (8.4) for any trial functions and the t dependence drops out for nonoverlapping trial functions H and K, $m(t)$, although it increases monotonically, must tend to a finite limit $m(\infty)$ as $t \to \infty$. A more detailed discussion of the radial equations will be given elsewhere.

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APPENDIX

In this appendix we study the symmetry-breaking minima of G-invariant polynomials, where G is a compact Lie group acting linearly on a finite-dimensional real vector space $\mathcal{E}$:

$$G \ni g \to \Delta(g) \in \mathcal{L}(\mathcal{E}) \tag{A1}$$

where $\mathcal{L}(\mathcal{E})$ is the space of linear operators on $\mathcal{E}$.

For this representation we only assume[13] that it does not contain the trivial representation, i.e., there is no nonzero vector in $\mathcal{E}$ invariant by G. Since G is compact, this representation is equivalent to an orthogonal representation; hence, without loss of generality, we assume

$$\Delta(g^{-1}) = \Delta(g)^T. \tag{A2}$$

We denote by Φ an arbitrary element of $\mathcal{E}$ and by $p^{(n)}(\Phi)$, a G-invariant nth-degree homogeneous polynomial on $\mathcal{E}$. It satisfies

$$p^{(n)}(\Delta(g)\Phi) = p^{(n)}(\Phi), \quad p^{(n)}(\lambda\Phi) = \lambda^n p^{(n)}(\Phi). \tag{A3}$$

Any linear combination of two such polynomials for a given n is again a $p^{(n)}$. So the most general G-invariant polynomial potential $V(\Phi)$ is of the form[14]

$$V(\Phi) = \sum_{n=0}^{N} p^{(n)}(\Phi). \tag{A4}$$

Since we are interested only in the extrema of V, the constant term is irrelevant. From our assumption of no invariant vectors on $\mathcal{E}$, V has no linear term, and its gradient $dV/d\Phi$ vanishes at the origin. To obtain symmetry breaking we do not want the extremum at the origin to be a minimum. Hence the quadratic term on at least one G-invariant subspace of $\mathcal{E}$ has to be of the form $-k(\Phi,\Phi)$, $k>0$, where (Φ,Φ) is a G-invariant, strictly positive orthogonal scalar product on $\mathcal{E}$. We assume that, on the whole space $\mathcal{E}$,

$$p^{(2)}=-\tfrac{1}{2}\mu^2(\Phi,\Phi),\quad (\Phi,\Phi)\geq 0,\quad (\Phi,\Phi)=0\rightarrow\Phi=0 \quad \text{(A5)}$$

and leave to the reader the straightforward generalization. The notation $\mu^2/2$ for the coefficient of the quadratic term is traditional in the literature on this subject, μ giving the mass scale. Furthermore, V has to be bounded below in order to have a lowest, finite-value minimum; this requires

$$p^{(N)}(\Phi)\geq 0 \text{ for every } \Phi\,, \quad \text{(A6)}$$

which implies that N is even. The lowest possible value of N is therefore 4. It happens that this is the only possible value for the quantum theory to be renormalizable. However, no such limitation exists for a purely classical field theory.

Given a $p^{(n)}(\Phi)$, one obtains by polarization an n-multilinear form

$$l^{(n)}(\Phi_1,\ldots,\Phi_n)=\frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\sum_{(k)}p^{(n)}\left(\sum_k\Phi_i\right), \quad \text{(A7)}$$

where $\sum_k$ means the sum over $n-k$ different vectors among the n variables of $l^{(n)}$ and the first $\sum_{(k)}$ means the sum over all possible terms. For example, for $n=3$,

$$l^{(3)}(\Phi_1,\Phi_2,\Phi_3)=\frac{1}{3!}\left[p^{(3)}(\Phi_1+\Phi_2+\Phi_3)-p^{(3)}(\Phi_1+\Phi_2)-p^{(3)}(\Phi_2+\Phi_3)-p^{(3)}(\Phi_3+\Phi_1)+p^{(3)}(\Phi_1)+p^{(3)}(\Phi_2)+p^3(\Phi_3)\right]. \quad \text{(A8)}$$

$l^{(n)}$ is completely symmetrical in its n variables and is G invariant. We can also define from $p^{(n)}$ a vector $t^{(n-1)}$ and a linear operator $T^{(n-2)}$ depending linearly and completely symmetrically on, respectively, $(n-1)$ and $(n-2)$ variables by

$$l^{(n)}(\Phi_1,\ldots,\Phi_n)=(\Phi_n,t^{(n-1)}(\Phi_1,\ldots,\Phi_{n-1}))=(\Phi_n,T^{(n-2)}(\Phi_1,\ldots,\Phi_{n-2})\Phi_{n-1}), \quad \text{(A9)}$$

i.e.,

$$T^{(n-2)}(\Phi_1,\ldots,\Phi_{n-2})\Phi_{n-1}=t^{(n-1)}(\Phi_1,\ldots,\Phi_{n-1})\,. \quad \text{(A9}')$$

Note that T is a symmetric operator and that t and T are covariant under G:

$$(T^T=T)\,, \quad \text{(A10)}$$

$$\Delta(g)t^{(n-1)}(\Phi_1,\ldots,\Phi_{n-1})=t^{(n-1)}(\Delta(g)\Phi_1,\ldots,\Delta(g)\Phi_{n-1})\,, \quad \text{(A10}')$$

$$\Delta(g)T^{(n-2)}(\Phi_1,\ldots,\Phi_{n-2})\Delta(g)^T=T^{(n-2)}(\Delta(g)\Phi_1,\ldots,\Delta(g)\Phi_{n-2})\,. \quad \text{(A10}'')$$

We finally remark that the gradient $dp^{(n)}/d\Phi$ and the Hessian $d^2p^{(n)}/d\Phi^2$ of $p^{(n)}$ are given by

$$\frac{dp^{(n)}}{d\Phi}=nt^{(n-1)}(\Phi,\ldots,\Phi)\,,\quad \frac{d^2p^{(n)}}{d\Phi^2}=n(n-1)T^{(n-2)}(\Phi,\ldots,\Phi)\,. \quad \text{(A11)}$$

Consider the simplest examples

$$p^{(2)}=(\Phi,\Phi)\,,\quad t^{(1)}=2\Phi,\quad T^{(0)}=2I\quad (I=\text{identity matrix})\,, \quad \text{(A12)}$$

$$p^{(4)}=(\Phi,\Phi)^2\,,\quad t^{(3)}=\tfrac{1}{3}[(\Phi_1,\Phi_2)\Phi_3+(\Phi_2,\Phi_3)\Phi_1+(\Phi_3,\Phi_1)\Phi_2]\,, \quad \text{(A13)}$$

$$T^{(2)}(\Phi_1,\Phi_2)=\tfrac{1}{3}[(\Phi_1,\Phi_2)I+|\Phi_1\rangle\langle\Phi_2|+|\Phi_2\rangle\langle\Phi_1|]\,, \quad \text{(A13}')$$

where $|\Phi_1\rangle\langle\Phi_2|$ is the dyadic operator whose action on Φ_3 is $\Phi_1(\Phi_2,\Phi_3)$. Hence

$$\frac{d}{d\Phi}(\Phi,\Phi)^2=4\Phi(\Phi,\Phi)\,,$$

$$\frac{d^2}{d\Phi^2}(\Phi,\Phi)^2=\tfrac{4}{3}(\Phi,\Phi)(I+2P_\Phi)\,, \quad \text{(A14)}$$

where P_Φ is the orthogonal projector on Φ, i.e., $P_\Phi=(\Phi,\Phi)^{-1}|\Phi\rangle\langle\Phi|$.

If we denote the general third- and fourth-degree invariants by $\theta(\Phi)$ and $\omega(\Phi)$, we can write them as

$$\theta(\Phi)=(b(\Phi),\Phi),\quad \omega(\Phi)=(t(\Phi),\Phi)\,, \quad \text{(A15)}$$

where for simplicity we have set the bilinear vector[15] $t^{(2)}(\Phi,\Phi)=b(\Phi)$ and the trilinear vector $t^{(3)}(\Phi,\Phi,\Phi)=t(\Phi)$. With this notational simplification the most general inhomogeneous fourth-degree Higgs potential has the form

$$V(\Phi)=\tfrac{1}{4}[\lambda(\Phi,\Phi)^2+\alpha\omega(\Phi)]+\frac{\beta}{3}\mu\theta(\Phi)-\frac{\mu^2}{2}(\Phi,\Phi), \tag{A16}$$

the invariant $\omega(\phi)$ is in general linearly independent from $(\Phi,\Phi)^2$, and further it is assumed to be positive since, if necessary, we can always add a term $k(\Phi,\Phi)^2$, $k>0$ to $\omega(\Phi)$ and call the new term $\omega(\Phi)$. Let χ_m and χ_M be, respectively, the minimum and maximum of $\omega(\Phi)$ on the unit sphere $(\Phi,\Phi)=1$ $(0\leq\chi_m<\chi_M)$. The condition that V is bounded below gives

$$0<\lambda+\alpha\chi \text{ for } \chi_m\leq\chi<\chi_M. \tag{A17}$$

From (A16), we obtain

$$\frac{dV}{d\Phi}=\alpha t(\Phi)+\beta\mu b(\Phi)+[\lambda(\Phi,\Phi)-\mu^2]\Phi \tag{A18}$$

$$=\{\alpha T(\Phi,\Phi)+\beta\mu\mathfrak{D}(\Phi)+[\lambda(\Phi,\Phi)-\mu^2]I\}\Phi, \tag{A19}$$

$$\frac{d^2V}{d\Phi^2}=3\alpha T(\Phi,\Phi)+2\beta\mu\mathfrak{D}(\Phi)+[\lambda(\Phi,\Phi)-\mu^2]I+2\lambda(\Phi,\Phi)P_\Phi, \tag{A20}$$

and $V(\Phi)$ is minimum for the Φ which satisfies

$$\left.\frac{dV}{d\Phi}\right|_{\phi=\phi_0}=0,\quad \left.\frac{d^2V}{d\Phi^2}\right|_{\phi=\phi_0}>0. \tag{A21}$$

Furthermore the spectrum of $d^2V/d\Phi^2|_{\phi=\phi_0}$ gives the squares of the masses for the Higgs fields Φ; however, the Φ's in the kernel of $d^2V/d\Phi^2$, i.e., those of the form $t(a)\Phi_0$ for all $a\in\mathcal{G}$ (they form the tangent plane to the orbit $G\cdot\Phi$) constitute the zero-mass Goldstone bosons. The gradients of the latter fields $t^\alpha\nabla_\mu\Phi$ become part of the gauge-field components A_μ^α which then acquires a mass in the Higgs phenomenon.

Let us first study the simple case $\beta=0$, which is the most general case when there are no third-degree invariants[16] on $\mathcal{E}$.

Let Φ_0 be a vector of $\mathcal{E}$ which minimizes V; we write

$$\Phi_0=c\phi \text{ with } (\phi,\phi)=1,\quad c>0. \tag{A22}$$

Then Eq. (A18) yields

$$c^2=\frac{\mu^2}{\lambda+\alpha\chi_0} \text{ with } \chi_0=\omega(\phi), \tag{A23}$$

which is consistent with (A17). We also obtain from (A16) the value of the minimum for the potential $V_{\min}$,

$$V_{\min}=-\frac{\mu^4}{4}\,\frac{1}{\lambda+\alpha\chi_0}. \tag{A24}$$

This shows that

$$\text{if } \alpha>0,\ \chi_0=\chi_m;\ \text{if } \alpha<0,\ \chi_0=\chi_M. \tag{A25}$$

The case $\alpha=0$ [which occurs when G is the orthogonal group O(n) and $\mathcal{E}$ is the n-dimensional space] is well known. In any case, the mass square in the radial direction is

$$\left(\phi,\frac{d^2V}{d\phi^2}\phi\right)=3(\lambda+\alpha\chi_0)c^2-\mu^2=2\mu^2. \tag{A26}$$

Finally, we need to compute the polynomial $W(\xi)$, used in Eq. (5.5′) of the text, which is defined by

$$\frac{dV}{d\phi}(\xi c\phi)=W(\xi)\mu^2\phi. \tag{A27}$$

We find from (A18) and (A23) that

$$W(\xi)=\xi^2-1. \tag{A27′}$$

Returning to the general case (A16) when both α and β are nonvanishing, we assume that the minimum is attained when the three vectors $t(\Phi)$, $b(\Phi)$, and Φ are collinear. More generally it might happen that $t(\Phi,\Phi,\Phi)$, $b(\Phi)$, and Φ form a two-plane and then there are two independent conditions for the minimum. We shall not consider the latter case here. The lower bound of V is therefore reached on the orbit of $\Phi_0=c\phi$ with

$$(\phi,\phi)=1,\quad b(\phi)=\phi_v\phi=\eta\phi,$$
$$t(\phi)=\chi_0\phi \text{ with } \chi_m\leq\chi_0<\chi_M; \tag{A28}$$

the extremum condition (A18) then gives

$$(\lambda+\alpha\chi_0)c^2+\beta\mu\eta c-\mu^2=0.$$

As we expect for a fourth-degree polynomial with a maximum at the origin, there are two solutions for c; that which corresponds to the lowest radial minimum and therefore to the lowest bound for V is (with the convention $c>0$)

$$c=\frac{\mu}{(\lambda+\alpha\chi)^{1/2}}[(1+\rho^2)^{1/2}+\rho],$$
$$\rho=\frac{-\beta\eta}{2(\lambda+\alpha\chi_0)^{1/2}}\geq0. \tag{A29}$$

Then $W(\xi)$, as defined in (A27) and used in Eq. (5.5′), can be written as

$$W(\xi)=(\xi-1)(\sigma^2\xi+1) \text{ with } \sigma=(1+\rho^2)^{1/2}+\rho\geq1. \tag{A30}$$

Note that

$$c=\frac{\mu}{(\lambda+\alpha\chi_0)^{1/2}}\sigma \tag{A31}$$

gives a generalization of (A23) when $\beta\neq0$. The generalizations of (A24) and (A26) are easily found to be

$$V_{\min} = -\frac{\mu^4}{\lambda + \alpha\chi_0}\,\frac{4(1+\rho\sigma)^2 - 1}{3} = -\frac{\mu^4}{\lambda + \alpha\chi_0}\,\frac{\sigma^2(\sigma^2+2)}{3} \tag{A32}$$

and the "radial" mass squared

$$\left(\phi, \frac{d^2V}{d\phi^2}\bigg|_{\phi_0}\phi\right) = 2\mu^2(1+\rho\sigma) = \mu^2(1+\sigma^2)\,, \tag{A33}$$

where we have used the identity $\sigma^2 = 1 + 2\rho\sigma$.

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$$\left(\begin{pmatrix} L & L & J \\ r & s & \mu \end{pmatrix}\right)$$

is not defined for odd J. For odd J, our equations turn out to be identities $0=0$.

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