Pole-dipole model of massless particles*

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Zero-rest-mass particles in a gravitational field are considered to have energy-momentum tensors T^{ij} satisfying $T^{ij}{}_{;j}=0$ and also $T^{i}{}_{i}=0$. The latter is satisfied by a massless electromagnetic pulse, and also by a neutrino. In analogy with the theory of material spinning particles in general relativity, moments of these equations are taken and cut off at the dipole level. The moments of the first equation yield the usual pole-dipole equations. The moments of the second equation yield the auxiliary conditions, which turn out to be $p_i v^i=0$ and $v_i S^{ki}=0$, where v^i is the velocity, p^i is the momentum, and S^{ki} is the spin. The special cases of arbitrary spin in flat space, zero spin in curved spaces, and the eikonal approximation are treated and shown to give null geodesic trajectories. However, such trajectories do not seem to be a necessary consequence in the general case.

I. INTRODUCTION

The equations of motion in general relativity for extended material particles with spin were first formulated by Mathisson,[1] using a singular energy-momentum tensor. These equations had to be supplemented by subsidiary conditions which in effect defined which point in the object was its mathematical center.

Since Mathisson, a considerable amount of work has been done on this subject, mainly having to do with the details of the derivation of the equations of motion and with the nature of the subsidiary conditions.

Papapetrou[2] worked with an assumed nonsingular localized energy-momentum tensor T^{ij} for a spinning particle, and obtained equations for the moments of T^{ij}. Tulczyjew[3] criticized the subsidiary conditions and came up with a different form. Dixon[4] found a way to make Papapetrou's type of argument covariant at each step of the derivation, and also found equations which referred to paths described by an arbitrary parameter rather than the world-line length. He generalized the equations to take into account quadrupole and higher terms.

The purpose of the present paper is to extend the method to take into account particles with zero rest mass, which we shall describe as massless particles for short. There are two such particles: the photon, or, say a localized pulse of electromagnetic energy and momentum, and the neutrino.

The method we use is essentially Papaetrou's; the reason is that the integrals as defined in his paper can be easily extended to the situation where the world line is along a null geodesic, which of course must be contemplated for massless particles. The integrals as defined by Dixon, for example, do not lend themselves easily to such an extension.[5]

In order to make the argument for massless particles, we must discuss both the equations of motion and the subsidiary condition. The equations of motion are obtained with virtually no deviation from the derivation of Papapetrou. However, the subsidiary conditions are derived from a simple property of the energy-momentum tensor. According to general relativity, the basic equation of motion is

$$T^{ij}{}_{;j}=0, \tag{1.1}$$

where the semicolon means of course covariant differentiation. In addition, for massless particles, the further condition

$$T^{i}{}_{i}=0 \tag{1.2}$$

is proposed.

It is well known that Eq. (1.2) is a property of electromagnetic systems without masses and will apply to photons. Equation (1.2) is also satisfied for neutrinos, as has been pointed out by Brill and Wheeler.[6] From our point of view, Eq. (1.2) is what is needed to *derive* the subsidiary equations of the theory. One of these conditions will turn out to be the analog of the equation $p_i p^i=0$ (p^i = momentum) which characterizes zero-rest-mass particles in special relativity.

It is appropriate to point out here that moment integrals of T^{ij} have been used by many people as characterizing a photon's energy and momentum, and even spin.[7] For example, the integral

$$\int d^3x[\vec{\mathrm{r}}\times(\vec{\mathrm{E}}\times\vec{\mathrm{B}})]_\mu=\int d^3x\,\epsilon_{\mu\alpha\beta}T^{\alpha 0}x^\beta, \tag{1.3}$$

where

$$T^{\alpha 0}=\epsilon^{\alpha\sigma\tau}E_\sigma B_\tau \quad (1.4)$$

can be shown[8] to lead to intrinsic spin. Here $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, and $\epsilon^{\alpha\sigma\tau}$ is the three-space alternating symbol.

In Sec. II, the material pole-dipole theory is discussed. In Sec. III, the theory for massless particles is described with some calculations reserved for the appendixes. Here the problem of whether or not the massless particles always travel in null geodesic trajectories is discussed. The remainder of the paper considers some special cases in which familiar results can be derived. The reduction to flat space is made in Sec. IV. The "pole approximation" (spin neglected) is in Sec. V, and the eikonal limit is in Sec. VI. Latin indices take on values 0, 1, 2, 3 and Greek indices take on values 1, 2, 3. The signature is -2 in what follows.

II. THE POLE-DIPOLE THEORY FOR MATERIAL PARTICLES

As mentioned in the Introduction, we shall follow the formalism of Papapetrou.[2] Let T^{ij} be the components of the energy-momentum tensor, considered localized so that a multipole expansion of the moments would converge, and consider integrals over the hypersurface Σ perpendicular to the time (0) axis. Papapetrou shows that the two combinations

$$p^i=\int_\Sigma T^{i0}(-g)^{1/2}d\Sigma+\Gamma^i{}_{jk}S^{k0}dx^j/dt, \quad (2.1)$$

$$S^{ij}=\int_\Sigma[\delta x^i(s)T^{j0}-\delta x^j(s)T^{i0}](-g)^{1/2}d\Sigma \quad (2.2)$$

transform as tensors. Here $\Gamma^i{}_{jk}$ is the Christoffel symbol defined at $x^i(s)$, and $x^i(s)$ is some center of the localized object; s is its pathlength parameter. $\delta x^i(s)$ is $x^i-x^i(s)$, where x^i is the arbitrary point in the particle, over which the integration is performed. Integrals with two factors of $\delta x^i(s)$ in the integrand are neglected in the pole-dipole approximation. p^i is interpreted as a generalized momentum,[3] and S^{ij} is the spin tensor.

By taking moments of the equation of motion, Eq. (1.1), Papapetrou derives the two equations

$$(D/Ds)[mv^i+v_jDS^{ij}/Ds]=-\tfrac{1}{2}R_{jkm}{}^iS^{jk}v^m, \quad (2.3)$$

$$DS^{ij}/Ds-v^iv_kDS^{kj}/Ds-v^jv_kDS^{ik}/Ds=0, \quad (2.4)$$

where $v^i=dx^i(s)/ds$, $R_{jkm}{}^i$ is the Riemann tensor, and

$$m=p^iv_i. \quad (2.5)$$

The quantity in square brackets in Eq. (2.3) is just p^i. m is interpreted as the mass of the object.

Equations (2.3) and (2.4) are the pole-dipole equations for a material particle. They do not form a complete system of equations for the 11 unknowns v^i, S^{ij}, and m. Equations (2.3) number four, but Eqs. (2.4) number only three, since multiplication of (2.4) by v_i and summing yields three identities. In addition to these seven equations, we also have $v^iv_i=1$, giving eight equations in 11 unknowns.

Thus three more equations are needed. Mathisson[1] chose these to be

$$v_iS^{ij}=0. \quad (2.6)$$

This has the significance that in the zero-velocity frame of reference, in which $v^i=(1,0,0,0)$, $S^{i0}=0$. But a glance at Eq. (2.2) shows that $S^{i0}=0$ means that $x^i(s)$ is the center of energy in that system. Thus Eq. (2.6) defines $x^i(s)$.

Papapetrou and Corinaldesi[9] used as the side conditions

$$S^{i0}=0, \quad (2.7)$$

which means that $x^i(s)$ is the center of energy in the frame of reference whose coordinate system is the x^i. (For a spinning object circling the earth, this would be the earth system.)

Tulczyjew[3] showed that Eq. (2.6) led to some nonphysical spiral-type motion in flat space, and suggested that

$$p_iS^{ij}=0 \quad (2.8)$$

be the side condition. This means that $x^i(s)$ is the center of energy in the zero-momentum system [the system in which $p^i=(1,0,0,0)$]. Dixon[4] adopted the Tulczyjew condition, and it seems now to be agreed upon that Eq. (2.8) is appropriate for material particles.

In extending the procedure so as to be systematically covariant in the derivation, Dixon came up with the equations of motion in the form

$$Dp^i/Dq=\tfrac{1}{2}R^i{}_{jkm}v^jS^{km}, \quad (2.9)$$

$$DS^{ij}/Dq=p^iv^j-p^jv^i, \quad (2.10)$$

where now q is an arbitrary parameter along the curve, and $v^i=dx^i(q)/dq$. Equations (2.9) and (2.10) can be derived by the Papapetrou procedure with no difficulty,[10] using the p^i's and S^{ij}'s as defined in Eqs. (2.1) and (2.2). In what follows, we shall find this formulation of the basic equations to be the most useful.

A useful consequence of Eq. (2.10) is

$$p^i=(p^0/v^0)v^i+(v^0)^{-1}DS^{i0}/Dq, \quad (2.11)$$

which gives a formal relation between p^i and v^i.

III. THE POLE-DIPOLE MODEL FOR MASSLESS PARTICLES

The plan here is to extend the discussion of the preceding section to cover particles characterized by an energy-momentum tensor satisfying not only Eq. (1.1) but also Eq. (1.2). Further, for such particles, we must anticipate the possibility that the trajectory lies along a null direction for which $v^i v_i = 0$. Thus we cannot use the world-line length s as a parameter describing the motion. Rather, we must use an arbitrary parameter, q, to be determined by convenience during the course of the calculation.

As for the basic pole-dipole equations, we cannot use Eq. (2.4) since that required $v^i v_i = 1$. However, we can use Eqs. (2.9), (2.10), and (2.11), which for completeness in this section we repeat here:

$$Dp^i/Dq = \tfrac{1}{2}R^i{}_{jkm}v^j S^{km}, \tag{3.1}$$

$$DS^{ij}/Dq = p^i v^j - p^j v^i. \tag{3.2}$$

Nothing in the derivation of these equations required the object to have a rest mass, or required $g_{ij}dx^i dx^j$ to be nonzero. They are therefore sufficiently general to relate to any localized object. Higher-order multipole terms would exist in a systematic expansion,[4] but for particles with only momentum and spin (intrinsic or otherwise), Eqs. (3.1) and (3.2) are accurate.

Just as for material particles, these equations must be supplemented by subsidiary conditions. The conditions are found from the first two moments of Eq. (1.2). In Appendix A it is shown that the moments of $T^i{}_i$, although not tensors, nevertheless are zero in all reference frames if they are zero in one, whence the moments of Eq. (1.2) have an invariant significance. In Appendix B it is shown that the first moment of Eq. (1.2) yields

$$v_i p^i = -da/dq \tag{3.3}$$

and the second moment yields

$$b^i \equiv v_k S^{ki} = a v^i, \tag{3.4}$$

where a is a scalar which can be identified in any coordinate system as

$$a = b^0/v^0. \tag{3.5}$$

Equations (3.3) and (3.4) are the new subsidiary conditions, replacing $v_i v^i = 1$ and $p_k S^{ki} = 0$, respectively, which are valid for material particles.

These subsidiary conditions can be simplified: First we show that a is a constant, then that by choice of an initial condition, the constant may always be chosen to be zero.

Multiply Eq. (3.4) by v_i. The left-hand side is zero by symmetry, so

$$a(v_i v^i) = 0. \tag{3.6}$$

Thus if $v_i v^i \neq 0$, then $a = 0$. But it may well be that $x^i(q)$ traces out a null geodesic. In this case multiply Eq. (3.2) by v_i. Use of Eqs. (3.3) and (3.4) gives after some manipulation

$$2v^j da/dq = -aDv^j/Dq + S^{ij}Dv_i/Dq - (v_i v^i)p^j. \tag{3.7}$$

If $x^i(q)$ is a null geodesic, then the right-hand side is zero, whence $da/dq = 0$.

Thus independent of what the path is, a is a constant, and Eq. (3.3) reduces to

$$v_i p^i = 0. \tag{3.8}$$

If the path is not a null geodesic then Eq. (3.6) shows that a is zero. We can argue that even if $v_i v^i = 0$, initial conditions can always be chosen to set $a = 0$. At the initial point, place an orthonormal tetrad $XYZT$, where X represents the space direction of the pulse, with $v_I = \eta_{IJ}v^J$, where η_{IJ} is the Minkowski metric and $v^I = (v^0, v^X, 0, 0)$. In this tetrad, from Eq. (3.5), $a = (v_X/v^0)S^{X0}$. Now $S^{X0} = 0$ would mean from Eq. (2.2) that $x^X(q)$ is the center-of-energy component in the X direction. Thus setting $a = 0$ has the invariant significance that initially $x^X(q)$ is the center-of-energy component in every tetrad that has X as the space direction of propagation. This will be true not only initially but everywhere along the curve. We propose then that an initial condition appropriate to the problem is that a is zero. Then Eq. (3.4) reduces to

$$v_k S^{ki} = 0. \tag{3.9}$$

Although there may be other choices, this is the only one that gives the same form whether or not the trajectory is a null geodesic, since Eq. (3.6) shows that a must be zero if the path is not null.

We therefore adopt Eqs. (3.8) and (3.9) to be the appropriate auxiliary conditions for massless particles. Equations (3.1), (3.2), (3.8), and (3.9) constitute 14 equations, counting only three in Eq. (3.9) since multiplication by v_i gives an identity. The number of unknowns is also 14 (four p^i's, four v^i's, and six S^{ij}'s). Thus to the extent that we have as many independent equations as unknowns, we have a determinate system.

Equation (3.8) is the generalization of $p_i p^i = 0$ customarily used in special relativity to define a massless particle. The form in (3.8) was found by Jauch and Watson[11] long ago for a photon in flat space but in a dielectric medium. The fact that we have found the same equation is another verification of the similarity of the effects of a gravitational field to those of a dielectric medium.[12] Note that Eq. (3.8) shows that the "m" of Eq. (2.5) is zero.

Equation (3.9) is Mathisson's original side condition, Eq. (2.6), back again, and is not the one used nowadays by most workers for material particles, Eq. (2.8). There is of course a major difference in the present argument compared to previous work: Equations (3.3) and (3.4) are obtained not on the basis of a definition of $x^i(q)$, but because they are necessary consequences of Eq. (1.2). Our only reference to an interpretation was in the evaluation of a by an initial condition for a null geodesic.

The objects under discussion in this paper can be described as "massless localized pulses of energy, momentum and spin." The equations, along with appropriate initial and boundary conditions, should therefore be capable of describing the propagation of both neutrinos and photons in a gravitational field. As regards photons, there cannot be any inconsistency between the pole-dipole equations and Maxwell's equations, since the former are based on Eqs. (1.1) and (1.2), both of which are consistent with the latter. In fact, the only way to discuss momentum and spin in classical electrodynamics is indeed to introduce Poynting's vector and the energy-momentum tensor.

Nevertheless, when it comes to photons, one does not usually rely on T^{ij} or its moments: One works directly from Maxwell's equations. Convincing arguments then lead to results that have been described by Synge this way[13]: "In so far as they may be regarded as shock waves, waves of light are null surfaces. Further, the bicharacteristics are null geodesics . . . ."

Although in Secs. V, VI, and VII below we show that if the space is flat, or if spin is neglected, or if the high-frequency limit is taken, the pole-dipole equations predict null geodesics, we have been unable to prove that null geodesics necessarily follow for the general case of spin in curved spaces. It even looks unlikely in view of the complexity of Eq. (3.1). Thus there seems at least the possibility of a deviation from the conclusion quoted in the paragraph above. But since the pole-dipole equations are consistent with Maxwell's equations, could the pole-dipole trajectories ever end up not being null geodesics? If they did, would this not invalidate the whole pole-dipole approach?

The pole-dipole trajectories may end up always being null geodesics, but there would seem to be no necessity for it. The reason is that the usual treatments of light propagation in a gravitational field are made either in the eikonal limit[14,15] or by neglecting spin[16]; both cases are limits in which the pole-dipole equations do predict null geodesics. Even the quotation two paragraphs earlier is a consequence of an argument which at the outset Synge describes as "a modern equivalent of an old dodge in optics, viz., the passage from 'physical optics' to 'geometrical optics' by considering periodic waves of high frequency,"[17] i.e., as the eikonal limit.

Thus it all depends on what light is considered to be. If light is defined to be the eikonal ray associated with the electromagnetic shock wave, then there is no question: The trajectories are null geodesics. If, however, it is defined as a localized pulse[18] of electromagnetic energy, momentum, and spin, then it should travel along trajectories as defined by the pole-dipole equations, which may, or possibly may not always be null geodesics.

To finish this section we quote one general result obtained by multiplying Eq. (3.2) by S_{ij}:

$$d(S_{ij}S^{ij})/dq = 0. \tag{3.10}$$

The spin scalar is conserved.

The remainder of this paper considers some special cases in which familiar results can be recovered: flat space, the "pole" approximation, and the eikonal limit.

IV. FLAT SPACE

We wish to show here that the trajectories are null geodesics, in other words, that the description of special relativity is consistent with the pole-dipole equations. In flat space, Eq. (3.1) reduces to $dp^i/dq = 0$. Without loss of generality we can use Cartesian coordinates and choose x to be the direction of momentum: The solution to (3.1) is

$$p^i = (p'^0, p'^x, 0, 0), \tag{4.1}$$

where the primes indicate constants of integration.

Equations (3.2) can now be integrated:

$$S^{0x} = p'^0(x - x_0) - p'^x c(t - t_0), \tag{4.2a}$$

$$S^{0y} = p'^0(y - y_0), \tag{4.2b}$$

$$S^{0z} = p'^0(z - z_0), \tag{4.2c}$$

$$S^{xy} = p'^x(y - y_0'), \tag{4.2d}$$

$$S^{yz} = p'^x K, \tag{4.2e}$$

$$S^{zx} = -p'^x(z - z_0'). \tag{4.2f}$$

Here $x_0 x_0'$, y_0, y_0', t_0, and K are constants of integration.

Equation (3.8) can also be integrated,

$$x = (p'^0/p'^x)ct + x_0', \tag{4.3}$$

where x_0' is another constant of integration.

Substitution of Eqs. (4.2) and (4.3) into the three independent equations of (3.9) gives

$$v^0 p'^0(-y_0' + y_0) - K v^z = 0, \tag{4.4}$$

$$v^0 p'^0(-z_0'+z_0)+Kv^y=0, \quad (4.5)$$

$$(x-x_0)dx/dq+(y-y_0)dy/dq+(z-z_0)dz/dq -c(t-t_0)d(ct)/dq=0. \quad (4.6)$$

Equation (4.6) can be integrated immediately:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2-c^2(t-t_0)^2=0. \quad (4.7)$$

In general a constant of integration appears on the right-hand side, but by suitable choice of the origin of t, this constant can be set equal to zero.

Equations (4.1)–(4.7) give the complete set of pole-dipole equations. Any further information must come from the initial or boundary conditions, which information of course is not contained in the equations themselves. Equations (4.1) give the momentum; Eqs. (4.2) give the spin components in terms of $xyzt$; Eq. (4.3) gives x as a function of t; and Eqs. (4.4)–(4.6) give in principle $y(q)$, $z(q)$, and $t(q)$.

Equation (4.7) appears to show that the velocity is a null trajectory, but this is not immediately a consequence since although Eq. (4.3) shows that x is linear in t, nothing yet shows that y and z are also linear. In fact, if $y-y_0 \sim ct\cos wt$ and $z-z_0 \sim ct\sin wt$, reminding one of the spiral motions of Tulczyjew,[3] then this is also consisten with Eq. (4.7).

To obtain the linear behavior of y and z we must integrate Eqs. (4.4) and (4.5):

$$z=At+B, \quad (4.8)$$

$$y=Ct+D, \quad (4.9)$$

where ABC and D are constants related to those of Eqs. (4.4) and (4.5); the relation is not important. However, Eqs. (4.8) and (4.9) are valid consequences of Eqs. (4.4) and (4.5) only if K is not zero. But nothing in the equations forces K not to be zero. K represents, from Eq. (4.2e), the transverse component of spin. Thus any object which initially has a transverse component of spin will have y and z satisfying Eqs. (4.8) and (4.9). We shall confine ourselves to objects that initially have this property.

With each component x, y, and z now linear in t, Eq. (4.7) shows that the velocity is c, and

$$v_i v^i=0. \quad (4.10)$$

This is about as far as one can go with the pole-dipole equations in flat space. There are, however, certain expected features which have not been demonstrated, namely that $p_i p^i=0$ and that p^i is parallel to v^i. These can only be shown by an appeal to boundedness. That is, if S^{ij} gets large without limit, the very premise of the whole calculation, namely that a pole-dipole expansion makes sense (i.e., converges), breaks down. Thus we shall consider as consistent with the assumptions of the theory only those solutions which do not blow up. If Eqs. (4.8) and (4.9) are substituted into Eqs. (4.2b) and (4.2c), then unless $A=C=0$, the spin components S^{0y} and S^{0z} will increase without limit as time goes on. Thus we require that $A=C=0$. But this means that $v^y=v^z=0$. Since v^i has already been shown to be null, this means that

$$v^i=(v'^0, v'^x, 0, 0), \quad v'^0=v'^x. \quad (4.11)$$

Equation (4.3) then shows that $p'^0=p'^x$. Thus $p^i = \text{const}\times v^i$, which is what we wished to show.

To finish this section, we shall say a few words about helicity. In texts on special relativity,[19] it is shown that the polarization four-vector

$$w^i=\tfrac{1}{2}\epsilon_{ijkm}S^{jk}p^m \quad (4.12)$$

for massless particles satisfies

$$w_i=\lambda p_i. \quad (4.13)$$

Here ϵ_{ijkm} is the alternating symbol times $(-g)^{1/2}$, and λ is a constant, the helicity.

The space parts $S^{\alpha\beta}$ of the spin tensor can be found from inserting Eq. (4.13) into (4.12):

$$p^0 S^{\alpha\beta}=\lambda\epsilon^{0\alpha\beta\gamma}p_\gamma - S^{\beta 0}p^\alpha + S^{\alpha 0}p^\beta. \quad (4.14)$$

Take an absolute derivative of this equation and subtract it from

$$DS^{\alpha\beta}/Dq=(p^\beta/p^0)DS^{\alpha 0}/Dq-(p^\alpha/p^0)DS^{\beta 0}/Dq, \quad (4.15)$$

which is obtained by substituting Eq. (2.11) into the space parts of (3.2). The result is

$$S^{\alpha\beta}Dp^0/Dq=\lambda\epsilon^{0\alpha\beta\gamma}Dp_\gamma/Dq - S^{\beta 0}Dp^\alpha/Dq + S^{\alpha 0}Dp^\beta/Dq. \quad (4.16)$$

If the helicity equations are valid in general curved spaces, then Eq. (4.16) should also be valid. Although we cannot prove that this equation is satisfied generally, it is easy to see that in flat space each term in Eq. (4.16) is zero, and the equation is satisfied. In fact, Eq. (4.13) is an integral of Eq. (4.15) in flat space.

V. POLE APPROXIMATION

The pole approximation is defined to be the one in which $S^{ij}=0$. We shall show in this section what the consequences are.

From Eqs. (3.1) and (3.2), we get

$$Dp^i/Dq=0, \quad (5.1)$$

$$p^i=(p^0/v^0)v^i \equiv M_0 v^i, \quad (5.2)$$

where M_0 is a scalar (since p^i and v^i are vectors), and from Eq. (2.2) can be written out as

$$M_0 = p^0/v^0 = (dx^0/dq)^{-1} \int T^{00}(-g)^{1/2} d\Sigma . \quad (5.3)$$

Substitution of (5.2) into (5.1) gives

$$v^i dM_0/dq + M_0 Dv^i/Dq = 0. \quad (5.4)$$

For material particles, Eq. (5.4) is enough to conclude that M_0 is a constant: Multiply (5.4) by v_i and sum. Since $v_i v^i$ is not zero, M_0 must be constant along the curve. For massless particles, however, $v_i v^i$ may be zero, and this argument breaks down.

However, we can accomplish the same end by utilizing the arbitrariness of q: Choose q to be such that dM_0/dq is zero. If the original q does not do this, transform to another q, call it q' in terms of which from (5.3)

$$M_0' = (dx^0/dq') \int T^{00}(-g)^{1/2} d\Sigma$$
$$= M_0(q) dq/dq' .$$

Here M_0' is the M_0 function when the new q' is used. Choose now q' to satisfy $M_0(q)dq/dq' = \text{const}$, i.e., $q' = \text{const} \times \int M_0(q) dq$.

With this result (using the symbol q again), Eq. (5.4) becomes

$$dM_0/dq = 0, \quad (5.5a)$$

$$Dv^i/Dq = 0. \quad (5.5b)$$

Further, use of (5.2) in (3.8) gives

$$v_i p^i = 0, \quad (5.6a)$$

$$v_i v^i = 0, \quad (5.6b)$$

$$p_i p^i = 0. \quad (5.6c)$$

Thus the path is a null geodesic.

To complete this section we shall derive the red-shift for massless particles in a time-orthogonal system of coordinates ($g_{0\alpha} = 0$, $\alpha = 1, 2, 3$). From Eqs. (5.5) and (5.6) we can show that q must be related to x^0 by

$$dx^0/dq = (g_{00})^{-1}. \quad (5.7)$$

The reason is that the $i = 0$ component of $Dv^i/Dq = 0$, Eq. (5.5), reduces in a time-orthogonal system to

$$d(g_{00} dx^0/dq)/dq = 0. \quad (5.8)$$

But from $v_i v^i = 0$, Eq. (5.6),

$$g_{00}(dx^0/d\sigma)^2 = 1, \quad (5.9)$$

where σ is defined from $ds^2 = g_{00}c^2 dt^2 - d\sigma^2$. The derivative of the square root of Eq. (5.9) is zero, a result compatible with Eq. (5.8) only if Eq. (5.7) is valid, using $dx^0 dq = (dx^0/d\sigma)(d\sigma/dq)$, and Eq. (5.9). An arbitrary constant has been absorbed into q.

Equation (5.7) used in Eq. (5.5) with (5.3) gives

$$dM_0/dq = 0 = (d/dq)\left[g_{00} \int T^{00}(-g)^{1/2} d\Sigma\right]. \quad (5.10)$$

We wish to show that this is a statement of the conservation of energy and contains the red-shift. If we write $ds^2 = g_{00}c^2dt^2 + \gamma_{\mu\nu}dx^\mu dx^\nu$, then $g = g_{00}\gamma$, where γ is the determinant of the $\gamma_{\mu\nu}$. Within the pole approximation, g_{00} can be brought in and out of the integral sign. Then Eq. (5.10) can be written

$$(d/dq)\left[(g_{00})^{1/2} \int T_0{}^0(-\gamma)^{1/2} d\Sigma\right] = 0, \quad (5.11)$$

where $T^{00} = (g_{00})^{-1} T_0{}^0$.

The conservation of energy for extended material particles and for an electromagnetic system is computed from the energy density[20]

$$e = (g_{00})^{1/2} T_0{}^0. \quad (5.12)$$

Thus the total energy is

$$\int e(-\gamma)^{1/2} d\Sigma = \int T^0{}_0(-g)^{1/2} d\Sigma$$
$$= (g_{00})^{1/2} \int T^0{}_0(-\gamma)^{1/2} d\Sigma . \quad (5.13)$$

But this is just what appears in the square brackets of Eq. (5.11), and is the M_0 from Eq. (5.3). Thus Eq. (5.10), $dM_0/dq = 0$, is a statement of the conservation of energy.

It is well known that the presence of the factor $(g_{00})^{1/2}$ generalizes an energy in the absence of a gravitational field to take into account the interaction with the field. Thus if $g_{00} = 1 + 2V/c^2$, where V is the gravitational potential,

$$(g_{00})^{1/2} mc^2 = mc^2(1 + V/c^2 + \cdots)$$
$$= mc^2 + mV + \cdots . \quad (5.14)$$

The second term is the usual interaction with the gravitational field. The first term is the isolated particle energy.

Thus in Eq. (5.13) the integral may be interpreted as the isolated particle energy, which if we identify with a frequency ν we can call $h\nu$. Thus

$$(d/dq)(g_{00})^{1/2} h\nu = 0. \quad (5.15)$$

Thus the particle frequency varies as $(g_{00})^{-1/2}$. This is the gravitational red-shift.

In this way we recover a number of familiar results in the case where the pole-dipole particle reduces to just a pole particle. It should be noted that Papapetrou[2] also considered the pole-particle limit. In his case M_0 corresponds to the rest mass of the particle. Although the rest mass of the massless particle is zero in the sense that $p_i v^i = 0$, nevertheless the quantity M_0 that appears in Eq. (5.2) has been made constant along the path, and denotes the constant of proportionality between p^i and v^i. So in many respects it plays the role of a rest mass.

Notice that the trajectories are null geodesics for massless particles of *all* frequencies in the pole approximation. Under this approximation then the geodesic hypothesis is strictly valid. Finally, Eq. (4.16) is satisfied.

VI. THE EIKONAL APPROXIMATION

The eikonal approximation is a perturbation expansion based on the smallness of the inverse frequency; that is, it is the high-frequency limit. High frequency refers to the propagating object. The medium in which it travels varies relatively slowly.

There is nothing, however, in the pole-dipole formalism that refers specifically to a frequency. In order to apply this approximation, therefore, we must go beyond the formalism and associate the frequency with the momentum four-vector p^i, as is customarily done. Thus we look to a perturbation expansion based on p^i being very large.

Equation (3.1) then gives as the zeroth-order equation

$$Dp^i/Dq = 0 \tag{6.1}$$

and Eq. (3.2) gives as its zeroth-order equation

$$p^i v^j - p^j v^i = 0 \tag{6.2a}$$

for which the general solution is

$$p^i = M v^i, \tag{6.2b}$$

where M is a scalar that can be identified from the equation with p^0/v^0.

Equation (3.2) then gives as its first-order equation

$$DS^{ij}/Dq = 0. \tag{6.3}$$

Equations (6.1)–(6.3) constitute the eikonal limit of the pole-dipole equations. Equations (6.1) and (6.2) are the same as in the pole approximation, and the argument of Sec. V can be applied here, leading again to conservation of momentum, null geodesics, the red-shift, and the validity of Eq. (4.16). Equation (6.3) shows that the spin tensor is parallel propagated in this limit.

VII. SUMMARY

In this paper we have applied the moment expansion of the energy-momentum tensor to the case of massless particles propagating in a gravitational field. Such moments exist since the energy-momentum tensor itself exists for such particles. Further, most of the results of the work of Papapetrou[2] can be maintained.[10] The only additional feature for massless particles is that $T^i{}_i = 0$. But it is just this equation (or rather its moments) that supplies the needed auxiliary conditions for the pole-dipole approximation.

These auxiliary conditions are first of all Eq. (3.8), $p_i v^i = 0$, which replaces $p_i p^i = 0$ as the defining equation for a massless particle in a gravitational field, and is the same equation found by Jauch and Watson for a photon in a dielectric medium.[11] The second auxiliary condition is Eq. (3.9), $v_i S^{ij} = 0$, which brings back Mathisson's original equation.[1]

The basic equations governing the motion are Eqs. (3.1), (3.2), (3.8), and (3.9). They seem at first glance to allow the possibility of nongeodesic trajectories (just as the corresponding equations do for material particles), but we have not yet located a special case in which that has been shown to be a rigorous consequence.

However, in the special cases of flat space (Sec. IV), or in which the spin is neglected (the "pole" approximation, Sec. V), or in which the frequency becomes large (the eikonal approximation, Sec. VI), it was possible to show that null geodesics are the trajectories, and that the usual helicity relations occur. Thus in these cases at least the geodesic hypothesis is a consequence of the pole-dipole equations.

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APPENDIX A: TRANSFORMATION PROPERTIES OF THE MOMENTS OF THE TRACE $T^i{}_i$

In this and the following appendix, we rely on the paper of Papapetrou,[2] and shall adopt his notation except for the following. Latin indices are used for 0, 1, 2, 3. Papapetrou's velocity $u^i = dx^i/ds$ is replaced by our $v^i = dx^i/dq$, and all derivatives are taken with respect to q, not s. In the equations we use, this replacement is justified. (For example, dx^i/dt can be written as v^i/v^0 just as well as by u^i/u^0 in general. For light in say flat space, it is essential since $ds = 0$.) Finally we shall use i, j, k, l to indicate the unprimed system, and $a, b, c, d, \ldots$ to represent a primed system of coordinates, when two coordinate systems appear in the same equation.

Papapetrou's paper uses quantities defined as

$$M^{ik}=v^0\int T^{ik}(-g)^{1/2}d\Sigma, \qquad (A1)$$

$$M^{jik}=-v^0\int \delta x^j T^{ik}(-g)^{1/2}d\Sigma, \qquad (A2)$$

where T^{ij} is the energy-momentum tensor and δx^i is the distance between the central point X^i and the arbitrary point x^i in the particle. These M quantities are not tensors.

We shall be interested on the other hand in the quantities

$$M^i{}_k=v^0\int T^i{}_k(-g)^{1/2}d\Sigma, \qquad (A3)$$

$$M^{ji}{}_k=-v^0\int \delta x^j T^i{}_k(-g)^{1/2}d\Sigma, \qquad (A4)$$

since from these we can form the trace $T^i{}_i$ which is zero in the case of electromagnetism:

$$M^i{}_i=0, \qquad (A5)$$

$$M^{ji}{}_i=0. \qquad (A6)$$

In Appendix B we shall show that Eqs. (A5) and (A6) lead to the auxiliary conditions of Eqs. (3.3) and (3.4) of the paper.

In this appendix, we wish to show that if $M^i{}_i$ and $M^{ki}{}_i$ are zero in one system of coordinates, then they are zero in all systems. Thus even though they are not tensors, they have this invariant property.

To do this, we relate the $M^i{}_k$'s to the M^{ik}'s by writing $T^i{}_k=g_{kj}T^{ij}$ in Eqs. (A3) and (A4) and expanding the g_{ij}'s:

$$g_{ik}(x)=g_{ik}(X)+g_{ik,n}(X)\delta x^n$$
$$=g_{ik}(X)+(\Gamma_{k,in}+\Gamma_{i,kn})\delta x^n. \qquad (A7)$$

Then from Eqs. (A3) and (A4) we get

$$M^{ji}{}_k=g_{kn}M^{jin}, \qquad (A8)$$

$$M^i{}_k=g_{kn}M^{in}-(\Gamma_{n,km}+\Gamma_{k,nm})M^{min} \qquad (A9)$$

to within the pole-dipole approximation.

The transformation properties of the objects in Eqs. (A8) and (A9) are obtained from Papapetrou's Eqs. (4.3) and (4.4) which we rewrite here as

$$M^{jik}=L^i{}_aL^k{}_b(L^j{}_c-v^jL^0{}_c/v^0)M'^{cab}, \qquad (A10)$$

$$M^{ik}=L^i{}_aL^k{}_bM'^{ab}-(L^i{}_{ac}L^k{}_b+L^i{}_aL^k{}_{bc})M'^{cab}$$
$$+(d/dq)(L^i{}_aL^k{}_bL^0{}_cM'^{cab}/v^0), \qquad (A11)$$

where the transformation is between the primed and unprimed systems, and where

$$L^i{}_a=\partial x^i/\partial x'^a,$$
$$L^i{}_{ab}=\partial^2x^i/\partial x'^a\partial x'^b=L^i{}_{ba}. \qquad (A12)$$

Using

$$g'_{ab}=g_{kn}L^k{}_aL^n{}_b, \qquad (A13)$$

we see immediately from Eqs. (A10) and (A8) that

$$M^{jk}{}_k=(L^j{}_c-v^jL^0{}_c/v^0)M'^{ca}{}_a. \qquad (A14)$$

Thus if $M'^{ca}{}_a$ is zero for all c in the primed system, then $M^{jk}{}_k$ is zero for all j in any other system.

For $M^i{}_i$, insert Eq. (A11) into Eq. (A9) after setting $k=i$ and summing. Using Eq. (A13) and the usual relation $d(fg)=fdg+gdf$, we reduce Eq. (A11), and use both it and Eq. (A10) in Eq. (A9). After some manipulation we find

$$M^i{}_i=M'^a{}_a+(d/dq)(L^0{}_cM'^{ca}{}_a/v^0)$$
$$-[(\partial g'_{ab}/\partial x'^c)-\Gamma'_{a,bc}-\Gamma'_{b,ac}]M'^{cab}. \qquad (A15)$$

The square bracket is Dg'_{ab}/Dx'^c which is zero. From Eq. (A15) we see that if $M'^a{}_a$ and $M'^{ca}{}_a$ are zero in the primed system, then $M^i{}_i$ is zero in any other system. This completes what we wished to show: If $M^i{}_i$ and $M^{ki}{}_i$ are zero in one system of coordinates, then the corresponding quantities are zero in any system of coordinates.

APPENDIX B: DERIVATION OF THE AUXILIARY CONDITIONS

In this appendix, we use Eqs. (A5) and (A6) to derive the auxiliary conditions (3.3) and (3.4) that help define the pole-dipole model. To do this, we use Papapetrou's Eqs. (3.8) and (3.10), which we rewrite as

$$2M^{ijk}=-(S^{ij}v^k+S^{ik}v^j)+(v^i/v^0)(S^{0j}v^k+S^{0k}v^j), \qquad (B1)$$

$$M^{ik}=v^iv^k(v^0)^{-2}M^{00}-(v^i/v^0)\Gamma^0{}_{mn}M^{kmn}-(v^i/v^0)d(M^{k00}/v^0)/dq-d(M^{ik0}/v^0)/dq-\Gamma^k{}_{mn}M^{imn}. \qquad (B2)$$

Using (B1) in (B2) and symmetrizing M^{ik}, we find after some calculation

$$M^{ik}=v^iv^k(v^0)^{-2}(M^{00}+\Gamma^0{}_{mn}v^nS^{m0})+\tfrac{1}{2}(v^i/v^0)dS^{k0}/dq+\tfrac{1}{2}(v^k/v^0)dS^{i0}/dq+\Gamma^0{}_{mn}(v^n/v^0)\tfrac{1}{2}(v^iS^{km}+v^kS^{im})$$
$$+\tfrac{1}{2}(d/ds)(v^kS^{i0}/v^0+v^iS^{k0}/v^0)+\tfrac{1}{2}(\Gamma^k{}_{mn}S^{im}+\Gamma^i{}_{mn}S^{km})v^n$$
$$+\tfrac{1}{2}(\Gamma^k{}_{mn}v^i+\Gamma^i{}_{mn}v^k)v^nS^{m0}/v^0. \qquad (B3)$$

Now from Eq. (A6), using Eqs. (A8) and (B1),

$$M^{ik}{}_{k}=0=-S^{ik}v_k+v^i v_k S^{0k}/v^0. \tag{B4}$$

This is Eq. (3.4) of the text. Further discussion of it can be found there.

The computation starting from Eq. (A5) is rather tedious. Equation (B3) is reduced by noticing from Eq. (2.1) that the first term is

$$v^i v^k p^0/v^0=\tfrac{1}{2}[v^i(p^k-(v^0)^{-1}DS^{k0}/Dq)+v^k(p^i-(v^0)^{-1}DS^{i0}/Dq)]. \tag{B5}$$

The last form appears because of Eq. (2.11) of Sec. II.

Using

$$DS^{i0}/Dq=dS^{i0}/dq+\Gamma^i{}_{mn}S^{m0}v^n+\Gamma^0{}_{mn}S^{im}v^n, \tag{B6}$$

we find from Eq. (B3)

$$M^{ik}=\tfrac{1}{2}(v^ip^k+v^kp^i)+\tfrac{1}{2}(d/dq)(v^kS^{i0}/v^0+v^iS^{k0}/v^0)+\tfrac{1}{2}v^n(\Gamma^k{}_{mn}S^{im}+\Gamma^i{}_{mn}S^{km}). \tag{B7}$$

Rewriting covariant derivatives on the v^kS^{i0} factors, using Eq. (B6), we obtain

$$\begin{aligned}M^{ik}=&\tfrac{1}{2}(v^ip^k+v^kp^i)-\tfrac{1}{2}(v^0)^{-2}(dv^0/dq)(v^kS^{i0}+v^iS^{k0})+\tfrac{1}{2}(v^0)^{-1}(D/Dq)(v^kS^{i0}+v^iS^{k0})\\&-\tfrac{1}{2}(v^0)^{-1}[(\Gamma^i{}_{mn}v^k+\Gamma^k{}_{mn}v^i)S^{m0}v^n+(\Gamma^k{}_{mn}S^{i0}+\Gamma^i{}_{mn}S^{k0})v^mv^n\\&+\Gamma^0{}_{mn}v^n(v^kS^{im}+v^iS^{km})]+\Gamma^k{}_{mn}S^{im}v^n+\Gamma^i{}_{mn}S^{km}v^n.\end{aligned} \tag{B8}$$

Now we go back to Eq. (A5), insert into it Eq. (A9), and then into that result Eq. (B8), where in Eq. (A9), Eq. (B1) is used. After considerable rearrangement we find

$$M^i{}_i=0=v^ip_i-(v^0)^{-2}(dv^0/dq)b^0+(v^0)^{-1}Db^0/Dq, \tag{B9}$$

where b^0 is the zeroth component of $b^i=v_kS^{ki}$. By defining $a=b^0/v^0$, we can convert this result to

$$v^ip_i+da/dq=0. \tag{B10}$$

Equation (B10) is Eq. (3.3) of the text.

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[4]W. Dixon, Gen. Relativ. Gravit. 4, 199 (1973); Proc. R. Soc. London A314, 499 (1970) and references contained therein.

[5]Dixon defines a surface formed by all the geodesics perpendicular to a timelike velocity that is parallel to the momentum four-vector. He remarks (on page 512 of the last of Ref. 4) that the velocity must not be too close to the null cone, otherwise the surface may not be everywhere spacelike. It is possible that a reworking of Dixon's definitions could lead to invariant definitions even when the velocity lies in a null direction. However, we have not attempted to do this, but have proceeded with the simpler, if not so rigorous or elegant treatment of Papapetrou.

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[10]In Ref. 2, replace everywhere $u^i=dx^i/ds$ by $v^i=dx^i/dq$. There is only one place in the argument where this cannot be done, namely, in proceeding from Eq. (5.2) to (5.3), since to do that, the relation $u^iu_i=1$ is required. To get our Eq. (2.10), substitute the equation below (5.3) of Ref. 2 into (5.2) of Ref. 2, using the definition in our Eq. (2.1), i.e., $p^i=(v^0)^{-1}(M^{00}+\Gamma^0{}_{mn}S^{m0}v^n)$. To get our Eq. (2.9), rewrite (3.13) of Ref. 2 in terms of p^i and S^{ij}, using our Eq. (2.10), which has just been established.

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[15]C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 22.5.

[16]E.g., Ref. 15, p. 388.

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