Scalar-metric and scalar-metric-torsion gravitational theories

S. J. Aldersley
Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
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The techniques of dimensional analysis and of the theory of tensorial concomitants are employed to study field equations in gravitational theories which incorporate scalar fields of the Brans-Dicke type. Within the context of scalar-metric gravitational theories, a uniqueness theorem for the geometric (or gravitational) part of the field equations is proven and a Lagrangian is determined which is uniquely specified by dimensional analysis. Within the context of scalar-metric-torsion gravitational theories a uniqueness theorem for field Lagrangians is presented and the corresponding Euler-Lagrange equations are given. Finally, an example of a scalar-metric-torsion theory is presented which is similar in many respects to the Brans-Dicke theory and the Einstein-Cartan theory.

I. INTRODUCTION

In previous papers[1,2] the author has employed the technique of dimensional analysis to prove uniqueness-type theorems for the field equations of metric and metric-torsion gravitational theories. In the present paper, these theorems will be extended by the introduction of a scalar field of the Brans-Dicke type. We shall begin with an examination of such scalar fields within the context of conventional scalar-metric theories and conclude with a generalization to theories of a scalar-metric-torsion type.

In the scalar theories we are considering, the reciprocal of the scalar field $\tilde{\phi}^{-1}$ is interpreted as representing a varying gravitational constant, i.e., a function with dimensions of length/mass (in units with $c=1$) which can be interpreted as representing a proportionality between mass and length in the weak-field limit. If K denotes the usual gravitational constant, then one can take the dimensionless scalar field $\phi \equiv K\tilde{\phi}$ as the scalar field. Furthermore, by choosing our units of mass so that $K=1$, we can and will measure any quantity which has units involving length, mass, or time in *length units*.

With the units chosen in the above manner, the technique of dimensional analysis is readily developed. The remaining freedom of a particular unit of length in terms of which physical measurements are to be made is called the choice of *scale* which is denoted by the symbol L. It can be shown that no generality is lost if we confine our attention to those charts (U,x) on the spacetime manifold in which the local coordinates $x^i \sim L^1$ (read "x^i has dimensions L to the power 1") and the metric components $g_{ij} \sim L^0$. The choice of scale may be changed without altering $c=K=1$ by multiplying the scale by a positive real factor (and thereby inducing a transformation on dimensioned quantities). The usefulness of dimensional analysis originates in the postulate that physical formulas must be invariant under such *scale transformations*. In particular, we have the following axiom for dimensional analysis in relativistic gravitational theories[1]:

Axiom. If the quantities $Q_1, Q_2, \ldots, Q_n$ *have dimensions* $L^{\alpha_1}, L^{\alpha_2}, \ldots, L^{\alpha_n}$, *respectively, and if they enter into a physical theory in a function of the form*

$$\Phi(Q_1, \ldots, Q_n)$$

which is such that $\Phi \sim L^{\alpha}$, *then under an arbitrary scale transformation with scale factor* λ *the relationship*

$$\lambda^{\alpha}\Phi(Q_1, \ldots, Q_n) = \Phi(\lambda^{\alpha_1}Q_1, \ldots, \lambda^{\alpha_n}Q_n)$$

holds.

In the function Φ, some of the quantities Q_i could be constants with dimension L^{α}, $\alpha \neq 0$. However, in our applications we shall consider as Q_i's only field functions (e.g., g_{ij}) and their derivatives, excluding such constants with dimension. One should note that in the above axiom we have assumed that the functional form of Φ does not depend upon the choice of scale.

II. SCALAR-METRIC THEORIES

Our main concern here is the field equations of scalar-metric theories[3] and our approach to these equations will be via variational principles. We assume that the "gravitational part" of the field equations are derivable from a Lagrangian $\mathcal{L}$ of the form

$$\mathcal{L} = \mathcal{L}(\phi; \phi_{,i}; \ldots; \phi_{,i_1 \cdots i_\alpha}; g_{ij}; g_{ij,k}; \cdots; g_{ij,k_1 \cdots k_\beta})$$

and that the field equations take the form

$$A^{ij} = 8\pi\sqrt{g}\, T^{ij} \quad (1)$$

and

$$B = 0, \quad (2)$$

where[4]

$$A^{ij} \equiv \frac{\delta \mathfrak{L}}{\delta g_{ij}}, \tag{3}$$

$$B \equiv \frac{\delta \mathfrak{L}}{\delta \phi}, \tag{4}$$

and T^{ij} is the usual energy-momentum tensor for matter and other nongravitational fields of general relativity.

The energy-momentum tensor has dimensions of L^{-2} provided the conventional interpretation is assumed.[1] Thus for dimensional consistency in (1) we require that

$$A^{ij} \sim L^{-2}.$$

It can be shown that[3,5] $A^b_{a|b} = \frac{1}{2}\phi_{,a}B$, from which we deduce that $B \sim L^{-2}$ upon noting $\phi \sim L^0$ (and $g_{ij} \sim L^0$). With the dimensions determined as above, we have the following:

Theorem I. Let A^{ij} be a class-C^3 tensor density concomitant of the form

$$A^{ij} = A^{ij}(\phi;\phi_{,i};\ldots;\phi_{,i_1\cdots i_\alpha};g_{ij};g_{ij,k};\ldots;g_{ij,k_1\cdots k_\beta})$$

and let B be a class-C^3 scalar density concomitant of the form

$$B = B(\phi;\phi_{,i};\ldots;\phi_{,i_1\cdots i_\delta};g_{ij};g_{ij,k};\ldots;g_{ij,k_1\cdots k_\sigma}).$$

Suppose that $A^{ij} \sim L^{-2}$, $B \sim L^{-2}$, and they both satisfy the axiom of dimensional analysis (here we have $g_{ij} \sim L^0$ and $\phi \sim L^0$). If there exists a Lagrangian $\mathfrak{L}$ for which A^{ij} and B are given by (3) and (4), respectively, then in a 4-space[6]

$$A^{ij} = \sqrt{g}\,[(h'' - f)\phi^{|i}\phi^{|j} - (h'' - \tfrac{1}{2}f)g^{ij}\phi_{|r}\phi^{|r} + h'(\phi^{|ij} - g^{ij}\phi^{|r}{}_{|r}) - hG^{ij}] \tag{5}$$

and

$$B = \sqrt{g}\,(h'R - f'\phi_{|r}\phi^{|r} - 2f\phi^{|r}{}_{|r}), \tag{6}$$

where $f = f(\phi)$ and $h = h(\phi)$ are arbitrary unitless functions, and prime denotes a derivative with respect to ϕ. Moreover, a Lagrangian which yields (5) and (6) as its Euler-Lagrange expressions is given by

$$\mathfrak{L} = \sqrt{g}\,(f\phi_{|i}\phi^{|i} + hR). \tag{7}$$

Proof. The axiom of dimensional analysis implies that $\forall \lambda > 0$,

$$\lambda^2 A^{ab}(\phi;\phi_{,i};\ldots;\phi_{,i_1\cdots i_\alpha};g_{ij};g_{ij,k};\ldots;g_{ij,k_1\cdots k_\beta}) = A^{ab}(\phi;\lambda\phi_{,i};\ldots;\lambda^\alpha\phi_{,i_1\cdots i_\alpha};g_{ij};\lambda g_{ij,k};\ldots;\lambda^\beta g_{ij,k_1\cdots k_\beta}),$$

and

$$\lambda^2 B(\phi;\phi_{,i};\ldots;\phi_{,i_1\cdots i_\delta};g_{ij};g_{ij,k};\ldots;g_{ij,k_1\cdots k_\sigma}) = B(\phi;\lambda\phi_{,i};\ldots;\lambda^\delta\phi_{,i_1\cdots i_\delta};g_{ij};\lambda g_{ij,k};\ldots;\lambda^\sigma g_{ij,k_1\cdots k_\sigma}).$$

Differentiating twice with respect to λ in both of the above identities and taking the limits as $\lambda \to 0^+$ we find that

$$A^{ab} = \Psi_1^{abij}\phi_{,i}\phi_{,j} + \Psi_2^{abij}\phi_{,ij} + \Psi_1^{abijkl}\phi_{,i}g_{kj,l} + \Psi^{abijklmn}g_{ij,k}g_{lm,n} + \Psi_2^{abijkl}g_{ij,kl}$$

and

$$B = \Phi_1^{ij}\phi_{,i}\phi_{,j} + \Phi_2^{ij}\phi_{,ij} + \Phi_1^{ijkl}\phi_{,i}g_{jk,l} + \Phi^{ijklmn}g_{ij,k}g_{lm,n} + \Phi_2^{ijkl}g_{ij,kl},$$

where $\Psi^{\because\because}$ and $\Phi^{\because\because}$ have the obvious symmetries and are concomitants of only g_{ij} and ϕ(i.e., they are independent of the partial derivatives of g_{ij} and ϕ). Using the replacement theorems of classical tensor calculus (see Ref. 7, p. 109) we reduce the above two equations to

$$A^{ab} = \Psi_1^{abij}\phi_{|i}\phi_{|j} + \Psi_2^{abij}\phi_{|ij} + \tfrac{2}{3}\Psi_2^{abijkl}R_{ljki}, \tag{8}$$

$$B = \Phi_1^{ij}\phi_{|i}\phi_{|j} + \Phi_2^{ij}\phi_{|ij} + \tfrac{2}{3}\Phi_2^{ijkl}R_{ljki}. \tag{9}$$

The coefficients $\Psi^{\because\because}$ and $\Phi^{\because\because}$ in (8) and (9) are easily seen to be tensor density concomitants of g_{ij} and ϕ.

Following Anderson[8] one can show that

$$\frac{\partial A^{ab}}{\partial \phi_{,ij}} = \frac{\partial B}{\partial g_{ab,ij}}, \quad \frac{\partial A^{ab}}{\partial g_{rs,tu}} = \frac{\partial A^{rs}}{\partial g_{ab,tu}}, \quad \frac{\partial A^{ab}}{\partial \phi_{,i}} = -\frac{\partial B}{\partial g_{ab,i}} + 2\left(\frac{\partial B}{\partial g_{ab,ij}}\right)_{,j}$$

which, when coupled with (8), (9), and the invariance identities for A^{ij} and B (viz., $\partial A^{ab}/\partial g_{i(j,kl)} = 0$ and $\partial B/\partial g_{i(j,kl)} = 0$; see Ref. 5), imply that

$$\Psi_2^{abij} = \Phi_2^{abij}, \tag{10}$$

$$2\Psi_1^{abik} = g^{k(a}\Phi_2^{b)i} - \tfrac{1}{2}g^{ki}\Phi_2^{ab} + \frac{2\partial\Phi_2^{abik}}{\partial\phi}\,, \tag{11}$$

$$\Psi_2^{abi(jkl)} = 0\,, \quad \Psi_2^{a(b|ij|kl)} = 0\,, \tag{12}$$

and

$$\Phi_2^{i(jkl)} = 0. \tag{13}$$

From (12), (13), and the invariance identities for tensorial concomitants of g_{ij} and ϕ we obtain (details of this and other computations can be found in Ref. 9)

$$\Phi_1^{ij} = a\sqrt{g}\, g^{ij}, \quad \Phi_2^{ij} = b\sqrt{g}\, g^{ij}, \quad \Phi_2^{abij} = c\sqrt{g}\,[g^{ab}g^{ij} - \tfrac{1}{2}(g^{ai}g^{bj} + g^{aj}g^{bi})]\,, \tag{14}$$

and

$$\Psi_2^{abijkl} = d\sqrt{g}\,(4g^{ab}g^{ij}g^{kl} - 2g^{ab}g^{ik}g^{jl} - 2g^{ab}g^{il}g^{jk} - 2g^{ai}g^{bj}g^{kl} - 2g^{aj}g^{bi}g^{kl} - 2g^{ak}g^{ij}g^{bl} - 2g^{al}g^{ij}g^{bk}$$
$$+ g^{ai}g^{bk}g^{jl} + g^{ai}g^{bl}g^{jk} + g^{aj}g^{bk}g^{il} + g^{aj}g^{bl}g^{ik} + g^{ak}g^{bi}g^{jl} + g^{ak}g^{bj}g^{il} + g^{al}g^{bi}g^{jk} + g^{al}g^{bj}g^{ik}), \tag{15}$$

where a, b, c, and d are arbitrary functions of ϕ.[10] From (14) and (11) we see that

$$2\Psi_1^{abik} = (2c' - \tfrac{1}{2}b)\, g^{ab}g^{ik} - (c' - \tfrac{1}{2}b)(g^{ai}g^{bk} + g^{ak}g^{bi}). \tag{16}$$

Substituting from (14), (15), (16), noting (10) into (8) and (9), we find that

$$A^{ab} = \sqrt{g}\,[\tfrac{1}{2}c'g^{ab}\phi_{|i}\phi^{|i} + \tfrac{1}{2}(c' - \tfrac{1}{2}b)(g^{ab}\phi_{|i}\phi^{|i} - 2\phi^{|a}\phi^{|b}) + c(g^{ab}\phi^{|i}{}_{|i} - \phi^{|ab}) + 8dG^{ab}] \tag{17}$$

and

$$B = \sqrt{g}\,(a\phi^{|i}\phi_{|i} + b\phi^{|i}{}_{|i} - cR). \tag{18}$$

From the identities $A^{ab}{}_{|b} = \tfrac{1}{2}\phi^{|a}B$ (see Ref. 3 or 5) and

$$\frac{\partial B}{\partial\phi_{,a}} = -\frac{\partial B}{\partial\phi_{,a}} + 2\left(\frac{\partial B}{\partial\phi_{,ab}}\right)_{,b}$$

(see Ref. 8) one can show that $c = 8d'$ and $a = \tfrac{1}{2}b'$. Setting $h = -8d$ and $-f = \tfrac{1}{2}b$ we can write (17) and (18) in the form (5) and (6), respectively. The Euler-Lagrange equations of (7) can be evaluated with the aid of Eqs. (3.32), (3.33), (3.34), and (3.35) of the paper by Horndeski and Lovelock (Ref. 3) (note that my definition of the Euler-Lagrange operator is the negative of theirs), and do in fact give (5) and (6).

Remark. When utilizing the technique of dimensional analysis it is essential that all dimensional dependences of the concomitants involved be stated explicitly, for otherwise one ignores such things as constants with dimension. For example, the specific functional form assumed for A^{ij} and B in theorem I implies that as far as scale and coordinate transformations are concerned A^{ij} and B depend only upon g_{ij}, ϕ, and their derivatives. For more on this physically important subject see Ref. 1.

Related to the results of theorem I is the following theorem which can be proved in a similar manner[9]:

Theorem II. If $\mathcal{L}$ is a class-C^3 scalar density concomitant of the form

$$\mathcal{L} = \mathcal{L}(\phi; \phi_{,i}; \ldots; \phi_{,i_1 \cdots i_\alpha}; g_{ij}; g_{ij,k}; \ldots; g_{ij,k_1 \cdots k_\beta}),$$

where $g_{ij} \sim L^0$, $\phi \sim L^0$, $\mathcal{L} \sim L^{-2}$ and satisfies the axiom of dimensional analysis, then in a 4-space

$$\mathcal{L} = \sqrt{g}\,[f_1(\phi)\phi_{|i}\phi^{|i} + f_2(\phi)R + f_3(\phi)\, g^{ij}\phi_{|ij}],$$

where f_1, f_2, and f_3 are arbitrary unitless functions of ϕ.

Remarks. (1) The restriction to 4-spaces is used to give the conventional interpretation of the energy-momentum tensor T^{ij} (see Ref. 1) and hence cannot be dropped from either of the theorems. Another assumption common to both theorems I and II is the differentiability condition on the concomitants. With this assumption, terms involving factors such as $(\phi_{,i}\phi_{,j}g^{ij})^{-1}$ are excluded from the above theorems. This places a restriction on the applications of those theorems to scalar-metric gravitational theories and must be considered when attempting to motivate the choice of field equations in such a theory.

(2) The Lagrangian $\mathcal{L}$ given by theorem II can be written in the form

$$\mathcal{L} = \sqrt{g}\,[\tilde{f}_1(\phi)\, g^{ij}\phi_{,i}\phi_{,j} + f_2(\phi)R] + [\sqrt{g}\, f_3(\phi)\, g^{ij}\phi_{,i}]_{,j}$$

and hence yields Euler-Lagrange expressions of the form (5) and (6). This result was to be expected in view of theorem I and the fact that the Euler-Lagrange expressions of a Lagrangian which satisfies the axiom of dimensional analysis will also satisfy this axiom. As a consequence of this latter point, we can take any Lagrangian $\mathcal{L}$ which yields (5) and (6) as Euler-Lagrange expressions and satisfies the hypotheses of theorem II and write $\tilde{\mathcal{L}}$ as

$$\tilde{\mathcal{L}} = \mathcal{L} + \mathcal{L}',$$

where $\mathcal{L}$ is given by (7) and the Euler-Lagrange expressions [(3) and (4)] of $\mathcal{L}'$ vanish identically (define $\mathcal{L}' \equiv \tilde{\mathcal{L}} - \mathcal{L}$). In this way, theorem II can effectively be deduced from theorem I.

III. SCALAR-METRIC-TORSION THEORIES

The torsion tensor in a metric-torsion gravitational theory[11] is that of a linear connection ∇ which preserves the metric tensor, i.e.,

$$\nabla g = 0.$$

From this equation one can deduce that the components of the linear connection are given by

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + S_{jk}{}^i - S_k{}^i{}_j + S^i{}_{jk}, \tag{19}$$

where

$$S_{jk}{}^i \equiv \tfrac{1}{2}(\Gamma_j{}^i{}_k - \Gamma_k{}^i{}_j).$$

The field equations in these theories are usually assumed to be the Euler-Lagrange equations of a suitably chosen Lagrangian of the form

$$\mathcal{L} + 2\mathcal{V},$$

where

$$\mathcal{V} = \mathcal{V}(\psi^\Omega; \psi^\Omega{}_{\|i}; g_{ij}),$$

the ψ^Ω are the matter fields, the bars denote covariant differentiation with respect to ∇, and $\mathcal{L}$ is a scalar density concomitant of the metric, torsion, and their derivatives.

By including the scalar field ϕ and its derivatives in $\mathcal{L}$ along with the corresponding Euler-Lagrange equations we obtain a scalar-metric-torsion theory. Here we shall consider scalar density Lagrangians and assume that our field equations are the Euler-Lagrange equations of such a Lagrangian. We take the Lagrangian $\mathcal{V}$ to be unchanged and regard the variational derivative

$$\sqrt{g}\, T^{ij} \equiv 2\frac{\delta \mathcal{V}}{\delta g_{ij}}$$

as the energy-momentum tensor in a manner analogous to its interpretation in general relativity. Since T^{ij} is usually assumed to have dimensions of L^{-2}, we must require $\mathcal{V}$ and hence $\mathcal{L}$ to have these units in order to preserve dimensional consistency.

Equation (19) implies that the torsion tensor $S_{ij}{}^k \sim L^{-1}$ and hence its derivatives have dimensions

$$S_{ij}{}^k{}_{,l_1 \cdots l_\gamma} \sim L^{-\gamma-1}.$$

The ideas in the above discussion lead us to consider the following theorem whose proof is similar to that of theorems I and II[9]:

Theorem III. If $\mathcal{L}$ is a class-C^3 scalar density concomitant of the scalar field ϕ, the metric g_{ij}, and the torsion $S_{ij}{}^k$ and their derivatives, and if $\mathcal{L} \sim L^{-2}$ and satisfies the axiom of dimensional analysis, then in a 4-space

$$\begin{aligned}\mathcal{L} = {}& a_1\sqrt{g}\, R + a_2\sqrt{g}\, g^{ij}\phi_{|i}\phi_{|j} + (a_3\sqrt{g}\, g^{ij}\phi_{|i})_{,j} + (a_4\sqrt{g}\, S^{kl}{}_k)_{,l} + (a_5\epsilon^{ijkl}S_{ijk})_{,l} \\ &+ a_6\sqrt{g}\, S_{kj}{}^k S_l{}^{jl} + a_7\sqrt{g}\, S_{ijk}S^{ijk} + a_8\sqrt{g}\, S_{ijk}S^{kij} + a_9 S_{ij}{}^k S_{lmk}\epsilon^{ijlm} \\ &+ a_{10}S_{ij}{}^k S_{kmn}\epsilon^{ijmn} + a_{11}\phi_{|i}S_{jkl}\epsilon^{ijkl} + a_{12}\sqrt{g}\, \phi_{|i}S^{il}{}_l,\end{aligned} \tag{20}$$

where a_δ, $\delta = 1, \ldots, 12$ are arbitrary unitless functions of ϕ. The Euler-Lagrange expressions of the Lagrangian $\mathcal{L}$ in (20) are

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta g_{ij}} = \sqrt{g}\Big[&(a_1'' - a_2)\phi^{|i}\phi^{|j} - (a_1'' - \tfrac{1}{2}a_2)g^{ij}\phi_{|r}\phi^{|r} + a_1'(\phi^{|ij} - g^{ij}\phi^{|r}{}_{|r}) - a_1 G^{ij} + \tfrac{1}{2}g^{ij}C^{rs}{}_t S_{rs}{}^t + C^{rsi}S_{rs}{}^j - 2C^{rj}{}_s S_r{}^{is} \\ &+ \tfrac{1}{2}a_{11}\frac{\epsilon^{abjk}}{\sqrt{g}}\phi_{|a}S_{bk}{}^i + \tfrac{1}{2}a_{11}\frac{\epsilon^{abki}}{\sqrt{g}}\phi_{|a}S_{bk}{}^j + \tfrac{1}{2}a_{12}(g^{ij}\phi_{|r}S^r{}_l{}^l - \phi^{|j}S^i{}_l{}^l - \phi^{|i}S^j{}_l{}^l)\Big],\end{aligned} \tag{21}$$

$$\frac{\delta\mathcal{L}}{\delta S_{lm}{}^k} = 2\sqrt{g}\left(C^{lm}{}_k + \tfrac{1}{2}a_{11}\phi_{|r}g_{ks}\frac{\epsilon^{lmrs}}{\sqrt{g}} + \tfrac{1}{2}a_{12}\phi^{|[l}\delta^{m]}_k\right), \tag{22}$$

where

$$C^{lm}{}_k \equiv a_6\delta^{[l}_k S_b{}^{m]b} + a_7 S^{lm}{}_k + a_8 S_k{}^{[lm]} + \frac{a_9}{\sqrt{g}}\epsilon^{lmbc}S_{bck} + \tfrac{1}{2}a_{10}\left(\frac{\epsilon^{lmab}}{\sqrt{g}}S_{kab} - \frac{\epsilon^{abc[m}}{\sqrt{g}}S_{ab}{}^{l]}g_{ck}\right), \tag{23}$$

and finally

$$\frac{\delta \mathfrak{L}}{\delta \phi} = \sqrt{g}\Big(a_1' R - a_2' \phi_{|i}\phi^{|i} - 2a_2 \phi^{|i}{}_{|i} + a_6' S_{kj}{}^{k} S_l{}^{jl} + a_7' S_{ijk} S^{ijk} + a_8' S_{ijk} S^{kij} + a_9' S_{ij}{}^{k} S_{lmk} \frac{\epsilon^{ijlm}}{\sqrt{g}} + a_{10}' S_{ij}{}^{k} S_{kmn} \frac{\epsilon^{ijmn}}{\sqrt{g}} + a_{11} \frac{\epsilon^{ijkl}}{\sqrt{g}} S_{ijk|l} - a_{12} S^{il}{}_{l|i}\Big). \tag{24}$$

The field equations one would obtain from Eqs. (21), (22), (23), and (24) will be quite unwieldy in general. Hence we consider a somewhat simplified example in order to determine what qualitative characteristics a scalar-metric-torsion theory could have. The "most natural" extension of the Brans-Dicke theory to one involving torsion would probably start with the Lagrangian[12]

$$\mathfrak{L} = \sqrt{g}\,\phi R(\nabla) + \frac{a\sqrt{g}}{\phi}\,\phi_{|i}\phi^{|i} \tag{25}$$

[one could also obtain this Lagrangian from the Einstein-Cartan Lagrangian $\sqrt{g}\,R(\nabla)$ by a "natural extension"]. If we decompose $R(\nabla)$ into the Ricci scalar of the metric g_{ij} and some torsion-dependent terms,[13] we would obtain the Lagrangian (20) with $a_1 = \phi$, $a_2 = a/\phi$, $a_4 = -4\phi$, $a_6 = -4\phi$, $a_7 = \phi$, $a_8 = -2\phi$, and $a_{12} = -4$. All other terms in (20) are then $a_3 = a_5 = a_7 = a_9 = a_{10} = a_{11} = 0$. Thus we can use (21), (22), (23), and (24) to obtain the field equations

$$G^{ij} - \frac{\phi^{|ij} - g^{ij}\phi^{|r}{}_{|r}}{\phi} + \frac{a\phi^{|i}\phi^{|j}}{\phi^2} - \frac{ag^{ij}}{2}\,\frac{\phi^{|r}\phi_{|r}}{\phi^2} + g^{ij}(4S_{rs}{}^{r}S^{ts}{}_{t} - S_{rst}S^{rst} + 2S_{rst}S^{trs}) - 4S_r{}^{ir}S_s{}^{js} - S^{lri}S_{lr}{}^{j} + 2S^{irl}S^{j}{}_{lr} + 2S^{ilr}S^{j}{}_{lr} + \frac{2}{\phi}(g^{ij}\phi_{|r}S^{rl}{}_{l} - \phi^{|i}S^{jl}{}_{l} - \phi^{|j}S^{il}{}_{l}) = \frac{1}{\phi}\,T^{ij}, \tag{26}$$

$$-S^{lm}{}_{k} + S_k{}^{lm} - S_k{}^{ml} + 2\delta^l_k S_b{}^{mb} - 2\delta^m_k S_b{}^{lb} + \frac{\phi^{|l}\delta^m_k - \phi^{|m}\delta^l_k}{\phi} = \frac{1}{\phi\sqrt{g}}\,\frac{\delta \mathfrak{V}}{\delta S_{lm}{}^{k}}, \tag{27}$$

and

$$R + \frac{a\phi_{|i}\phi^{|i}}{\phi^2} - \frac{2a\phi^{|i}{}_{|i}}{\phi} - 4S_{kj}{}^{k}S_l{}^{jl} + S_{ijk}S^{ijk} - 2S_{ijk}S^{kij} + 4S^{il}{}_{l|i} = 0. \tag{28}$$

In Eqs. (26), (27), and (28) we find a fairly complicated coupling between the torsion tensor and the derivative $\phi_{|i}$. Of particular significance is the $S^{il}{}_{l|i}$ term in Eq. (28). It seems to indicate that when the gravitational "constant" varies ($\phi_{|i} \neq 0$), then torsion will propagate "outside" of spinning matter. This is in sharp contrast with the Einstein-Cartan theory in which torsion enters only algebraically (see Ref. 11) and hence vanishes outside spinning matter. Furthermore, the Einstein Cartan theory reduces to general relativity outside spinning matter, hence we might expect that the Eqs. (26), (27), and (28) would imply the Brans-Dicke theory for such matter. However, this does not occur here. For if we impose the condition $\delta\mathfrak{V}/\delta S_{ij}{}^{k} = 0$ (i.e., no spin angular momentum) in (27) we can solve it to obtain

$$S_{ijk} = \frac{1}{4\phi}(\phi_{|j}g_{ik} - \phi_{|i}g_{jk}). \tag{29}$$

Now, $\delta\mathfrak{V}/\delta S_{ij}{}^{k} = 0$ implies that $\partial\mathfrak{V}/\partial S_{ij}{}^{k} = 0$ and hence $\mathfrak{V} = \mathfrak{V}(\Psi^{\Omega}; \Psi^{\Omega}{}_{|i}; g_{ij})$, which is the form of the matter Lagrangian in general relativity and the Brans-Dicke theory. Thus T^{ij} in (26) becomes the usual general-relativistic energy-momentum tensor. Substituting from (29) into (26) and (28) we find that

$$G^{ij} - \frac{1}{\phi}(\phi^{|ij} - g^{ij}\phi^{|r}{}_{|r}) + (a + \tfrac{3}{2})\frac{\phi^{|i}\phi^{|j}}{\phi^2} - \tfrac{1}{2}ag^{ij}\frac{\phi_{|r}\phi^{|r}}{\phi^2} = \frac{1}{\phi}\,T^{ij} \tag{30}$$

and

$$R + (a + \tfrac{3}{2})\left(\frac{\phi_{|r}\phi^{|r}}{\phi^2} - \frac{2\phi^{|r}{}_{|r}}{\phi}\right) = 0. \tag{31}$$

Taking g_{ij} times Eq. (30), defining $T \equiv T^i{}_i$, and using Eq. (31) we obtain the following equation for ϕ:

$$-2a\phi^{|i}{}_{|i} + 3\frac{\phi_{|r}\phi^{|r}}{\phi} = T. \tag{32}$$

Equations (30), (31), and (32) are quite similar to those which arise in the usual Brans-Dicke theory. However, torsion enters nontrivially, affecting Eq. (32) in particular, which is now a quasilinear second-order partial-differential equation for ϕ. If we take $a = -\omega$, the Dicke constant, we see that Eqs. (30) and (31) also differ from their counterparts in the Brans-Dicke theory by, essentially, modifications of the numerical coefficients of terms in $\phi^{|i}$ and $\phi^{|i}{}_{|i}$. Whether or not these apparently minor modifications will lead to any significant difference from the usual Brans-Dicke theory or from general relativity remains to be seen. The main point of (30), (31), and (32) is that torsion plays a significant role here even in vacuum, which contrasts markedly with its role in metric-torsion theories such as the Einstein-Cartan theory.

So far our discussion has centered upon mathematical generalizations of metric-torsion (or alternatively scalar-metric) gravitational theories. The question then arises as to what physical interpretation one should give to the apparent coupling between the torsion and scalar fields which occurs in the above generalizations. One possible interpretation is based upon the observation that for nonconstant ϕ, the derivative $\phi_{,i}$ defines a vector field on spacetime. Hence the interaction characterized in vacuum by Eq. (29) could be interpreted as the reaction of torsion to the spin angular momentum of the vector field $\phi_{,i}$.

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[1]S. J. Aldersley, Phys. Rev. D 15, 370 (1977).

[2]S. J. Aldersley, Gen. Relativ. Gravit. (to be published).

[3]For more information on scalar-metric theories see G. W. Horndeski and D. Lovelock, Tensor 4, 79 (1972) and the references contained therein.

[4]
$$\frac{\delta \mathcal{L}}{\delta g_{ij}} \equiv \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \frac{d^{\alpha}}{dx^{i_1} \cdots dx^{i_\alpha}} \frac{\partial \mathcal{L}}{\partial g_{ij,i_1 \cdots i_\alpha}}$$

the sum stops when α equals the order of $\mathcal{L}$ in derivatives of g_{ij}. Similarly

$$\frac{\delta \mathcal{L}}{\delta \phi} \equiv \sum_{\alpha=0}^{\infty} (-1)^{\alpha} \frac{d^{\alpha}}{dx^{i_1} \cdots dx^{i_\alpha}} \frac{\partial \mathcal{L}}{\partial \phi_{,i_1 \cdots i_\alpha}}$$

(summation over repeated Latin indices).

[5]G. W. Horndeski, Util. Math. 9, 3 (1976).

[6]The components of the curvature tensor, Ricci tensor, and Einstein tensor of the Levi-Civita connection (of g_{ij}) are given by

$$R_{ijk}{}^{l} \equiv \left\{ {l \atop j\ \ k} \right\}_{,i} - \left\{ {l \atop i\ \ k} \right\}_{,j} + \left\{ {r \atop j\ \ k} \right\} \left\{ {l \atop i\ \ r} \right\} - \left\{ {r \atop i\ \ k} \right\} \left\{ {l \atop j\ \ r} \right\},$$

$$R_{ij} \equiv R_{lij}{}^{l}, \text{ and } G_{ij} \equiv R_{ij} - \tfrac{1}{2} g_{ij} R,$$

respectively, where $R \equiv g^{ij} R_{ij}$ denotes the curvature scalar. Throughout this paper indices will be lowered and raised by means of g_{ij} and its matrix inverse g^{ij}. Round brackets on indices denote symmetrization and square brackets denote antisymmetrization of indices, e.g., $(ij) = \frac{1}{2}(ij + ji)$ and $[ij] = \frac{1}{2}(ij - ji)$.

[7]T. Y. Thomas, *The Differential Invariants of Generalized Spaces* (Cambridge Univ. Press, Cambridge, 1934).

[8]In I. M. Anderson, Aeq. Math. (to be published) (see in particular the remark made following Theorem 2 in this paper), the following is proven: Given a Lagrangian $\mathcal{L}$ of the form

$$\mathcal{L} = \mathcal{L}(\rho_A; \rho_{A,i}; \ldots; \rho_{A,i_1 \cdots i_\alpha}),$$

where ρ_A denotes a set of field functions, e.g., $\rho_A = (\phi, g_{ij})$, then

$$\sum_{p=r}^{\infty} (-1)^p \binom{p}{r} \frac{d^p}{dx^{i_{r+1}} \cdots dx^{i_p}} \frac{\partial}{\partial \rho_{A,i_1 \cdots i_p}} \left(\frac{\delta \mathcal{L}}{\delta \rho_B} \right) = \frac{\partial}{\partial \rho_{B,i_1 \cdots i_r}} \left(\frac{\delta \mathcal{L}}{\delta \rho_A} \right)$$

for $r = 0, 1, 2, \ldots,$ where

$$\frac{\delta \mathcal{L}}{\delta \rho_A} \equiv \sum_{p=0}^{\infty} (-1)^p \frac{d^p}{dx^{i_1} \cdots dx^{i_p}} \frac{\partial \mathcal{L}}{\partial \rho_{A,i_1 \cdots i_p}}.$$

(Note the sums over p stop at finite values of p due to the finite order of $\mathcal{L}$ in ρ_A.)

[9]Detailed proofs of theorems I, II, and III are given in S. J. Aldersley, report (unpublished) available from the Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Candaa.

[10]The construction of these concomitants in essentially the same as that of constructing the corresponding concomitants of only g_{ij}. The results given in (14) and (15) are derived for this latter case in D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles* (Wiley, New York, 1975), p. 317.

[11]The "prototype" metric-torsion theory is the Einstein-Cartan theory. For information and references see F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).

[12]$R(\nabla)$ is obtained by substituting $\overline{\Gamma_j{}^i{}_k}$ given in (19) for $\{{}_j{}^i{}_k\}$ in the definitions given in Ref. 7.

[13]$R(\nabla) = R(\{{}_j{}^i{}_k\}) + 4S^{rl}{}_{l|r} - 4S^{rl}{}_{l} S_{rs}{}^{s} - 2S^{lkr} S_{krl} - S^{krl} S_{krl}.$