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Nonexistence of kinematic constants for Lorentz-invariant Newtonian mechanics

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For Poincaré-invariant Newtonian equations of motion for two interacting particles, there is no function of the particle velocities that is a constant of motion over a range of values of the velocities and relative position and is the sum of a nonzero term for each particle which is a rotational-vector function of the velocity of that particle.

INTRODUCTION

A recent paper[1] showed how Lorentz-invariant Newtonian equations of motion for two interacting particles can be constructed, by making global Lorentz transformations, from a specification of the relative acceleration as a function of the relative position and relative velocity at zero center-of-mass velocity, if the center-of-mass acceleration is assumed to be zero at zero center-of-mass velocity. For example, this includes all parity-invariant Poincaré-invariant Newtonian equations of motion for two identical particles.[1]

For three or more interacting particles, it was pointed out[2] that, although Poincaré-invariant Newtonian equations of motion can be constructed in the same way, on the assumption that for the entire system of particles the center-of-mass acceleration is zero when the center-of-mass velocity is zero, a problem arises if we require separability or cluster decomposition. For this implies that when the center-of-mass velocity for the entire system is zero, and all particles except two are widely separated in space, the center-of-mass acceleration of the remaining two-particle system must be zero, for a range of values of the center-of-mass velocity of the two-particle system depending on the masses and velocities of the other particles. It is known that for Poincaré-invariant Newtonian equations of motion for two interacting particles, the center-of-mass acceleration cannot be zero over a range of values of the velocities and relative position.[3]

Everything said so far is in terms of the ordinary nonrelativistic center of mass. The same is true if the total relativistic kinematic particle momentum and its time derivative are used in place of the center-of-mass velocity and acceleration. Lorentz-invariant Newtonian equations of motion can be constructed for any number of interacting particles by assuming that when the total relativistic kinematic particle momentum is zero then its time derivative is also zero. But systems of two or three interacting particles are not obtained from a larger system of this kind when the other particles are widely separated in space. For Poincaré-invariant Newtonian equations of motion for either two or three interacting particles, the total relativistic kinematic particle momentum cannot be a constant of motion for a range of values of the momenta and relative positions.[4-6]

In the method that has been outlined[1] for constructing Lorentz-invariant Newtonian equations of motion, we could use any translation-invariant rotational-vector quantity in place of the center-of-mass velocity or the total relativistic kinematic particle momentum, assuming that when it is zero its time derivative also is zero. Here we will show that there is no such quantity that we can form simply in a kinematic way, as a sum of contributions from individual particles, to construct Lorentz-invariant Newtonian equations of motion so that a system of two interacting particles can be separated out from a larger system. More specifically, we show that for Poincaré-invariant Newtonian equations of motion for two interacting particles, there are no two nonzero translation-invariant rotational-vector functions, one for each particle, depending only on the variables of that particle, whose sum is a constant of motion over a range of values of the velocities and relative position.

KINEMATIC CONSTANTS FOR TWO PARTICLES

Consider a classical mechanical system of two particles described by positions $\vec{x}_1$ and $\vec{x}_2$ and velocities $\vec{v}_1$ and $\vec{v}_2$. Suppose we have a translation-invariant rotational-vector constant of motion that is a sum of contributions from individual particles. It must be of the form

$$\sum_{n=1}^{2} f_n(v_n{}^2)\vec{v}_n \tag{1}$$

with functions f_1 and f_2 of the squares of the velocities, because translation-invariant functions of the positions would involve both particles together in $\vec{x}_1 - \vec{x}_2$.

We wish to show that if a (sufficiently differentiable) function of this form (1), with both f_1 and f_2 nonzero,[7] is a constant of motion (for a range of values of the velocities and relative position), then Poincaré invariance implies that the individual particle velocities $\vec{v}_1$ and $\vec{v}_2$ are both constants of motion, which means there is no interaction. The idea is simply that every Lorentz transform of the constant of motion (1) must be a constant of motion, and the only way they can all be constant is for the individual particle velocities to be constant.

In general, if $\vec{W}$ is a translation-invariant constant of motion, so is every Lorentz transform of $\vec{W}$; using bracket-generator symbols $[\ ,H]$ for time derivatives, $[\ ,\vec{P}]$ for space-translation derivatives, and $[\ ,\vec{K}]$ for Lorentz-transformation derivatives,[8] and using the bracket relations of the Poincaré group,[8] we see that if

$$[W_l, H]=0$$

and

$$[W_l, P_j]=0$$

for $l,j=1,2,3$, then

$$\begin{aligned}[[W_l,K_k],H]&=[[W_l,H],K_k]+[W_l,[K_k,H]]\\&=[W_l,P_k]=0\end{aligned} \tag{2}$$

and

$$\begin{aligned}[[W_l,K_k],P_j]&=[[W_l,P_j],K_k]+[W_l,[K_k,P_j]]\\&=[W_l,\delta_{kj}H]=0\end{aligned} \tag{3}$$

for $l,k,j=1,2,3$. (We use units such that $c=1$.)

Taking the translation-invariant constant of motion (1) for $\vec{W}$, we thus obtain another constant of motion

$$\left[\sum_{n=1}^{2} f_n v_{nl}, K_k\right]=\sum_{n=1}^{2}\left(f_n[v_{nl},K_k]+f_n' 2\sum_{i=1}^{3} v_{ni}[v_{ni},K_k]v_{nl}\right) \tag{4}$$

for $l,k=1,2,3$ where f_n' is the derivative of f_n with respect to v_n^2. For a Lorentz transformation with velocity $\tanh\epsilon$ in the kth direction, the jth component of the transformed position of the nth particle, that is the position at time zero in the transformed frame, is

$$x_{nj}+\epsilon x_{nk} v_{nj}$$

to first order in ϵ, given that $\vec{x}_n$ and $\vec{v}_n$ are the position and velocity at time zero in the original frame.[9-11] This means that

$$[x_{nj},K_k]=x_{nk}v_{nj} \tag{5}$$

for $n=1,2$ and $j,k=1,2,3$. The first-order part of the similarly transformed velocity is[9-11]

$$\begin{aligned}[v_{nj},K_k]&=[[x_{nj},H],K_k]\\&=[[x_{nj},K_k],H]-[x_{nj},[K_k,H]]\\&=[x_{nk}v_{nj},H]-[x_{nj},P_k]\\&=x_{nk}\dot{v}_{nj}+v_{nk}v_{nj}-\delta_{jk}\end{aligned} \tag{6}$$

for $n=1,2$ and $j,k=1,2,3$. (Here and in the following a dot means a time derivative.) Using this formula for $[\ ,K_k]$, we find that the constant of motion (4) is

$$\sum_{n=1}^{2}\left[f_n x_{nk}\dot{v}_{nl}+f_n v_{nk}v_{nl}-f_n\delta_{kl}+f_n' 2\vec{v}_n\cdot\dot{\vec{v}}_n x_{nk}v_{nl}+f_n' 2(v_n^2-1)v_{nk}v_{nl}\right]. \tag{7}$$

Taking the time derivative, we get

$$\begin{aligned}\sum_{n=1}^{2}\big[&f_n x_{nk}\ddot{v}_{nl}+2f_n v_{nk}\dot{v}_{nl}+f_n\dot{v}_{nk}v_{nl}-f_n' 2\vec{v}_n\cdot\dot{\vec{v}}_n\delta_{kl}+f_n' 4\vec{v}_n\cdot\dot{\vec{v}}_n x_{nk}\dot{v}_{nl}+f_n' 8\vec{v}_n\cdot\dot{\vec{v}}_n v_{nk}v_{nl}\\&+f_n' 2(v_n^2-1)v_{nk}\dot{v}_{nl}+f_n' 2(v_n^2-1)\dot{v}_{nk}v_{nl}+f_n' 2(\vec{v}_n\cdot\ddot{\vec{v}}_n+\dot{\vec{v}}_n^{\,2})x_{nk}v_{nl}\\&+f_n'' 4(\vec{v}_n\cdot\dot{\vec{v}}_n)^2 x_{nk}v_{nl}+f_n'' 4(v_n^2-1)(\vec{v}_n\cdot\dot{\vec{v}}_n)v_{nk}v_{nl}\big]=0\end{aligned} \tag{8}$$

for $k,l=1,2,3$.

We write the position variables in terms of

$$\vec{X}=\tfrac{1}{2}(\vec{x}_1+\vec{x}_2)$$

and

$$\vec{x}=\vec{x}_1-\vec{x}_2.$$

Then we find that the coefficient of X_k in Eq. (8) is just twice the second time derivative of the constant of motion (1), which is zero. Let $\vec{e}$ be a vector perpendicular to $\vec{x}$. We multiply Eq. (8) by e_k and sum over $k=1,2,3$ to get

$$\sum_{n=1}^{2}\{[f_n''4(v_n^2-1)+f_n'8]\vec{v}_n\cdot\dot{\vec{v}}_n(\vec{e}\cdot\vec{v}_n)\vec{v}_n+[f_n'2(v_n^2-1)+f_n](\vec{e}\cdot\dot{\vec{v}}_n)\vec{v}_n+[f_n'2(v_n^2-1)+2f_n](\vec{e}\cdot\vec{v}_n)\dot{\vec{v}}_n-f_n'2(\vec{v}_n\cdot\dot{\vec{v}}_n)\vec{e}\}=0. \tag{9}$$

Suppose $\vec{e}$ is also perpendicular to $\vec{v}_1\times\vec{v}_2$. Then the dot product of Eq. (9) with $\vec{v}_1\times\vec{v}_2$ yields

$$\sum_{n=1}^{2}[f_n'(v_n^2-1)+f_n](\vec{e}\cdot\vec{v}_n)\dot{\vec{v}}_n\cdot\vec{v}_1\times\vec{v}_2=0. \tag{10}$$

Taking the dot product with $\vec{v}_1\times\vec{v}_2$ of the time derivative of the constant of motion (1), we have also

$$\sum_{n=1}^{2}f_n\dot{\vec{v}}_n\cdot\vec{v}_1\times\vec{v}_2=0. \tag{11}$$

This and Eq. (10) imply that

$$\{[f_1'(v_1^2-1)+f_1](1/f_1)\vec{v}_1-[f_2'(v_2^2-1)+f_2](1/f_2)\vec{v}_2\}\cdot\vec{e}\,f_1\dot{\vec{v}}_1\cdot\vec{v}_1\times\vec{v}_2=0 \tag{12}$$

provided neither f_1 nor f_2 is zero. Since in the first factor, the first term is a function only of $\vec{v}_1$ and the second a function only of $\vec{v}_2$, and $\vec{v}_1$ and $\vec{v}_2$ can be varied independently in their plane without changing $\vec{e}$, which is in that plane, it follows that either

$$f_n'(v_n^2-1)+f_n=0 \tag{13}$$

for $n=1,2$, or

$$\dot{\vec{v}}_n\cdot\vec{v}_1\times\vec{v}_2=0 \tag{14}$$

for $n=1,2$. In either case, taking the dot product of Eq. (9) with $\vec{v}_1\times\vec{v}_2$, we now see that

$$\sum_{n=1}^{2}f_n'2\vec{v}_n\circ\dot{\vec{v}}_n\vec{e}\cdot(\vec{v}_1\times\vec{v}_2)=0$$

for any vector $\vec{e}$ perpendicular to $\vec{x}$, so

$$\sum_{n=1}^{2}f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n=0, \tag{15}$$

which means that

$$\sum_{n=1}^{2}f_n \tag{16}$$

is a constant of motion.

Taking the translation-invariant constant of motion (16) in place of W_l, we get another translation-invariant constant of motion which, using Eq. (6) again, we find to be

$$\left[\sum_{n=1}^{2}f_n,K_k\right]=\sum_{n=1}^{2}f_n'2\sum_{i=1}^{3}v_{ni}[v_{ni},K_k]$$
$$=2\sum_{n=1}^{2}[f_n'\vec{v}_n\cdot\dot{\vec{v}}_nx_{nk}+f_n'(v_n^2-1)v_{nk}]. \tag{17}$$

Taking half of this vector plus the original vector constant of motion (1), we see that for $k=1,2,3$,

$$\sum_{n=1}^{2}\{f_n'\vec{v}_n\cdot\dot{\vec{v}}_nx_{nk}+[f_n'(v_n^2-1)+f_n]v_{nk}\} \tag{18}$$

is a constant of motion. If we write the position variables in terms of $\vec{X}$ and $\vec{x}$, we find that the coefficient of X_k is just half the time derivative (15) of the constant of motion (16), which is zero, so we can write the constant of motion (18) as

$$\tfrac{1}{2}(f_1'\vec{v}_1\cdot\dot{\vec{v}}_1-f_2'\vec{v}_2\cdot\dot{\vec{v}}_2)x_k+\sum_{n=1}^{2}[f_n'(v_n^2-1)+f_n]v_{nk}. \tag{19}$$

If Eq. (13) holds for $n=1,2$, then

$$(f_1'\vec{v}_1\cdot\dot{\vec{v}}_1-f_2'\vec{v}_2\cdot\dot{\vec{v}}_2)\vec{x}$$

is a constant of motion, but the direction of $\vec{x}$ cannot be constant except when $\dot{\vec{x}}=\vec{v}_1-\vec{v}_2$ is collinear with $\vec{x}$, so

$$f_1'\vec{v}_1\cdot\dot{\vec{v}}_1-f_2'\vec{v}_2\cdot\dot{\vec{v}}_2=0. \tag{20}$$

(We will not concern ourselves with singular accelerations that are zero for almost all values of the positions and velocities.) From (15) and (20), we have

$$f_n'\vec{v}_n\cdot\dot{\vec{v}}_n=0 \tag{21}$$

for $n=1,2$. On the other hand, if Eq. (13) holds for $n=1,2$, then

$$f_n=m_n(1-v_n^2)^{-1}, \tag{22}$$

with nonzero constants m_n, for $n=1,2$. This implies, first, that $f_n'\neq0$, so from Eq. (21) we see that

$$\vec{v}_n\cdot\dot{\vec{v}}_n=0 \tag{23}$$

for $n=1,2$. It also implies that

$$f_n''4(v_n^2-1)+f_n'8=0$$

for $n=1,2$. From the latter, together with Eqs. (13) and (15), we see that Eq. (9) reduces to

$$-\sum_{n=1}^{2}f_n(\vec{e}\cdot\dot{\vec{v}}_n)\vec{v}_n=0,$$

from which it follows that

$$\vec{e}\cdot\dot{\vec{v}}_n=0$$

for any vector $\vec{e}$ perpendicular to $\vec{x}$, which means that $\dot{\vec{v}}_n$ is collinear with $\vec{x}$, for $n=1,2$. From this and Eq. (23) it follows that $\dot{\vec{v}}_n=0$ for $n=1,2$.

We are thus finished with the case of Eq. (13). It remains to consider only the alternative that Eq. (14) holds for $n=1,2$.

If we use $\vec{X}$ and $\vec{x}$ for the position variables in the constant of motion (7), we find that the coefficient of X_k is just the time derivative of the constant of motion (1), which is zero. Then, adding the constant of motion (16) multiplied by δ_{kl}, we see that

$$\tfrac{1}{2}x_k(f_1\dot{v}_{1l}-f_2\dot{v}_{2l}+f_1'2\vec{v}_1\cdot\dot{\vec{v}}_1v_{1l}-f_2'2\vec{v}_2\cdot\dot{\vec{v}}_2v_{2l}) + \sum_{n=1}^{2}[f_n+f_n'2(v_n^2-1)]v_{nk}v_{nl} \tag{24}$$

is a constant of motion for $k,l=1,2,3$. The time derivative of this constant (24) contains terms proportional to x_k, $v_{1k}-v_{2k}$, v_{1k}, v_{2k}, $\dot{v}_{1k}$, and $\dot{v}_{2k}$, but Eq. (14) holding for $n=1,2$ means that $\vec{v}_1$, $\vec{v}_2$, $\dot{\vec{v}}_1$, and $\dot{\vec{v}}_2$ are all in the same plane, so the coefficient of x_k must be zero by itself, which means that

$$f_1\dot{v}_{1l}-f_2\dot{v}_{2l}+f_1'2\vec{v}_1\cdot\dot{\vec{v}}_1v_{1l}-f_2'2\vec{v}_2\cdot\dot{\vec{v}}_2v_{2l} \tag{25}$$

is a constant of motion. Comparing this with the time derivative of the constant of motion (1), which is constant because it is zero, we see that

$$f_n\dot{v}_{nl}+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nl} \tag{26}$$

is a constant of motion for $n=1,2$ and $l=1,2,3$.

Similarly, the time derivative of the constant of motion (19) contains terms proportional to x_k, $v_{1k}-v_{2k}$, v_{1k}, v_{2k}, $\dot{v}_{1k}$, and $\dot{v}_{2k}$, but Eq. (14) holding for $n=1,2$ means that $\vec{v}_1$, $\vec{v}_2$, $\dot{\vec{v}}_1$, and $\dot{\vec{v}}_2$ are all in the same plane, so the coefficient of x_k must be zero by itself, which means that

$$f_1'\vec{v}_1\cdot\dot{\vec{v}}_1-f_2'\vec{v}_2\cdot\dot{\vec{v}}_2 \tag{27}$$

is a constant of motion. Comparing this with Eq. (15), we see that

$$f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n \tag{28}$$

is a constant of motion for $n=1,2$.

For the Lorentz transform of the acceleration we use the bracket relations of the Poincaré group, the translation invariance of the velocity, and Eq. (6) to find

$$\begin{aligned}[\dot{v}_{nj},K_k]&=[[v_{nj},H],K_k]\\&=[[v_{nj},K_k],H]-[v_{nj},[K_k,H]]\\&=[x_{nk}\dot{v}_{nj}+v_{nj}v_{nk}-\delta_{jk},H]-[v_{nj},P_k]\\&=x_{nk}\ddot{v}_{nj}+2\dot{v}_{nj}v_{nk}+v_{nj}\dot{v}_{nk}.\end{aligned} \tag{29}$$

We use this and Eq. (6), and take the translation-invariant constant of motion (28) in place of the W_l of Eqs. (2) and (3), to work out another translation-invariant constant of motion

$$\begin{aligned}[f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n,K_k]&=2\vec{v}_n\cdot\dot{\vec{v}}_nf_n''2\vec{v}_n\cdot[\vec{v}_n,K_k]+f_n'2\dot{\vec{v}}_n\cdot[\vec{v}_n,K_k]+f_n'2\vec{v}_n\cdot[\dot{\vec{v}}_n,K_k]\\&=f_n''(2\vec{v}_n\cdot\dot{\vec{v}}_n)^2x_{nk}+f_n''2\vec{v}_n\cdot\dot{\vec{v}}_n2(v_n^2-1)v_{nk}+f_n'2\dot{\vec{v}}_n^2x_{nk}+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nk}-f_n'2\dot{v}_{nk}\\&\quad+f_n'2\vec{v}_n\cdot\ddot{\vec{v}}_nx_{nk}+2f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nk}+f_n'2v_n^2\dot{v}_{nk}\end{aligned} \tag{30}$$

for $n=1,2$ and $k=1,2,3$. Since this is translation invariant, the terms involving x_{nk} must vanish, so we have

$$f_n''(2\vec{v}_n\cdot\dot{\vec{v}}_n)^2+f_n'2\dot{\vec{v}}_n^2+f_n'2\vec{v}_n\cdot\ddot{\vec{v}}_n=0 \tag{31}$$

for $n=1,2$. Then the constant of motion (30) is

$$[f_n''2(v_n^2-1)+3f_n']2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nk}+f_n'2(v_n^2-1)\dot{v}_{nk}. \tag{32}$$

Finally we take the translation-invariant constant (26) for W_l, and use Eqs. (6) and (29) again to compute yet another translation-invariant constant of motion

$$\begin{aligned}[f_n\dot{v}_{nl}+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nl},K_k]&=f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nx_{nk}\dot{v}_{nl}+f_n'2(v_n^2-1)v_{nk}\dot{v}_{nl}+f_nx_{nk}\ddot{v}_{nl}+f_n2v_{nk}\dot{v}_{nl}+f_n\dot{v}_{nk}v_{nl}\\&\quad+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nx_{nk}\dot{v}_{nl}+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nk}v_{nl}-f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n\delta_{kl}+[f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n,K_k]v_{nl}.\end{aligned} \tag{33}$$

For this to be translation invariant the terms involving x_{nk} must vanish, so

$$f_n\ddot{v}_{nl}+2f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n\dot{v}_{nl}=0 \tag{34}$$

for $n=1,2$ and $l=1,2,3$. Then, since the coefficient of δ_{kl} is just the constant (28), Eq. (33) yields a new constant of motion

$$G_{nk}v_{nl}+(f_n\dot{v}_{nk}+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_nv_{nk})v_{nl}+[f_n'2(v_n^2-1)+2f_n]v_{nk}\dot{v}_{nl} \tag{35}$$

for $n=1,2$ and $k,l=1,2,3$, where G_{nk} is the constant (30) or (32). Recognizing also the constant (26) in ( ), we take the time derivative of the constant (35), use Eq. (34) to eliminate $\ddot{v}_{nl}$, and get

$$\{[2f_n''2(v_n^2-1)2\vec{v}_n\cdot\dot{\vec{v}}_n+4f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n-2(f_n'^2/f_n)2(v_n^2-1)2\vec{v}_n\cdot\dot{\vec{v}}_n]v_{nk}+[f_n'4(v_n^2-1)+3f_n]\dot{v}_{nk}\}\dot{v}_{nl}=0 \tag{36}$$

for $n=1,2$ and $k,l=1,2,3$.

Using Eq. (34) to eliminate the $\ddot{\vec{v}}_n$ in Eq. (31), we obtain

$$f_n''(2\vec{v}_n\cdot\dot{\vec{v}}_n)^2+f_n'2\dot{\vec{v}}_n^{\,2}-2(f_n'^2/f_n)(2\vec{v}_n\cdot\dot{\vec{v}}_n)^2=0. \tag{37}$$

If the quantity in [ ] multiplying $\dot{v}_{nk}$ in Eq. (36) were zero, we would have

$$f_n'=\tfrac{3}{4}f_n(1-v_n^2)^{-1}. \tag{38}$$

Then

$$f_n''=\tfrac{21}{16}f_n(1-v_n^2)^{-2}. \tag{39}$$

Substituting Eqs. (38) and (39) into Eq. (37), we find

$$\dot{\vec{v}}_n^{\,2}=-\tfrac{1}{8}(2\vec{v}_n\cdot\dot{\vec{v}}_n)^2(1-v_n^2)^{-1} \tag{40}$$

which, for speeds less than that of light, $v_n^2<1$, implies that $\dot{\vec{v}}_n$ is zero. Then from Eq. (36) we see that if $\dot{\vec{v}}_n$ is not zero it is collinear with $\vec{v}_n$. Therefore in the time derivative of the original constant (1)

$$\sum_{n=1}^{2}(f_n\dot{\vec{v}}_n+f_n'2\vec{v}_n\cdot\dot{\vec{v}}_n\vec{v}_n),$$

the part for each n must be zero separately, so $f_1\vec{v}_1$ and $f_2\vec{v}_2$ are each constant. If we start at the beginning with these two separate constants instead of the total constant (1), then in place of the constant (16) we find that f_1 and f_2 are each separately constant. Then $\vec{v}_1$ and $\vec{v}_2$ are each constant. This completes the proof.

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