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Interacting sine-Gordon solitons and classical particles: A dynamic equivalence

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Motivated by the work of Fogel *et al.*, who showed that the dynamics of the classical sine-Gordon soliton in the presence of external perturbations is essentially the dynamics of a Newtonian particle, we have looked for such particlelike behavior in those cases where sine-Gordon solitons interact with each other rather than with an applied field. We find that, although a direct soliton-particle equivalence is not possible in these cases, there is an indirect representation in terms of the poles of the Hamiltonian density when the latter is assumed to be a function of a complex space variable. These poles have well-defined motions in the complex space plane, and we show that there is a strong correlation between the dynamics of the poles, considered as classical particles, and those of the interacting solitons they represent. Thus, all the qualitative features of the two-soliton (and/or antisoliton) interactions, as well as analytic expressions for the forces between them, are predicted by analyzing the motions of the poles of the Hamiltonian density in the complex space plane. If this "particle" picture is used to identify the individual solitons in a two-soliton profile, then it turns out that the solitons move at a constant speed, but with a changing rest mass. Hence, solitons do not maintain their "free" mass identity when interacting with each other. A few analogies with quantum behavior are also discussed.

I. INTRODUCTION

The sine-Gordon equation (SGE) is one of several classical quasilinear partial differential equations, the Korteweg–de Vries and Boussinesq equations are other well-known examples, which are currently being used to model the dynamics of nonlinear wave phenomena, that is, the generation, propagation, and interaction of waves in media which are nonlinear and dispersive, but not dissipative.[1] These equations all represent theories of single scalar fields in one space and one time dimension and, interestingly enough, they all possess exact soliton solutions and some of the associated formal properties (infinite number of conservation laws, Bäcklund transformations, inverse method of solution, separable Hamiltonians, etc.) The importance of this is easily seen if we recall that the soliton is the nonlinear, dispersive equivalent of the linear, nondispersive wave packet. The analogy with particle behavior is strong and hence this family of equations should make an ideal starting point for nonperturbative theories of particles in which the particles appear as pulselike excitations in nonlinear wave fields, as, for example, in the Born-Infeld model of the electron.[2]

What distinguishes the SGE from the other members of this family is that it is, essentially, the only Lorentz-covariant member and hence is the "natural" equation to use for a fully relativistic, quantum theory of elementary particles. Its use in this context was first proposed by Skyrme,[3] who, after rediscovering its soliton modes,[4] suggested that in a quantized field theory these modes appear as fermions with interactions of the Thirring-model type. This conjecture has recently been proved by Coleman,[5] who showed, within the framework of perturbation theory, that the quantum SGE is completely equivalent to the charge-zero sector of the massive Thirring model, modulo an identification of couplings and fields. Even more interesting, from the classical point of view, is the work of Dashen *et al.*,[6] who quantize soliton solutions of the classical SGE (using a field-theoretic version of the WKB method) and obtain results in agreement with Coleman even for strong coupling.[7] This is both surprising and suggestive, since the WKB method is a semiclassical approximation which effectively turns a quantum problem into a classical one, whereas Coleman's proof is based entirely on the quantum theory and is exact.

Now the fact that the semiclasscal analysis, mentioned above, shows that the soliton mode of the classical SGE corresponds to a particle in the quantum theory — the quantum soliton — leads us to ask whether this particlelike behavior is present at the classical level itself. In particular, we can ask (a) whether the dynamics of the soliton is equivalent to that of a classical Newtonian particle and (b) whether the N-soliton solution can be represented as an N-body interaction of classical particles. As far as (a) is concerned, a partial answer has been provided in a recent paper by Fogel *et al.*[8] They show that the dynamics of the translation mode of the classical soliton, or, what is the same thing, the motions of its center of mass in the presence of external perturbations, are essentially those of a Newtonian particle moving in external fields of force. Thus, the answer to (a) seems to be affirmative. The N-soliton solution, however, is quite a different matter. Firstly, since there are no static (i.e., permanent profile) solutions for $1<N<\infty$,[9] the individual solitons in an N-soliton solution can only be identified in the asymptotic limit of time $\to\pm\infty$. For all finite times their identities are lost in the ever-changing total profile. Secondly, the motion of a soliton in the presence of another one is an extremely large, rather than a small, effect and hence cannot be treated in perturbation theory.

Thus, the problem of the translation mode for the N-soliton case can be posed as follows: Can we obtain an alternative representation of the N-soliton solution in which the centers of mass of the individual solitons are clearly identifiable at all times and the dynamics of their interactions with each other is made explicit? This paper is a report on an attempt to obtain such a representation. The main results of the investigation are as follows:

(1) Associated with the two-soliton solution is an invariant which can be used to construct the general two-soliton interaction from a linear superposition of two single solitons. The method can then be extended to construct the N-soliton solution ($N>2$) from a linear superposition of N single solitons by considering the constituent solitons in pairs. Thus, the N-soliton interaction may be thought of as a combination of two-soliton interactions and hence is somewhat similar to the N-body interaction in classical electrodynamics and Newtonian gravitation.

(2) If the space coordinate is extended to complex values, then the Hamiltonian densities of the one- and two-soliton solutions develop an infinite set of regularly spaced poles along lines parallel to the imaginary axis. The single-soliton solution has one line of poles, while the two-soliton solution has two lines which are symmetrically positioned when the solitons are in their center-of-velocity frame. The real parts of these poles are time-dependent and hence, as time develops, they move across the complex plane along lines parallel to the real axis. If we analyze the dynamics of this motion we see that it corresponds to that of a classical particle moving in a potential which is constant for the single soliton, repulsive for the soliton-soliton interaction and attractive for the soliton-antisoliton interaction. Thus the poles of the complex Hamiltonian give us a localized or particlelike representation of free and interacting solitons.

(3) If this "particle" representation is used to identify the individual solitons in the two-soliton profile then their dynamical behavior can be interpreted as follows: Interacting solitons move in a fashion so as to preserve their speeds, but change their rest mass in a well-defined way (the rest-mass energy is transformed into interaction energy and vice versa). Thus solitons do not maintain their "free" mass identity when interacting with each other.

The analysis leading to these results is given in Secs. II, III, and IV below. Section V contains our concluding remarks and some conjectures.

II. GENERAL FORMALISM

The SGE is the equation of motion for a theory of a single, dimensionless scalar field ϕ, in one space and one time dimension, whose dynamics is determined by the Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}(\phi_t^{\,2}-c^2\phi_x^{\,2})+\frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)-\mu. \tag{2.1}$$

Here c is a limiting velocity while m, λ, and μ are real parameters. ϕ_t and ϕ_x are the partial derivatives of ϕ with respect to t and x, respectively. In the terminology of quantum field theory, m is the mass associated with the normal modes of the linearized theory (i.e., the Klein-Gordon equation), while λ/m^2 is a dimensionless, coupling constant that measures the strength of the interaction between these normal modes. In classical theory, of course, m is proportional to the characteristic frequency of these normal modes.

If we now scale the variables so that

$$x\to x/m\,,\quad t\to t/m\,,\quad \phi\to m\phi/\sqrt{\lambda} \tag{2.2}$$

and, as is conventional, set $c=1$, then the Lagrangian density becomes

$$\mathcal{L}=\frac{m^4}{2\lambda}[(\phi_t^{\,2}-\phi_x^{\,2})+2\cos\phi]-\mu \tag{2.3}$$

with the corresponding Hamiltonian density being given by

$$\mathcal{H}=\frac{m^4}{2\lambda}(\phi_t^{\,2}+\phi_x^{\,2}-2\cos\phi)+\mu. \tag{2.4}$$

By choosing

$$\mu=m^4/\lambda \tag{2.5}$$

the minimum energy of the theory is made zero and (2.4) can be written as

$$\mathcal{H}=\frac{m^4}{2\lambda}[\phi_t^{\,2}+\phi_x^{\,2}+2(1-\cos\phi)]\,. \tag{2.6}$$

Except for the scaling factor m^4/λ, this is the expression normally employed for the Hamiltonian density in classical SGE theory. However, in classical physics there is no natural scale of energy and hence this factor is irrelevant. On the other hand, in quantum theory there is a natural scale of energy, namely Planck's constant $\hbar$, and hence the factor becomes important.[5] Note that, in this connection, the semiclassical analysis mentioned in the Introduction is a small-$\hbar$ approximation.

If we use the Lagrangian density written in terms of the scaled variables then the SGE takes the form

$$\phi_{xx} - \phi_{tt} = \sin\phi \,, \tag{2.7}$$

with an obvious notation for the second partial derivatives. We shall work in terms of the scaled variables in the bulk of what follows [i.e., on the basis of Eq. (2.7)], but shall, on occasion, return to the original variables of (2.1).

The SGE is obviously invariant under Lorentz transformations as well as translations in space and time. In addition, it possesses some discrete symmetries, since it is invariant under the transformations

$$x \to -x, \quad t \to -t, \text{ and } \phi \to 2n\pi \pm \phi, \quad n \in Z \,. \tag{2.8}$$

The last symmetry is due to the periodic nature of the potential $(1 - \cos\phi)$.

III. N-SOLITON SOLUTIONS AND CANONICAL FRAMES

The vacuum solutions of the SGE are given by

$$\phi = 2n\pi \,, \quad n \in Z \tag{3.1}$$

and correspond to the infinite number of discrete and degenerate absolute minima of the potential $(1 - \cos\phi)$. However, since we have adjusted the vacuum energy to zero these solutions are rather trivial. The first nontrivial finite-energy solution which cannot be reached from the vacuum by perturbation is the soliton

$$\phi = 4 \tan^{-1}[\exp(\pm\gamma\xi + \alpha)] \,, \tag{3.2}$$

where $\xi = (x - ut)$, $\gamma = (1 - u^2)^{-1/2}$, and α is a real parameter. The physical picture is that of a solitary kink traveling in the positive x direction with a constant speed u, while γ is the Lorentz factor of the observer's frame and α is the phase of the wave at the origin. The amplitude of the soliton is bounded by 0 and 2π and the plus and minus signs correspond to profiles which increase and decrease with increasing x, respectively. It is conventional to call the first profile a soliton and the second an antisoliton.

Since the soliton moves at a constant speed we can make a transformation to its rest frame. This is accomplished by means of a Lorentz boost and a translation leading to the static solution

$$\phi = 4 \tan^{-1} e^{\pm x} \,, \tag{3.3}$$

where x is now the space coordinate in the rest frame. In terms of the unscaled variables this solution takes the form

$$\phi = \frac{4m}{\sqrt{\lambda}} \tan^{-1} e^{\pm mx} \,, \tag{3.4}$$

and so we see that the "width" of the kink profile is proportional to $1/m$. Increasing m sharpens the kink and decreasing m flattens it. Note that decreasing λ also reduces the soliton width.

Although the single soliton is interesting to study in its own right, it is essentially a free entity. It corresponds to a free particle and the most we can do with it is to investigate its behavior in the presence of weakly coupled fields as was done, for example, in Ref. 8. Thus, as far as particle physics is concerned, both classical and otherwise, the more interesting solutions of the SGE are those that tend asymptotically to a linear combination of single solitons. Such solutions can then be thought of as an interaction of solitons and, using this picture, it is then only one more step to see if they can indeed be represented as interacting particles.

The expressions we need to carry through this analysis can be found in a paper by Hirota,[10] who has written down solutions of the SGE that tend asymptotically to N solitons, each with a different speed (N-soliton solution). However, Hirota's formulas are somewhat involved and it is difficult to see how the solution is constructed in terms of interacting solitons. Thus, our first task is to rewrite Hirota's solutions in a form which makes this feature transparent. We do this in two stages. Firstly, we analyze Hirota's two-soliton solution in a preferred or "canonical" frame of reference. This leads to an invariant which characterizes the interaction. We then construct the N-soliton solution from a linear superposition of N solitons by considering them in pairs and using the corresponding canonical frame invariants.

To get an idea of what we mean by a canonical frame, consider the single soliton. Here the standard or canonical frame is the rest frame and from the static solution (3.3) all other forms of the single-soliton solution can be generated by operating with elements of the symmetry groups of the SGE. Similarly, all weak-coupling perturbations of the soliton are best understood in this frame; indeed perturbation about a static solution is sometimes essential.[7]

For the two-soliton solution the choice of a canonical frame is again unambiguous and unique. Since this solution does not have a permanent profile, symmetry dictates that we choose the center-of-velocity frame. However, for the N-soliton solution, with $N > 2$, there is no unique choice of frame and the best one can do is to consider the solitons in pairs via their corresponding center-of-velocity frames. Our first step, therefore, is to look at the two-soliton solution.

Hirota's formula for the two-soliton solution is

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{2}\exp(\gamma_i\xi_i+\alpha_i)}{1+\exp[B_{12}+\sum_{i=1}^{2}(\gamma_i\xi_i+\alpha_i)]}, \tag{3.5}$$

where $\xi_i=x-u_it$, $\gamma_i=(1-u_i^2)^{-1/2}$, the α_i's are phase factors, and

$$\exp(B_{12})=\frac{(\gamma_1-\gamma_2)^2-(\gamma_1u_1-\gamma_2u_2)^2}{(\gamma_1+\gamma_2)^2-(\gamma_1u_1+\gamma_2u_2)^2}. \tag{3.6}$$

With one notable exception this solution tends asymptotically to the sum of two solitons, each with velocity u_i in the x direction. The exception is when $u_1=u_2$. In this case the solution degenerates into a single soliton. To transform this solution to its center-of-velocity frame we first make the translations

$$\begin{aligned} x\to x'&=x+\frac{\gamma_2u_2(\alpha_1+\ln u)-\gamma_1u_1(\alpha_2+\ln u)}{\gamma_1\gamma_2(u_2-u_1)},\\ t\to t'&=t+\frac{\gamma_2(\alpha_1+\ln u)-\gamma_1(\alpha_2+\ln u)}{\gamma_1\gamma_2(u_2-u_1)}, \end{aligned} \tag{3.7}$$

where

$$u=\frac{1-u_1u_2-[(1-u_1^2)(1-u_2^2)]^{1/2}}{u_1-u_2} \tag{3.8}$$

is the common speed of the solitons in their canonical frame.

After translation, (3.5) reduces to

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{2}\exp(\gamma_i\xi_i')}{u[1-\exp(\gamma_i\xi_i')]}. \tag{3.9}$$

We now make the Lorentz transformations

$$\begin{aligned} x'\to x''&=\gamma'(x'-vt'),\\ t'\to t''&=\gamma'(t'-vx'), \end{aligned} \tag{3.10}$$

where the boost velocity v is given by

$$v=\frac{1+u_1u_2-[(1-u_1^2)(1-u_2^2)]^{1/2}}{u_1+u_2}, \tag{3.11}$$

and $\gamma'=(1-v^2)^{-1/2}$. A simple computation now leads to the canonical form for the two-soliton solution

$$\tan(\tfrac{1}{4}\phi)=\frac{\exp[\gamma(x-ut)]+\exp[\gamma(x+ut)]}{u[1-\exp(2\gamma x)]}, \tag{3.12}$$

where $\gamma=(1-u^2)^{-1/2}$ and we have omitted the double primes since the context is unambiguous. In terms of hyperbolic functions (3.12) can be written as

$$\tan(\tfrac{1}{4}\phi)=-\frac{\cosh\gamma ut}{u\sinh\gamma x} \tag{3.13}$$

or, since $\phi+2\pi$ is also a solution, in the well-known form

$$\tan(\tfrac{1}{4}\phi)=\frac{u\sinh\gamma x}{\cosh\gamma ut}. \tag{3.14}$$

Equations (3.12), (3.13), and (3.14) are canonical forms for the two-soliton solution and any two-soliton state can be studied by using one of these forms. It is then a simple matter to transform back into the original state. However, the important point about the analysis given above is the emergence of the canonical invariant $\exp(B_{12})=-u^2$. It gives us, in effect, a simple way of introducing the interaction into a linear superposition of the individual solitons. Thus, consider the two solitons

$$\tan(\tfrac{1}{4}\phi)=\exp(\gamma_i\xi_i+\alpha_i)\quad(i=1,2), \tag{3.15}$$

where the notation is the same as before. A linear superposition of these solitons is given by

$$\phi=\sum_{i=1}^{2}\phi_i, \tag{3.16}$$

leading to the expression

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{2}\tan(\tfrac{1}{4}\phi_i)}{1-\prod_{i=1}^{2}\tan(\tfrac{1}{4}\phi_i)}. \tag{3.17}$$

This represents the profile of two noninteracting solitons. We now "switch on" the interaction by introducing the invariant u^2 into the product term in the denominator giving

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{2}\tan(\tfrac{1}{4}\phi_i)}{1-u^2\prod_{i=1}^{2}\tan(\tfrac{1}{4}\phi_i)}, \tag{3.18}$$

which is the same as Hirota's two-soliton formula (3.5) once the identification of $-u^2$ with $\exp(B_{12})$ is made.

This method of switching on the interaction can be extended to the N-soliton solution. In this case Hirota's general formula is

$$\tan(\tfrac{1}{4}\phi)=g/f, \tag{3.19}$$

where

$$\begin{aligned} f&=\sum_{\mu=0,1}^{(e)}\exp\Big[\sum_{i<j}^{N}B_{ij}\mu_i\mu_j+\sum_{i=1}^{N}\mu_i(\gamma_i\xi_i+\alpha_i)\Big],\\ g&=\sum_{\mu=0,1}^{(o)}\exp\Big[\sum_{i<j}^{N}B_{ij}\mu_i\mu_j+\sum_{i=1}^{N}\mu_i(\gamma_i\xi_i+\alpha_i)\Big], \end{aligned} \tag{3.20}$$

while $\sum_{\mu=0,1}^{(e)}$ and $\sum_{\mu=0,1}^{(o)}$ imply summation over all possible combinations $\mu_1=0,1;\ \mu_2=0,1;\ldots;\mu_N=0,1$ under the conditions $\sum_{i=1}^{N}\mu_i$ even and $\sum_{i=1}^{N}\mu_i$ odd, respectively. If we set $\exp(B_{ij})=-1$ for all i and j in Eqs. (3.20), then (3.19) will just represent a linear sum of N noninteracting solitons. In the case of three solitons, for example, this is just

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{3}\tan(\tfrac{1}{4}\phi_i)-\prod_{i=1}^{3}\tan(\tfrac{1}{4}\phi_i)}{1-\sum_{i\neq j}\tan(\tfrac{1}{4}\phi_i)\tan(\tfrac{1}{4}\phi_j)}. \tag{3.21}$$

Introducing the interactions between pairs of solitons leads to the expression

$$\tan(\tfrac{1}{4}\phi)=\frac{\sum_{i=1}^{3}\tan(\tfrac{1}{4}\phi_i)-(u_{12}u_{23}u_{31})^2\prod_{i=1}^{3}\tan(\tfrac{1}{4}\phi_i)}{1-\sum_{i\neq j}u_{ij}^2\tan(\tfrac{1}{4}\phi_i)\tan(\tfrac{1}{4}\phi_j)}, \tag{3.22}$$

where u_{ij} is the common speed of the ith and jth solitons in their center-of-velocity frame. Equating $-u_{ij}^2$ with $\exp(B_{ij})$ then makes Eq. (3.22) identical to Hirota's three-soliton solution. This example shows quite clearly that the N-soliton solution can be thought of as pairwise interactions determined by the various canonical invariants $-u_{ij}^2$. Furthermore, and more importantly, it shows that if a particlelike representation can be found for the two-soliton case then it should be extendible to the N-soliton interaction.

IV. INTERACTION PICTURES

A. Conventional interpretation

Since the soliton is the nonlinear, dispersive equivalent of the linear, nondispersive wave packet, it seems natural to associate such amplitudes directly with particles. In such a picture the rest mass of the associated particle is given by the energy of the soliton computed in its canonical or rest frame. If we use the Hamiltonian density

$$\mathcal{H}=\tfrac{1}{2}[\phi_t^2+\phi_x^2+2(1-\cos\phi)], \tag{4.1}$$

where we have omitted the irrelevant scaling factor m^4/λ of Eq. (2.6), and substitute the canonical solution (3.3) this leads to an energy

$$\begin{aligned}E&=\int_{-\infty}^{\infty}\mathcal{H}\,dx\\&=\int_{-\infty}^{\infty}4\,\mathrm{sech}^2x\,dx\\&=8.\end{aligned} \tag{4.2}$$

Thus, the rest mass of the classical soliton particle is 8. In terms of the unscaled variables it would, of course, be $8m^3/\lambda$. If we perform the same calculation for a soliton moving with a speed u, then its energy turns out to be 8γ [$\gamma=(1-u^2)^{-1/2}$], which is the same as that of a particle of rest mass 8 moving with a speed u. This shows that the identification of a soliton with a particle is consistent at least as far as the free translation mode is concerned.

The next step in building up the particle picture is to investigate the behavior of the soliton under weak-field perturbations, and this has been done in a recent paper by Fogel *et al.*[8] They show that the soliton maintains its integrity to a high degree in the presence of weak applied fields and scattering potentials and, furthermore, that the dynamics of its interaction with these perturbations is essentially the dynamics of a classical Newtonian particle. These results show that the domain in which a soliton has particlelike behavior can be extended to include both free and weakly perturbed translation modes.

This direct equivalence between the dynamics of a classical particle and the translation mode of a weakly perturbed soliton leads us to ask whether the soliton maintains these properties, not in the presence of other weak external fields, but rather in the presence of another soliton. In other words, is the dynamics of the two-soliton interaction essentially the same as that between two classical particles, each of rest mass 8? In looking at this problem we immediately come up against two difficulties. The first is that, unlike particles, interacting solitons do not accelerate or decelerate. However, this shortcoming can be overcome, to a certain extent, by assuming that solitons in interaction lose or gain momentum by changing their rest masses rather than their speeds. This preserves the particlelike behavior of the soliton, but is somewhat unsatisfactory since the associated particle has to have a variable rest mass. The second difficulty, however, is more serious and concerns the interaction energy of the two solitons. Thus, consider the canonical form of the two-soliton solution as given by Eq. (3.14). This is

$$\phi_{SS}=4\tan^{-1}\left(\frac{u\sinh\gamma x}{\cosh\gamma ut}\right). \tag{4.3}$$

Substituting into the Hamiltonian density (4.1) gives the energy of the two-soliton solution as

$$\begin{aligned}E_{SS}&=\int_{-\infty}^{\infty}\mathcal{H}(\phi_{SS})dx\\&=\int_{-\infty}^{\infty}\frac{8u^2}{\Delta^2}[\gamma^2u^2\sinh^2\gamma x\sinh^2\gamma ut\\&\qquad+\cosh^2\gamma ut(\sinh^2\gamma x+\gamma^2\cosh^2\gamma x)]dx,\end{aligned} \tag{4.4}$$

where $\Delta=\cosh^2\gamma ut+u^2\sinh^2\gamma x$. Since the SGE represents a conservative system, this integral can be evaluated at $t=0$ and leads to the result $E_{SS}=16\gamma$ which, of course, is the same as the energy of two free solitons each moving with a speed u.

This result could have been foreseen if we had recalled that the two-soliton solution tends asymptotically to a linear sum of two free solitons. On the other hand, it also means that the interaction energy of the solitons cannot be extracted from calculations of the type given above.

The same situation arises with the soliton-antisoliton scattering solution

$$\phi_{SA} = 4\tan^{-1}\left(\frac{u\cosh\gamma x}{\sinh\gamma ut}\right), \tag{4.5}$$

but in the case of the "breather"

$$\phi_B = 4\tan^{-1}\left\{\frac{\sin[vt/(1+v^2)^{1/2}]}{v\cosh[x/(1+v^2)^{1/2}]}\right\}, \tag{4.6}$$

which can be thought of as a bound soliton-antisoliton pair whose relative separation oscillates in time with period $\tau = 2\pi(1+v^2)^{1/2}/v$ with v real, the energy turns out to be $16/(1+v^2)^{1/2}$. This is less than that of two free solitons and indicates that the forces between a soliton and an antisoliton are attractive. However, once again there seems to be no way of extracting the potential; the soliton and antisoliton are always within a finite distance of each other and one has no information at all about their speeds.

Having said this, one should mention a paper by Rubinstein[11] in which he puts forward a formula for the forces acting between solitons. By considering the two-soliton and the soliton-antisoliton scattering solutions as small perturbations on a linear sum of the corresponding profiles, he predicts that the forces acting between them are given by the expression

$$F \simeq 32 N_1 N_2 e^{-2q}, \tag{4.7}$$

where $N_i = +1$ for a soliton, -1 for an antisoliton, and $2q$ is the relative separation between the profiles, which he assumes to be large compared to their widths. However, we have serious misgivings about Rubinstein's approach to the problem and our views can best be summed up as follows: Firstly, if his analysis is valid at all it is only valid in the asymptotic region where the solitons are virtually free. For all finite times his basic assumptions are violated. Secondly, he assumes the existence of a two-soliton solution in which the solitons are at rest relative to each other. But, as we have mentioned in the Introduction and elsewhere, there is no static two-soliton solution of the SGE. Finally, his interpretation of the force is rather ambiguous since what he has is an impulse rather than a force. In view of these remarks, we feel that Rubinstein's analysis does not provide a solution to the difficulties of calculating the soliton-soliton interaction potentials and so the problem still remains.

This inability to extract the potentials, and hence the forces, between interacting solitons and antisolitons is surely a severe shortcoming of the direct particle-soliton equivalence. The existence of the breather solution and a consideration of the phase shifts of the scattering solutions indicate that the soliton-soliton interaction is repulsive and the soliton-antisoliton interaction is attractive, but there seems to be no way of getting a quantitative estimate of these features, that is, it does not seem to be possible to divide the total Hamiltonian into a free and interaction part.

B. Poles of the Hamiltonian density

The discussion given in subsection A shows quite clearly that we cannot directly represent the two-soliton solution as an interaction between classical particles with fixed rest masses. The simple picture we had with the single soliton has not been repeated. This leads to one of two conclusions: Either no such interacting particle interpretation is possible or else it is implicit and has to be obtained via an indirect representation. Now the success obtained with the semiclassical analysis and the quantum SGE, mentioned in the Introduction, leads us to believe that the latter is true, i.e., the necessary information is contained in the Hamiltonian density and the problem is to extract it. One way of doing this is to consider the Hamiltonian density to be a function of a complex, rather than a real, space variable. Then, for a given solution of the SGE, the poles of this complex Hamiltonian density provide us with a possible link between the soliton amplitudes and localized point particles. We shall show that there is, indeed, a strong correlation between the dynamics of these poles, considered as classical particles, and that of the solitons they represent. Thus, all the qualitative features of the two-soliton (and/or antisoliton) solutions, as well as analytic expressions for the forces between them, are predicted by studying the motion of the poles of the Hamiltonian density in the complex space plane.

We begin with the N-soliton solution in the form given by Hirota,

$$\phi = 4\tan^{-1}(g/f). \tag{4.8}$$

Substituting into Eq. (4.1) gives us a Hamiltonian density

$$\mathcal{H} = \frac{8}{(f^2+g^2)^2}[(fg_x - gf_x)^2 + (fg_t - gf_t)^2 + g^2 f^2]; \tag{4.9}$$

assuming that the space variable is complex, rather than real, allows $\mathcal{H}$ to develop poles at points in the complex-space plane where

$$f = \pm i g \,. \tag{4.10}$$

Now f and g are time-dependent and hence, as time develops, these poles will move across the complex-space plane in a well-defined manner. Let us analyze the dynamics of this motion for the one- and two-soliton solutions.

Consider first the one-soliton solution which, in terms of the complex variable z, is

$$\phi_S = 4 \tan^{-1} \exp[\gamma(z - ut)] \,. \tag{4.11}$$

Using Eq. (4.10) the poles of $\mathcal{H}$ corresponding to (4.11) occur at

$$z = ut + \frac{i\pi}{2\gamma}(1+2n) \quad (n = 0, \pm 1, \pm 2, \text{etc.}) \,. \tag{4.12}$$

Thus, there is an infinite set of regularly spaced poles lying on a line parallel to the imaginary axis, all moving with the same constant speed u along lines parallel to the real axis and in the direction of increasing x. Any one of these poles then gives us a localized representation of the single soliton and predicts correctly its motion as a free particle with fixed speed u.

For two solitons the solution in the canonical or center-of-velocity frame is given by

$$\phi_{SS} = 4 \tan^{-1}\left(\frac{u \sinh \gamma z}{\cosh \gamma u t}\right). \tag{4.13}$$

Thus, the poles of $\mathcal{H}$ occur where

$$\sinh \gamma z = \pm \frac{i}{u} \cosh \gamma u t \,. \tag{4.14}$$

Solving for the real and imaginary parts of z gives us respectively

$$X = \pm \frac{1}{\gamma} \ln\left[\frac{\cosh \gamma u t + (\cosh^2 \gamma u t - u^2)^{1/2}}{u}\right] \tag{4.15a}$$

and

$$y = \frac{\pi}{2\gamma}(1+2n) \quad (n = 0, \pm 1, \pm 2, \text{etc.}) \,. \tag{4.15b}$$

This time there are two sets of poles lying on lines parallel to the imaginary axis and symmetrically placed on either side of it. These poles move in pairs (i.e., two with the same imaginary part) along straight-line trajectories parallel to the real axis and any such pair then gives us a localized representation of the two interacting solitons. We can now use Eq. (4.15a) to compute the velocities and accelerations of these representative pole particles. We find

$$\dot{X} = \pm \frac{u \sinh \gamma u t}{(\cosh^2 \gamma u t - u^2)^{1/2}} \tag{4.16}$$

and

$$\ddot{X} = \pm \frac{u^2 \cosh \gamma u t}{\gamma(\cosh^2 \gamma u t - u^2)^{3/2}} \,, \tag{4.17}$$

where the plus signs in Eqs. (4.16) and (4.17) go together and likewise for the minus signs. Analyzing the motion represented by these equations and Eq. (4.15a) we have (see Fig. 1) the following:

(1) As $t \to -\infty$, $X \to \mp\infty$, $\dot{X} \to \pm u$, and $\ddot{X} \to 0$.

(2) At $t = 0$, $X = \mp(1-u^2)^{1/2} \cosh^{-1}(1/u)$, $\dot{X} = 0$, and $\ddot{X} = \mp \gamma^2 u^2$.

(3) As $t \to \infty$, $X \to \mp\infty$, $\dot{X} \to \mp u$, and $\ddot{X} \to 0$.

Thus, the two pole particles start out ($t = -\infty$) at opposite ends of the real axis and move toward each other with steadily decreasing speed. At $t = 0$ they come to a stop at a distance of $2(1 - u^2)^{1/2} \cosh^{-1}(1/u)$ apart — the distance of closest approach. They then turn around and accelerate away from each other back to where they came from. The force between them is obviously repulsive and its magnitude can easily be computed from Eqs. (4.15a), (4.16), and (4.17). It turns out to be

$$F = 8\gamma^2 \operatorname{sech}^2 \gamma X \,, \tag{4.18}$$

where $|2X|$ is the distance between the pole particles in the canonical frame and their rest masses are taken to be the same as that of solitons, i.e., 8. The particles thus move in a potential given by (see Fig. 2)

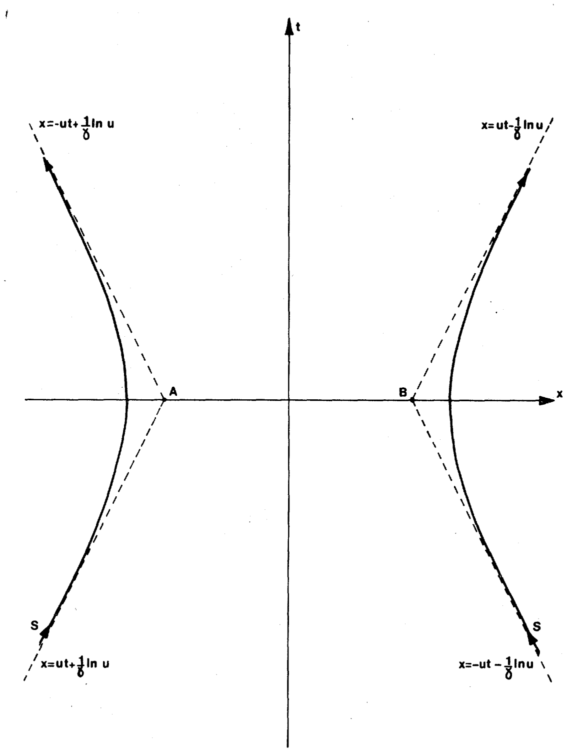

FIG. 1. Pole-particle trajectories for the soliton-soliton (SS) solution. Note that the magnitude of the phase shift for the solution is given by the intersection of the asymptotes and the x axis, i.e., $AB = (2/\gamma)|\ln u|$.

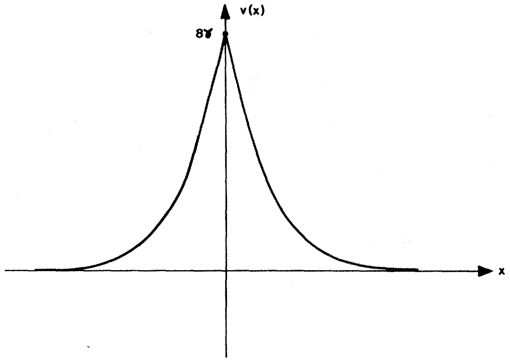

FIG. 2. Potential for the soliton-soliton interaction in the pole-particle picture.

$$V(X) = 8\gamma(1 - |\tan\gamma X|) . \quad (4.19)$$

Thus, the representation of the two-soliton solution in terms of the poles of its complex Hamiltonian density gives us a particle picture which reproduces all the qualitative features of the two-soliton interaction. It confirms the conjecture that the interaction is repulsive and provides us with a potential that is valid for all times and not just in the asymptotic region. However, if we consider the asymptotic form taken by the force acting on a pole particle we find, from Eq. (4.18),

$$F \approx 32\gamma^2 e^{-\gamma|2X|} , \quad (4.20)$$

which only agrees with Rubinstein's value [Eq. (4.7)] when $\gamma = 1$, i.e., when $u = 0$. But, as we pointed out, there is no two-soliton solution of the SGE with $u = 0$.

We turn now to the soliton-antisoliton scattering solution. In its canonical or center-of-velocity frame this takes the form

$$\phi_{SA} = 4\tan^{-1}\left(\frac{u\cosh\gamma z}{\sinh\gamma ut}\right). \quad (4.21)$$

Thus, the poles of $\mathcal{H}$ occur where

$$\cosh\gamma z = \pm\frac{i}{u}\sinh\gamma ut , \quad (4.22)$$

leading to

$$X = \pm\frac{1}{\gamma}\ln\left[\frac{\sinh\gamma ut + (\sinh^2\gamma ut + u^2)^{1/2}}{u}\right], \quad (4.23a)$$

$$y = \frac{\pi}{2\gamma}(1 + 2n) \quad (n = 0, \pm1, \pm2, \text{etc.}) . \quad (4.23b)$$

Again, as in the case of two solitons, there are pairs of pole particles which move along straight lines parallel to the real axis with velocities and accelerations given by

$$\dot{X} = \pm\frac{u\cosh\gamma ut}{(\sinh^2\gamma ut + u^2)^{1/2}} , \quad (4.24)$$

$$\ddot{X} = \mp\frac{u^2\sinh\gamma ut}{\gamma(\sinh^2\gamma ut + u^2)^{3/2}} , \quad (4.25)$$

where, in this instance, the plus sign in (4.24) goes with the minus sign in (4.25) and vice versa. Analyzing this motion we have (see Fig. 3) the following:

(1) As $t \to -\infty$, $X \to \mp\infty$, $\dot{X} \to \pm u$, and $\ddot{X} \to 0$.
(2) At $t = 0$, $X = 0$, $\dot{X} = \pm1$, and $\ddot{X} = 0$.
(3) As $t \to +\infty$, $X \to \pm\infty$, $\dot{X} \to \pm u$, and $\ddot{X} \to 0$.

The particles thus accelerate toward each other and meet at the origin of the real axis at $t = 0$ and with the maximum possible speed, i.e., $|\dot{X}| = 1$. They then pass through each other and separate at a steadily decreasing speed which tends to u as $t \to \infty$. The interaction is obviously attractive and the force between them has a magnitude

$$F = 8\gamma^2 \operatorname{csch}^2\gamma X , \quad (4.26)$$

corresponding to a potential (see Fig. 4)

$$V(X) = 8\gamma(1 - |\coth\gamma X|) . \quad (4.27)$$

Thus, once again, the poles of the Hamiltonian density give us a particle representation which correctly reproduces the characteristic features of the soliton-antisoliton scattering solution. The attractive nature of the interaction is confirmed and the form of the potential specified.

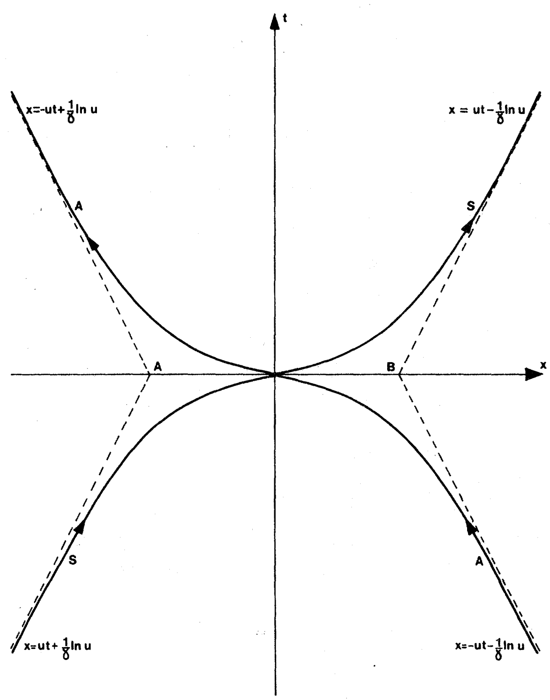

FIG. 3. Pole-particle trajectories for the soliton-antisoliton (SA) solution. Note that the magnitude of the phase shift for the solution is given by the intersection of the asymptotes and the x axis, i.e., $AB = (2/\gamma)|\ln u|$.

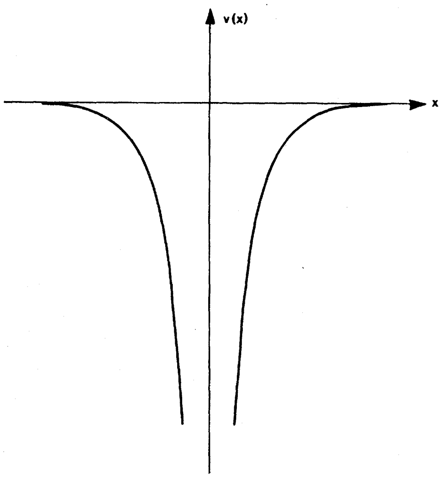

FIG. 4. Potential for the soliton-antisoliton interaction in the pole-particle picture.

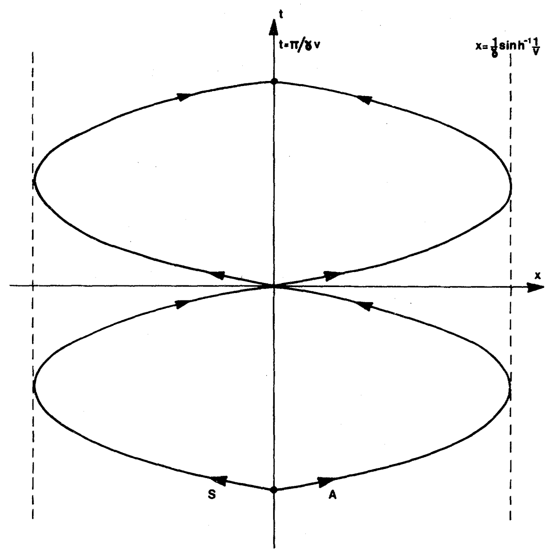

FIG. 5. Pole-particle trajectories for the breather solution. Note for the breather $\gamma = 1/(1+v^2)^{1/2}$.

Finally, we look at the "breather" solution. Running through the relevant details we have the following:

(1) The solution

$$\phi_B = 4\tan^{-1}\left(\frac{\sin\sigma v t}{v\cosh\sigma z}\right), \tag{4.28}$$

where $\sigma = (1+v^2)^{-1/2}$, $v \in R$.

(2) The pole positions

$$X = \pm\frac{1}{\sigma}\ln\left[\frac{\sin\sigma v t + (\sin^2\sigma v t + v^2)^{1/2}}{v}\right], \tag{4.29a}$$

$$y = \frac{\pi}{2\sigma}(1+2n) \quad (n = 0, \pm 1, \pm 2, \text{etc.}). \tag{4.29b}$$

(3) The velocities and accelerations of the particles

$$\dot{X} = \pm\frac{v\cos\sigma v t}{(\sin^2\sigma v t + v^2)^{1/2}}, \tag{4.30}$$

$$\ddot{X} = \mp\frac{v^2\sin\sigma v t}{\sigma(\sin^2\sigma v t + v^2)^{3/2}}. \tag{4.31}$$

(4) The attractive force and potential

$$F = 8\sigma^2\operatorname{csch}^2\sigma X, \tag{4.32}$$

$$V(X) = 8\sigma(1 - |\coth\sigma X|). \tag{4.33}$$

As expected, the motion is periodic and the particles oscillate about the origin of the real axis (see Fig. 5) with a period

$$\tau = 2\pi/\sigma v = \frac{2\pi(1+v^2)^{1/2}}{v}, \tag{4.34}$$

and an amplitude

$$A = (1+v^2)^{1/2}\sinh^{-1}(1/v). \tag{4.35}$$

They reach a maximum speed of one when they meet and pass through each other at the origin, and minimum speed of zero at the boundaries of the potential well. Thus, all the properties of the soliton-antisoliton bound state are correctly reproduced by the pole-particle representation.

Although the results obtained above are, on the whole, rather encouraging, they do have one unsatisfactory feature. This occurs in the case of the soliton-antisoliton interaction where the representative particles pass through each other with the speed of light while the forces between them become infinite. In our picture this is inevitable, since the soliton-antisoliton potential [Eqs. (4.27) and (4.33)] is always attractive and the forces become stronger the closer the particles are to each other. This singular behavior is similar to that of the inverse square law in classical electrodynamics and Newtonian gravitation, and here, as there, the only way of circumventing the difficulty is to modify the small-distance behavior of the potential. This can be done either by putting in a cutoff or by introducing a repulsive core. However, this would result in changes to the soliton-antisoliton and hence to the SGE itself.

The extension of the pole-particle analysis to the N-soliton solutions when $N > 2$ is hampered by technical difficulties. In these cases the Hamiltonian density is much more involved than in the case $N = 2$, and we have not been able (as yet) to solve the problem of explicitly extracting the pole positions. However, the results of Sec. III lead us to conjecture that the particles in these cases would move in a "relativistic" superposition of the two-particle potentials obtained above.

C. Interacting solitons as variable rest-mass particles

The representation discussed in subsection B above can now be used to characterize the dynamics of a two-soliton interaction in terms of the constituent solitons themselves rather than in terms of the representative pole particles. Thus, consider the two-soliton solution given by Eq. (4.13). Since the pole particles are assumed to have a fixed rest mass of 8, the Hamiltonian for either one of a representative pair can be written as

$$H_{SS}=8\gamma'+8\gamma(1-|\tanh\gamma x|)\,, \qquad (4.36)$$

where γ is the same as before and $\gamma'=(1-\dot{X}^2)^{-1/2}$ with $\dot{X}$ being given by Eq. (4.16). Substituting for $\dot{X}$ and using Eq. (4.15a) gives us

$$\gamma'=\gamma|\tan\gamma X|\,. \qquad (4.37)$$

Thus Eq. (4.36) becomes

$$H_{SS}=m_{SS}\gamma+m_{SS}\gamma(|\coth\gamma X|-1)\,, \qquad (4.38)$$

where $m_{SS}=8|\tan\gamma X|$. Now this Hamiltonian can be interpreted as that of a particle of rest mass m_{SS}, moving with a fixed speed u in a potential $\gamma(|\coth\gamma X|-1)$ (Fig. 6). Since the solitons themselves move with a fixed speed u, it seems natural to associate this behavior directly with that of the individual solitons in the two-soliton interaction. If this is done then the following picture emerges: Interacting solitons are extended classical particles which move in a fashion so as to preserve their speeds, but change their rest mass in a well-defined way. During the interaction there is a continuous transformation of rest-mass energy into interaction energy, and vice versa, so that the momentum of the soliton changes continuously although its speed remains fixed. Thus solitons do not maintain their free mass identity when interacting with each other.

A similar analysis for the soliton-antisoliton scattering solution, Eq. (4.21), leads to the Hamiltonian

$$H_{SA}=m_{SA}\gamma+m_{SA}\gamma(|\tanh\gamma X|-1)\,, \qquad (4.39)$$

where again a variable-mass particle interpretation is possible, but this time the effective rest mass, m_{SA}, of the soliton (antisoliton) varies as $|\coth\gamma X|$ and the potential well $\gamma(|\tanh\gamma X|-1)$ has a finite depth (Fig. 6). Notice that the singular behavior of the pole-particle representation for this case is reproduced here in the effective mass term, which diverges as $X\to 0$. However, a quick glance at Eq. (4.39) shows that the Hamiltonian, which represents the total energy of the interacting soliton (antisoliton), remains fixed at the finite value of 8γ throughout the motion.

The analysis in the case of the breather, Eq. (4.28), is not as direct as those discussed above. Going through the same procedure as before gives us the Hamiltonian

$$H_B=8\sigma|\coth\sigma X|+8\sigma|\coth\sigma X|(|\tanh\sigma X|-1)\,, \qquad (4.40)$$

which, since $\sigma=(1+v^2)^{-1/2}$ is not a Lorentz factor, can no longer be interpreted in terms of a variable-mass particle. However, if we assume that the soliton (antisoliton) in the breather moves at a constant speed, then from the period and amplitude of the oscillatory motion, Eqs. (4.34) and (4.35), we can compute this speed. We get

$$u_B=\frac{2}{\pi}v\,\sinh^{-1}(1/v)\,. \qquad (4.41)$$

Rewriting Eq. (4.40) in the form

$$H_B=m_B\gamma_B+m_B\gamma_B(|\tanh\sigma X|-1)\,, \qquad (4.42)$$

where

$$\gamma_B=(1-u_B{}^2)^{-1/2}$$

and

$$m_B=8(\sigma/\gamma_B)|\coth\sigma X|\,, \qquad (4.43)$$

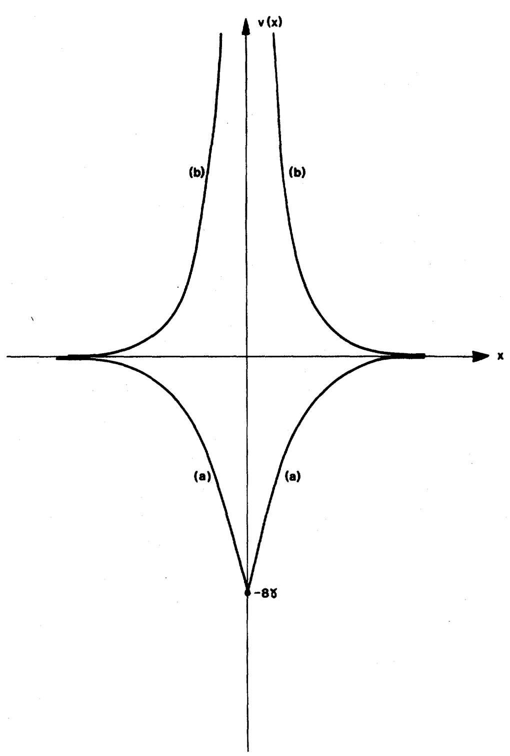

FIG. 6. (a) Potential for the soliton-antisoliton interaction in the variable-mass-particle picture. (b) Potential for the soliton-soliton interaction in the variable-mass-particle picture.

then gives us a Hamiltonian with the required interpretation. Thus, we find that the soliton (antisoliton) in a breather moves as an extended particle with variable rest mass m_B, and constant speed u_B, in a potential similar to that of the scattering solution.

A comparison of the results established in this section with those of the preceding section reveals a striking feature. There is a kind of duality between the two particle representations. In the pole-particle picture the particles are localized, have fixed rest masses, and move with varying speeds. On the other hand, in the soliton-particle picture the particles are extended, have varying rest masses, and move with constant speeds. Furthermore, the shapes of the potentials for the two representatives are also interchanged (cf. Fig. 6 with Figs. 2 and 4). It would seem that, just as in the quantum theory of the SGE, the original Hamiltonian density has two equally valid descriptions in terms of classical particles: the pole particles and the soliton particles. Whether this is linked with the quantum results or not is a matter of conjecture and might be worth exploring.

V. CONCLUDING REMARKS AND SOME CONJECTURES

The main result of our investigation is that, in the case of the classical, two-dimensional SGE, the constitutent solitons of the two-soliton solutions can be represented by relativistic Newtonian particles moving in well-defined fields of force. There are two representations which, in a sense, are dual to each other. The first is an indirect representation in terms of the poles of the complex Hamiltonian density and leads to particles of fixed mass, but variable speed. The second is a direct representation and leads to particles of fixed speed but variable mass. We have also shown how N-soliton interactions can be interpreted in terms of two-soliton interactions leading to the conjecture that the same behavior should occur in the particle pictures. In other words, the N-body problem should reduce to a set of two-body problems.

Apart from the explicit insight that these particle representations give into the nature of the soliton interaction potentials, they also have other advantages. For example, they can be used as a computational tool in the perturbation theory of the classical SGE, since this analysis is obviously easier to carry out in a particle picture (cf. Ref. 8). However, there are difficulties in translating the results back to the corresponding Hamiltonian density — it is always easier to find the poles of a given complex function rather than to construct the function from its poles — but we are investigating this problem and hope to publish our results in the near future.

Another important area in which the particle representations should prove useful is that of finding solutions to the SGE in the more realistic cases of two and three space dimensions. Knowing the effective potentials in one space dimension gives us a specific starting point for constructing them in higher dimensions. Then, by analyzing particle trajectories in these higher-dimensional potentials, we should, in principle, be able to work out the corresponding solutions via their Hamiltonian densities. The analysis may be extremely difficult, but it is possible.

Now, in addition to the classical ramifications of the particle picture, there are some interesting and suggestive quantum connections. For example, the conjecture that the quantum sine-Gordon soliton is a fermion seems to be reflected by the fact that there is no solution of the SGE in which two solitons move at the same speed and in the same direction. Some insight into this property of the SGE can be gained by looking at the corresponding situation in the pole-particle representation. Here the particles have zero speed and acceleration for all time and hence they always remain an infinite distance apart. In other words, this solution can never get started. Thus, even classical SGE solitons seem to exhibit some of the features of fermionic behavior.

Even more striking is the existence of infinite sets of poles for the complex Hamiltonian densities. If we take the imaginary part of the complex space seriously then, in the case of the single soliton, the corresponding pole particles have imaginary angular momenta with magnitudes given by $4\pi u(1+2n)$, where $n=0, \pm 1, \pm 2$, etc., and u is the speed of the soliton. If we plot these poles in a complex angular momentum plane then they all lie on the imaginary axis, regularly spaced at intervals of $8\pi u$. In this case, of course, their positions are fixed, but for the two-soliton solutions the corresponding poles move up and down this imaginary axis in a well-defined manner as time goes from $-\infty$ to $+\infty$. This is reminiscent of Regge-pole theory,[12] where the angular momentum is considered to be a complex function of the energy in the t channel and the pole positions vary continuously with t.

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