Possible experimental test of local commutativity

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A direct experimental test for local commutativity (i.e., $[A,B]=0$ if A and B are associated with mutually spacelike regions) is proposed. It consists of a comparison of the distribution of outcomes of A measurements in the two cases where A alone is measured (no measurement apparatus for B present) and where both are measured on the same sample, in short, a test of the compatibility of the two observation procedures for A and B. The implication compatibility→commutativity (substantially proved by von Neumann, but questioned by Park and Margenau) is proved in a stronger form, with close attention paid to the distinction between different procedures corresponding to a given self-adjoint operator.

I. INTRODUCTION

The assumption of local commutativity[1,2] (LC) (two observables A and B commute if they are associated with space-time regions which are spacelike) is widely accepted, although no direct experimental evidence seems to exist. Since there are some theoretical doubts[3] about its validity, a direct experimental test may be useful.

The proposed test uses two observation procedures on the same sample, each associated with different space-time regions R_1 and R_2 as, for instance, in the Einstein-Podolsky-Rosen (EPR) and hidden-variable experiments.[4] The relative frequency of an outcome λ in R_1 in such an experiment should be equal to that obtained in a parallel experimental run in which no measurement apparatus is present in R_2.

In the following, we first summarize some reasons for lingering doubts about the validity of local commutativity and then we prove that compatibility (of which the above experiment was an instance) implies local commutativity.

The reasons usually given for assuming LC are plausible but not entirely cogent. They combine the relativistic postulate of maximal signal velocity with the quantum-mechanical idea that noninterference between two observation procedures implies commutativity of the respective operators. It would be more satisfactory to formulate maximal signal velocity and noninterference in a precise and operational way and then deduce LC by using the axioms of quantum mechanics. Such an attempt has been made. Haag and Kastler[5] have defined operationally a property of the algebra $\mathfrak{A}$ of observables (later called local independence by Schlieder[6]) which is the precise version of maximal signal velocity, but it has been shown that it does not, by itself, imply LC.[7] Whether other axioms of relativistic quantum theory, added to local independence, imply LC remains an open question.[8,9]

On a more pragmatic level, the strict validity of local commutativity has been questioned because it leads to the well-known divergence of field theory, and attempts to construct divergence-free, relativistic, but nonlocal theories have been considered.[3] While some of these theories can be tested by their observable consequences, it is not clear whether a failure is due to nonlocality or to the specific theory.

The implication compatibility → commutativity has been substantially proved by von Neumann,[10] but there are some aspects of the situation not covered by him. Examples are nonsimultaneous measurements and measurements with a finite timelike duration, necessary in relativistic theory. Also, the validity of his conclusion has been recently denied by Park and Margenau,[11] so that a proof from explicitly stated prime principles is necessary. Since a good deal of doubt and disagreement in the previous literature on the subject is due to a somewhat allusive presentation, we try to be literal.

It is trivial that there are many procedures used to measure a physical quantity, a mercury thermometer and a thermocouple, for instance. However, previous studies have assumed that the operationally defined procedures (physical quantities for von Neumann,[10] observables for Park and Margenau[11]) are in a one-to-one relation to the self-adjoint operators. We find it necessary to distinguish between different observation procedures that correspond to a given self-adjoint operator because the operational property of compatibility cannot be ascribed to all procedures that map onto a self-adjoint operator.

II. ASSUMPTIONS AND THEOREMS

State-preparing and observation procedures are considered here to be instruction booklets or programs for robots. For our purposes it is necessary to include as procedures all "limits" of sequences of actual procedures. We distinguish between procedures which differ in irrelevant details—e.g. a red ammeter and a green ammeter. The outcome of carrying out a state-preparing procedure followed by an observation procedure is a (symbol for a) real number. Repetitions of an observation act α on samples, each produced by a procedure s, produce an infinite random[12] sequence $\psi_{s\alpha}$ of outcomes with limit mean value $\bar{M}\psi_{s\alpha}$.

An observation procedure will not, in general, use the same physical apparatus for different state-preparing procedures. For instance, thermocouples rather than mercury thermometers are used to deal with high temperatures. But there must be instructions included for building different pieces of equipment so that an observation procedure can be used for all state-preparing procedures.

For the purposes of this paper a minimum amount of theoretical development is necessary: Accordingly only that which is needed is given explicitly here. More details can be found elsewhere.[7] Define equivalence relations $\sim_{\mathcal{O}}$ and $\sim_{\mathcal{S}}$ on $\mathcal{O}$ and on $\mathcal{S}$ by

$$\alpha \sim_{\mathcal{O}} \beta \text{ if and only if } \bar{M}\psi_{s\alpha} = \bar{M}\psi_{s\beta} \tag{1}$$

for all s in $\mathcal{S}$ and

$$s \sim_{\mathcal{S}} t \text{ if and only if } \bar{M}\psi_{s\alpha} = \bar{M}\psi_{t\alpha} \tag{2}$$

for all α in $\mathcal{O}$. Let $\mathcal{O}/\sim_{\mathcal{O}}$ and $\mathcal{S}/\sim_{\mathcal{S}}$ denote the sets of equivalence classes of elements of $\mathcal{O}$ and $\mathcal{S}$, respectively.

Assumption (1).

(a) For each α in $\mathcal{O}$ and each Borel function f: $R \to R$ the procedure $f(\alpha)$ "do α and give as output f (outcome so obtained)" is in $\mathcal{O}$.[13]

(b) For each α, β in $\mathcal{O}$ there is a procedure γ in $\mathcal{O}$ such that $\bar{M}\psi_{s\gamma} = \bar{M}\psi_{s\alpha} + \bar{M}\psi_{s\beta}$ for all s in $\mathcal{S}$.

Assumption (2). $\mathcal{O}/\sim_{\mathcal{O}}$ is identified with the set $\mathcal{A}_{sa}$ of all self-adjoint operators in a von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space.[14]

The linear operations on $\mathcal{A}$ are related to the operations of assumption (1) as follows: Let Φ: $\mathcal{O} \to \mathcal{A}_{sa}$ be a many-one map from $\mathcal{O}$ onto $\mathcal{A}_{sa}$ which preserves equivalence classes.

Assumption (3).

(a) For each Borel function f and each α in $\mathcal{O}$,

$$\Phi(f(\alpha)) = f(\Phi(\alpha)). \tag{3}$$

(b) For all α, β, γ in $\mathcal{O}$,

$$[\text{for all } s \in \mathcal{S}(\bar{M}\psi_{s\alpha} + \bar{M}\psi_{s\beta} = \bar{M}\psi_{s\gamma})] \leftrightarrow \Phi(\alpha) + \Phi(\beta) = \Phi(\gamma).$$

Let S' be the set of all maps from $\mathcal{A}_{sa}$ to the set R of real numbers. Define Ψ: $\mathcal{S} \to S'$ to be the map given by

$$\Psi(s)(\Phi(\alpha)) = \bar{M}\psi_{s\alpha} \tag{4}$$

for all s in $\mathcal{S}$ and α in $\mathcal{O}$. It can be shown from the above assumptions that for each s, $\Psi(s)$ is a *state* on $\mathcal{A}_{sa}$ (i.e., a positive, linear, norm-1 function from $\mathcal{A}_{sa}$ to R) and that Ψ preserves equivalence classes of $\mathcal{S}/\sim_{\mathcal{S}}$. We extend (uniquely) $\Psi(s)$: $\mathcal{A}_{sa} \to R$ to a state ω_s: $\mathcal{A} \to C$.

So far, the extended Ψ is a map into the set S of all states over $\mathcal{A}$ (note that S is a subset of S') and is not necessarily onto. This is taken care of by the next assumption.

Assumption (4). For every state ρ in S there is an s in $\mathcal{S}$ such that (the extended) $\Psi(s) = \rho$.

It then follows from these assumptions that $\mathcal{S}/\sim_{\mathcal{S}}$ can be identified with the set S of all states over $\mathcal{A}$.

To define *compatibility* (a generalization of von Neumann's simultaneous measurability),[10] consider a pair of procedures α, β, in $\mathcal{O}$ such that they can both be applied to a given sample. It is by no means obvious that this is possible for all pairs of procedures in the equivalence classes $\Phi^{-1}[\Phi(\alpha)]$ and $\Phi^{-1}[\Phi(\beta)]$ of two given operators $\Phi(\alpha)$ and $\Phi(\beta)$. For instance, if $\Phi(\alpha)$ is the position of a particle at $t=0$ and $\Phi(\beta)$ is that at $t=1$, a procedure that measures the position by absorbing the particle on a photographic plate at $t=0$ prevents the performance of a similar procedure at $t=1$. On the other hand, two procedures $\alpha \in \Phi^{-1}[\Phi(\alpha)]$ and $\beta \in \Phi^{-1}[\Phi(\beta)]$ that use γ-ray microscopes at $t=0$ and $t=1$, respectively, are at least candidates for compatibility.

For these pairs of procedures α, β one obtains from repetitions of the compound measurement of α and β on a sample prepared by s, an infinite random sequence $\psi_{s\alpha,\beta}$ of pairs of real numbers. Let $\psi_{s\alpha'}$ and $\psi_{s\beta'}$ denote the sequences associated with the respective α measurement and β measurement (i.e., $\psi_{s\alpha,\beta} = \psi_{s\alpha'} \times \psi_{s\beta'}$). Let $\psi_{s\alpha}$ and $\psi_{s\beta}$ denote the sequences obtained for repetitions of usual observation acts (i.e., only one procedure used for each sample).

The definition of compatibility we shall use is similar to that given by Park and Margenau[11] and is essentially the same as that given elsewhere.[15] Two procedures α and β are defined to be *compatible* if $\psi_{s\alpha'}$ and $\psi_{s\alpha}$ have the same limit relative frequency of occurrence of outcomes and similarly for $\psi_{s\beta'}$ and $\psi_{s\beta}$. More precisely, this means that α and β are compatible if

$$\bar{M}F_{[a,b)}\psi_{s\alpha'} = \bar{M}F_{[a,b)}\psi_{s\alpha} \tag{5}$$

and

$$\bar{M}F_{[a,b)}\psi_{s\beta'} = \bar{M}F_{[a,b)}\psi_{s\beta} \tag{6}$$

for each half-open interval $[a, b)$ of the real line and for all $s \in \mathcal{S}$. $\bar{M}F_{[a,b)}\psi$ is the limit relative frequency of finding an element of ψ in $[a, b)$.

In this definition, nothing is assumed about the time at which the observations are performed; in particular, it is not assumed that they are instantaneous (which would probably be incompatible with relativistic quantum theory).

We now define two new observation procedures $\alpha \oplus \beta$ and $\alpha \ominus \beta$ by the instructions "do α and β on the same sample and add ($\oplus$) or subtract ($\ominus$) the outcomes." As noted above, $\alpha \oplus \beta$ and $\alpha \ominus \beta$ are defined only for those pairs α, β which are candidates for compatibility.

The outcome sequences $\psi_{s\alpha\oplus\beta}$ and $\psi_{s\alpha\ominus\beta}$ are related to $\psi_{s\alpha'}$ and $\psi_{s\beta'}$ by

$$\psi_{s\alpha\oplus\beta} = \psi_{s\alpha'} + \psi_{s\beta'} , \tag{7}$$

$$\psi_{s\alpha\ominus\beta} = \psi_{s\alpha'} - \psi_{s\beta'} , \tag{8}$$

respectively. We note also that if $\psi_{s\alpha,\beta}$, the original sequence of pairs of outcomes, is random then one can show that $\psi_{s\alpha\oplus\beta}$ and $\psi_{s\alpha\ominus\beta}$ are random.

Theorem 1. If α is compatible with β, then

$$\text{(a)}\quad \Phi(\alpha \oplus \beta) = \Phi(\alpha) + \Phi(\beta) , \tag{9}$$

$$\text{(b)}\quad \Phi(\alpha \ominus \beta) = \Phi(\alpha) - \Phi(\beta) . \tag{10}$$

Proof. We give an explicit proof for (a) only; that for (b) is similar. By Eq. (7)

$$\bar{M}\psi_{s\alpha\oplus\beta} = \bar{M}(\psi_{s\alpha'} + \psi_{s\beta'})$$
$$= \bar{M}\psi_{s\alpha'} + \bar{M}\psi_{s\beta'} \quad \text{for all } s \in \mathcal{S} .$$

By hypothesis and from the definition of compatibility, $\bar{M}\psi_{s\alpha'} = \bar{M}\psi_{s\alpha}$ and $\bar{M}\psi_{s\beta'} = \bar{M}\psi_{s\beta}$. Thus one has

$$\bar{M}\psi_{s\alpha\oplus\beta} = \bar{M}\psi_{s\alpha} + \bar{M}\psi_{s\beta}$$
$$= \Psi(s)(\Phi(\alpha)) + \Psi(s)(\Phi(\beta))$$
$$= \Psi(s)(\Phi(\alpha) + \Phi(\beta)) .$$

By Eq. (4) and assumption (3), one has

$$\Psi(s)(\Phi(\alpha\oplus\beta)) = \Psi(s)(\Phi(\alpha) + \Phi(\beta))$$

for all $s \in \mathcal{S}$. Since (the extended) Ψ is onto S [assumption (4)] and the states of $\mathfrak{A}$ separate the elements of $\mathfrak{A}$, one has $\Phi(\alpha \oplus \beta) = \Phi(\alpha) + \Phi(\beta)$. Q.E.D.

In preparation for the next theorem, one notes that by a strict interpretation of the definition of compatibility, α is never compatible with α, because any apparatus used for α clearly interferes with an identical apparatus in the same space-time region. However, by convention we shall denote the compound procedure for α *and* α on the same sample by the instructions, "do α and duplicate the outcome." By this convention, α is compatible with α and with $f(\alpha)$ for any Borel function f.

Theorem 2. (von Neumann). If α and β are compatible, then

$$[\Phi(\alpha), \Phi(\beta)] = 0 .$$

Proof. By the above convention, and by the hypothesis, $\alpha \oplus \beta$ and $\alpha \ominus \beta$ are compatible and are in $\mathcal{O}$.

Let $\alpha \odot \beta$ denote the procedure which does α and β on the same sample but multiplies the outcomes. One clearly has $\alpha\odot\beta = \beta\odot\alpha$ and

$$4(\alpha\odot\beta) = (\alpha \oplus \beta)^2 - (\alpha \ominus \beta)^2 ,$$

where $\alpha\odot\alpha$ is represented by α^2 for each α in $\mathcal{O}$. Thus one has, by theorem 1 and Eq. (3),

$$\Phi(\alpha\odot\beta) = \tfrac{1}{4}[(\Phi(\alpha\oplus\beta))^2 - (\Phi(\alpha\ominus\beta))^2]$$
$$= \frac{\Phi(\alpha)\circ\Phi(\beta) + \Phi(\beta)\circ\Phi(\alpha)}{2} . \tag{11}$$

By the above definition, the outcome sequence $\psi_{s\alpha\odot\beta}$ is given by

$$\psi_{s\alpha\odot\beta}(j) = \psi_{s\alpha'}(j)\cdot\psi_{s\beta'}(j)$$

for each j. By iteration, the procedure $(\alpha\odot\beta)\odot\beta$ has an outcome sequence $\psi_{s(\alpha\odot\beta)\odot\beta}$ given by

$$\psi_{s(\alpha\odot\beta)\odot\beta}(j) = (\psi_{s\alpha'}(j)\cdot\psi_{s\beta'}(j))\cdot\psi_{s\beta'}(j)$$

for each j. Similarly, $\alpha\odot(\beta\odot\beta)$ has an outcome sequence given by

$$\psi_{s\alpha\odot(\beta\odot\beta)}(j) = \psi_{s\alpha'}(j)\cdot(\psi_{s\beta'}(j))^2 .$$

By the associativity of multiplication of real numbers, one has $\psi_{s(\alpha\odot\beta)\odot\beta} = \psi_{s\alpha\odot(\beta\odot\beta)}$. Thus for each s in $\mathcal{S}$

$$\bar{M}\psi_{s(\alpha\odot\beta)\odot\beta} = \bar{M}\psi_{s\alpha\odot(\beta\odot\beta)}$$

and

$$(\alpha\odot\beta)\odot\beta \sim_0 \alpha\odot(\beta\odot\beta) .$$

or

$$\Phi((\alpha\odot\beta)\odot\beta) = \Phi(\alpha\odot(\beta\odot\beta)) . \tag{12}$$

By Eq. (11) one then has that

$$(\Phi(\alpha)\times\Phi(\beta))\times\Phi(\beta) = \phi(\alpha)\times(\Phi(\beta)\times\Phi(\beta)) , \tag{13}$$

where as a temporary notation $\Phi(\alpha)\times\Phi(\beta)$ denotes the symmetrized product

$$\tfrac{1}{2}[\Phi(\alpha)\circ\Phi(\beta) + \Phi(\beta)\circ\Phi(\alpha)] .$$

Now the operations + and $\times$ together with scalar multiplication define a real commutative Jordan algebra[16] on $\mathfrak{A}_{sa}$. By the theorem[16] on Jordan algebras relating commutativity and associativity,

$$[\Phi(\alpha), \Phi(\beta)] = 0 . \quad \text{Q.E.D.}$$

III. DISCUSSION

Consider two operators associated with spacelike regions R_1 and R_2—e.g. the spin components of two particles located in the regions R_1 and R_2, respectively, and assume that two corresponding procedures α and β are known. Then the experimental compatibility of α and β would corroborate, by theorem 2, the assumption of local commutativity. The test of compatibility consists in measuring, with the given state-preparing procedure, the outcome sequences $\psi_{s\alpha'}$ and $\psi_{s\beta'}$ by use of both instruments together and comparing $\psi_{s\alpha'}$ with the outcome sequence $\psi_{s\alpha}$ for only one spin (α), the instrument for measuring the other spin (β) having been removed. The two sequences $\psi_{s\alpha}$ and $\psi_{s\alpha'}$ should have the same limit frequencies of outcomes. For finite sequences, an approximate equality of the distribution of outcomes is corroborative. Similar arguments hold for $\psi_{s\beta}$ and $\psi_{s\beta'}$.

Conversely, one can ask: If the sequences $\psi_{s\alpha}$ and $\psi_{s\alpha'}$ are noticeably nonequivalent, should this be counted as evidence for the failure of local commutativity? Thus far, we have only proved that compatibility implies commutativity, but not the converse. In fact, the converse is not plausible without further restrictions, because a clumsy instrument measuring β correctly may well interfere with α and destroy compatibility. Hence, we consider another condition on the measuring procedure: *gentleness*, such that (commutativity + gentleness → compatibility). An example for a procedure α which is clearly not gentle for β was mentioned in Sec. II: a destructive position measurement at $t=0$ which makes another measurement at $t=1$ impossible. The best-known nontrivial example for the nonexistence of two mutually gentle procedures $\alpha \in \Phi^{-1}(A)$ and $\beta \in \Phi^{-1}(B)$ is Heisenberg's thought experiment on the simultaneous observation of the operators P and Q. The gentleness condition is also used in Pool's statement[17] of the projection axiom.[10]

Although no one has been able to give an operational definition, there are some cases where it is intuitively evident. The first case is that of the observation of two operators A and $B=f(A) \in \mathfrak{A}_{sa}$, where f is a suitable real-valued function. By assumption (1), there exist two procedures α and $f(\alpha)$ in $\mathcal{O}$ which differ only by a *software* instruction, so that only one material interaction process happens when the two procedures are carried out. Manifestly, α and $f(\alpha)$ are mutually gentle, and this fact might seem to be useful. Indeed, in the Abelian algebra generated by A and B, there exists[18] an element $C \in \mathfrak{A}_{sa}$ and two real-valued functions G and H such that

$$A=G(C) \quad \text{and} \quad B=H(C) ,$$

so that the existence of two mutually gentle procedures $\alpha=G(\gamma)$ in $\Phi^{-1}(A)$ and $\beta=H(\gamma)$ in $\Phi^{-1}(B)$ [where $\gamma \in \Phi^{-1}(C)$] is guaranteed. However, this fact is not useful for our purpose because the procedures in $\Phi^{-1}(C)$ are not known in general. We can assume realistically only the knowledge of reasonable subsets of procedures $\Phi^{-1}(A)$ and $\Phi^{-1}(B)$, such as the spin measurements of two particles in the same sample, in a given space-time region.

However, there is a simple case in which gentleness is reasonably certain: when the two apparatuses belonging to α and β are materially located and active in spacelike regions R_1 and R_2. Fortunately, this is not difficult to carry out, and we can conclude that the failure to find compatibility in an experiment with gentleness as described would indeed be strong evidence against the assumption of local commutativity.

We add a comment on the reason for the discrepancy between our conclusions and those of Park and Margenau. Park and Margenau[11] define what in our language would be the procedure $\alpha \oplus \beta$ for two procedures that are compatible only for *some* states. Explicitly, the procedures considered by Park and Margenau are such that the relations (5) and (6) of Sec. II are valid only for some state-preparing procedures $s \in \mathcal{S}$. We feel that it is not consistent to attribute a mutual relation to two elements α and β of $\mathcal{O}$ on the ground of a numerical relation between expectation values or distributions valid for some states. More generally, if the sequences $\psi_{s\alpha}$ and $\psi_{s\beta}$ have some mutual relations for all s, one can (as a matter of convenience), *define* corresponding relations between α and $\beta \in \mathcal{O}$, but not if the relation is true only for some $s \in \mathcal{S}$. This is the analog of inferring from the relation between expectation values

$$\mathrm{Tr}\rho A + \mathrm{Tr}\rho B = \mathrm{Tr}\rho C \qquad (14)$$

for all ρ that the operators A, B, C have the relation

$$A+B=C , \qquad (15)$$

but the inference fails if Eq. (14) is valid only for *some* density matrices ρ.

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[12]Randomness is an indispensable requirement on these sequences, but since we do not use it explicitly here the reader is referred to P. Benioff, Phys. Rev. D 7, 3603 (1973) and P. Benioff and H. Ekstein, Nuovo Cimento B (to be published), for a more extensive discussion.

[13]For many f the procedure $f(\alpha)$ as described must be a limit of a sequence of procedures $f_n(\alpha)$, $n=1,2,\ldots$ and thus, in a strict sence, cannot actually be carried out. Otherwise for some α the difficulty mentioned elsewhere [P. Benioff, Found. Phys. 5, 251 (1975)] arises here.

[14]For technical reasons, unbounded operators such as p and q must be excluded. However, all reasonable bounded functions of these are included.

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