Scalar form factor in K_{l3} decay

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The $(3,\bar{3})+(\bar{3},3)$ model of chiral-symmetry breaking is used to investigate the leading terms in the expansion of the scalar K_{l3} form factor $f_0(t)$ in powers of t. The results are compared with those of Dashen and Weinstein and a modification of the work of the latter due to Dashen, Li, Pagels, and Weinstein.

A theorem concerning the expansion of the scalar K_{l3} decay form factor $f_0(t)$ in powers of $t\,[=-q^2=-(p_K-p_\pi)^2]$ was derived by Dashen and Weinstein[1] some time ago. $f_0(t)$ is defined by

$$(m_K{}^2-m_\pi{}^2)f_0(t)=(m_K{}^2-m_\pi{}^2)f_+(t)+tf_-(t)\,, \quad (1)$$

where $f_\pm(t)$ are the usual form factors describing K_{l3} decay. Dashen and Weinstein showed that

$$(m_K{}^2-m_\pi{}^2)f_0(t)=(m_K{}^2-m_\pi{}^2)+\frac{1}{2}\left(\frac{f_K}{f_\pi}-\frac{f_\pi}{f_K}\right)t+O(t^2)\,. \quad (2)$$

This theorem was subsequently modified by Dashen, Li, Pagels, and Weinstein[2] in order to take account of threshold effects.[3]

The corrected result showed that

$$(m_K{}^2-m_\pi{}^2)f_0'(t)=\frac{1}{2}\left(\frac{f_K}{f_\pi}-\frac{f_\pi}{f_K}\right)+O(\epsilon)\,, \quad (3)$$

where t is evaluated at the unphysical point $t=m_\pi{}^2+m_K{}^2$. The leading term on the right-hand side of Eq. (3) is of order $\epsilon\ln\epsilon$ where the parameter ϵ sets the scale of chiral-symmetry breaking. This term was shown[2] to be independent of the structure of the chiral-symmetry-breaking term in the Hamiltonian. The correction term of order ϵ is model dependent and was claimed[2] to be an order of magnitude less important than the leading term within the framework of the $(3,\bar{3})+(\bar{3},3)$ model of chiral-symmetry breaking.[4]

In this note[5] we adopt this model to find an expression for the expansion of the scalar K_{l3} decay form factor $f_0(t)$ in powers of t. It is found that there is a not insignificant correction to the result[2] expressed in Eq. (3). We begin by defining the three-point functions $\Gamma(k^2,p^2,q^2)$, $f_\pm^K(k^2,p^2,q^2)$:

$$\Delta_K(q^2)\Delta_K(k^2)\Delta_\pi(p^2)\Gamma(k^2,p^2,q^2)=\iint d^4x\,d^4y\,e^{ik\cdot x}e^{-ip\cdot y}\langle 0|\,T\{\phi_{K^-}(x)\,\phi_{\pi^0}(y)\,\phi_{K^+}(0)\}\,|0\rangle \quad (4)$$

and

$$\Delta_K(k^2)\Delta_\pi(p^2)[f_+^K(k^2,p^2,q^2)(k+p)_\mu+f_-^K(k^2,p^2,q^2)(k-p)_\mu]=\iint d^4x\,d^4y\,e^{ik\cdot x}e^{-ip\cdot y}\langle 0|\,T\{\phi_{K^-}(x)\,\phi_{\pi^0}(y)\,V_\mu^{4+i5}(0)\}\,|0\rangle\,. \quad (5)$$

$f_\pm^K(k^2,p^2,q^2)$ and $f_\pm^\pi(k^2,p^2,q^2)$ may be defined similarly.

On multiplying Eq. (5) by $iq_\mu=i(k-p)_\mu$, the following relation between the various three-point functions is obtained:

$$f_+^K(k^2,p^2,q^2)(k^2-p^2)+f_-^K(k^2,p^2,q^2)\,q^2=f_K m_K{}^2\Delta_K(q^2)\,\Gamma(k^2,p^2,q^2)+\left(\frac{z_\pi}{z_K}\right)^{1/2}(k^2+m_K{}^2)-\left(\frac{z_K}{z_\pi}\right)^{1/2}(p^2+m_\pi{}^2)\,. \quad (6)$$

The current-divergence relation[6]

$$\partial_\mu V_\mu^{4+i5}(0)=if_K m_K{}^2\phi_{K^+}(0) \quad (7)$$

has been used together with the commutation relations derived from the $(3,\bar{3})+(\bar{3},3)$ model of chiral-symmetry breaking. z_π and z_K are the wave-function renormalization constants.

It is well known[7] that if we impose the smoothness assumption that $\Gamma(k^2,p^2,q^2)$ be no more than a quadratic function of the momenta, together with the assumption[7] that $f_+^K(k^2,p^2,0)$ be a constant equal to $f_+^K(0,0,0)$ (f_+^K for short) over a certain range of k^2 and p^2, then it follows that $\Delta_K{}^{-1}(q^2)$ is no more than a quadratic function of momentum.

Likewise if we impose similar constraints on $f_K^+(0,p^2,q^2)$ and $f_\pi^+(k^2,0,q^2)$, then it follows that $\Delta_K^{-1}(k^2)$ and $\Delta_\pi^{-1}(p^2)$ are no more than quadratic functions of momentum. In addition, we have the usual results[7]

$$f_+^\kappa=(f_K{}^2+f_\pi{}^2-f_\kappa{}^2)/2f_Kf_\pi\,, \tag{8a}$$

$$f_+^K=(f_K{}^2-f_\pi{}^2-f_\kappa{}^2)/2f_\kappa f_\pi\,, \tag{8b}$$

$$f_+^\pi=(f_K{}^2-f_\pi{}^2+f_\kappa{}^2)/2f_Kf_\kappa\,. \tag{8c}$$

Setting $k^2+m_K{}^2=0$, $p^2+m_\pi{}^2=0$, and $q^2=0$ in Eq. (6) we find

$$f_+^\kappa(m_K{}^2-m_\pi{}^2)=f_\kappa\Gamma(m_K{}^2,m_\pi{}^2,0)\,. \tag{9a}$$

Similarly, we may conclude from equations which are cyclic permutations of Eq. (6) relating $f_\pm^K(k^2,p^2,q^2)$ and $f_\pm^\pi(k^2,p^2,q^2)$ to $\Gamma(k^2,p^2,q^2)$ that

$$f_+^K(m_\kappa{}^2-m_\pi{}^2)=f_K\,\Gamma(0,m_\pi{}^2,m_\kappa{}^2) \tag{9b}$$

and

$$f_+^\pi(m_\kappa{}^2-m_K{}^2)=f_\pi\Gamma(m_K{}^2,0,m_\kappa{}^2)\,. \tag{9c}$$

In order to apply these results to the scalar form factor in K_{l3} decay we recall that

$$\begin{aligned}\langle\pi^0|\,\partial_\mu V_\mu^{4+i5}(0)\,|K^-\rangle &= i\,[f_+(q^2)(m_K{}^2-m_\pi{}^2)-q^2f_-(q^2)]\\ &= i(m_K{}^2-m_\pi{}^2)f_0(q^2)\,,\end{aligned} \tag{10}$$

where the on-shell form factors $f_\pm(q^2)$ are related to the off-shell form factors $f_\pm^\kappa(k^2,p^2,q^2)$ by setting the pion and kaon on the mass shell in the latter. We may also write

$$\begin{aligned}\langle\pi^0|\,\partial_\mu V_\mu^{4+i5}(0)\,|K^-\rangle &= if_\kappa m_\kappa{}^2\langle\pi^0|\,\phi_{\kappa^+}(0)\,|K^-\rangle\\ &= if_\kappa m_\kappa{}^2\Delta_\kappa(q^2)\,\Gamma(m_K{}^2,m_\pi{}^2,q^2)\,,\end{aligned} \tag{11}$$

making use of the current divergence relation Eq. (7). Combining Eqs. (10) and (11) we have

$$(m_K{}^2-m_\pi{}^2)f_0(q^2)=f_\kappa\frac{m_\kappa{}^2}{m_\kappa{}^2+q^2}\Gamma(m_K{}^2,m_\pi{}^2,q^2)\,. \tag{12}$$

The constant term in the expansion yields the result

$$(m_K{}^2-m_\pi{}^2)f_0(0)=f_\kappa\Gamma(m_K{}^2,m_\pi{}^2,0) \tag{13}$$

$$=(m_K{}^2-m_\pi{}^2)f_\kappa^+\,, \tag{14}$$

making use of Eq. (9a). From this we get the standard relation

$$f_0(0)=f_+(0)=\frac{f_K{}^2+f_\pi{}^2-f_\kappa{}^2}{2f_Kf_\pi}\,, \tag{15}$$

assuming there is a nonsingular behavior of $f_-(q^2)$ at $q^2=0$. The constant term [Eq. (14)] in the expansion is correct[8] to order ϵ^3. This may be compared with Ref. 1, where appeal to the Ademollo-Gatto theorem was used to discard the order ϵ^3 contribution.

Next we evaluate the derivative of the scalar form factor at $t(=-q^2)=0$ and the unphysical point $t=m_\pi{}^2+m_K{}^2$ in order to compare our result with Ref. 2. Differentiating Eq. (12) we find

$$(m_K{}^2-m_\pi{}^2)f_0(t)=f_\kappa\frac{m_\kappa{}^2}{(m_\kappa{}^2-t)^2}\times\Gamma(m_K{}^2,m_\pi{}^2,m_\kappa{}^2)\,. \tag{16}$$

The on-shell vertex function $\Gamma(m_K{}^2,m_\pi{}^2,m_\kappa{}^2)$ can be well approximated by $\Gamma(m_K{}^2,0,m_\kappa{}^2)$ because of the small pion mass [approximate SU(2)×SU(2) symmetry]. In this approximation, we may use Eq. (9c) to rewrite Eq. (16) as

$$(m_K{}^2-m_\pi{}^2)f_0'(t)=\frac{f_\kappa}{f_\pi}f_+^\pi(m_\kappa{}^2-m_K{}^2)\frac{m_\kappa{}^2}{(m_\kappa{}^2-t)^2} \tag{17}$$

$$=\frac{(f_K{}^2-f_\pi{}^2+f_\kappa{}^2)}{2f_Kf_\pi}\frac{(m_\kappa{}^2-m_K{}^2)m_\kappa{}^2}{(m_\kappa{}^2-t)^2}\,, \tag{18}$$

making use of Eq. (8c). From Eq. (18), the above-mentioned derivatives are

$$(m_K{}^2-m_\pi{}^2)f_0'(t)\,|_{t=0}=\frac{f_K{}^2-f_\pi{}^2+f_\kappa{}^2}{2f_Kf_\pi}\frac{(m_\kappa{}^2-m_K{}^2)}{m_\kappa{}^2} \tag{19}$$

and

$$(m_K{}^2-m_\pi{}^2)f_0'(t)\,|_{t=m_\pi{}^2+m_K{}^2}=\frac{f_K{}^2-f_\pi{}^2+f_\kappa{}^2}{2f_Kf_\pi}\times\frac{m_\kappa{}^2}{(m_\kappa{}^2-m_K{}^2)}\,, \tag{20}$$

where we have neglected m_π on the right-hand side of Eqs. (19) and (20) in order to be consistent with the earlier approximation.

From Eqs. (19) and (20) we may make the following observations:

(a) To order ϵ,[8] the derivative is the same at $t=0$, $t=m_\pi{}^2+m_K{}^2$, namely

$$\frac{1}{2}\left(\frac{f_K}{f_\pi}-\frac{f_\pi}{f_K}\right).$$

(b) There are correction terms of order ϵ^2 and higher which differ at the two points. This means that the slope of the scalar form factor will not be uniform in the physical region.

(c) Taking[9] $f_K/f_\pi=1.26$ and $f_\kappa{}^2/f_\pi{}^2=0.09$ and[10] $m_\kappa=1250$ MeV, we find numerically

$$(m_K{}^2-m_\pi{}^2)f_0'(t)\,|_{t=0}=0.23 \tag{19$'$}$$

and

$$(m_K{}^2 - m_\pi{}^2) f_0'(t)\,|_{t=m_\pi{}^2+m_K{}^2} = 0.32\,, \qquad (20')$$

whereas

$$\frac{1}{2}\left(\frac{f_K}{f_\pi} - \frac{f_\pi}{f_K}\right) = 0.23\,, \qquad (21)$$

in (presumably) fortuitous numerical agreement with the slope evaluated at $t=0$ [Eq. (19′)]. Note, however, that the value [Eq. (20′)] we find for the slope at the unphysical point $t=m_\pi{}^2+m_K{}^2$ is about 40% larger than the modified[2] Dashen-Weinstein result [Eq. (21)]. This is at some variance with the statement in Ref. 2 that model-dependent corrections within the framework of the $(3,\overline{3})+(\overline{3},3)$ model of chiral-symmetry breaking are an order of magnitude less important than the leading term

$$\frac{1}{2}\left(\frac{f_K}{f_\pi} - \frac{f_\pi}{f_K}\right).$$

A similar conclusion was reached by Auvil and Pritchett[11] using different arguments from this paper.

The constant term in the expansion, Eq. (14), differs numerically very little from the Dashen-Weinstein result. We find[9]

$$f_+^K = \frac{f_K{}^2 + f_\pi{}^2 - f_\kappa{}^2}{2 f_K f_\pi} = 0.99\,, \qquad (22)$$

which is very close to the SU(3) result used by them.

In terms of the usual slope parameter λ_0 defined by

$$\lambda_0 = m_\pi{}^2 f_0'(t)\,, \qquad (23)$$

we find $\lambda_0 = 0.019$ at $t=0$ and $\lambda_0 = 0.027$ at $t=m_K{}^2 + m_\pi{}^2$ indicating roughly the extent of the variation of the slope of the scalar form factor to be expected in the physical region $0 \leq t \leq (m_K - m_\pi)^2$. The current experimental situation is unsettled[10] with

$$\lambda_0 = -0.009 \pm 0.007 \ (K_{\mu 3}^+)$$

and

$$\lambda_0 = 0.021 \pm 0.006 \ (K_{\mu 3}^0)\,.$$

The value of λ_0 determined from $K_{\mu 3}^0$ experiments is in good agreement with theoretical expectations in contrast to the value of λ_0 determined from $K_{\mu 3}^+$ experiments.

In summary we have used the $(3,\overline{3})+(\overline{3},3)$ model of chiral-symmetry breaking together with the usual hard-meson treatment of vertex functions to investigate the leading terms in the expansion of the scalar K_{l3} decay form factor $f_0(t)$ in powers of t:

$$(m_K{}^2 - m_\pi{}^2) f_0(t) = \frac{f_K{}^2 + f_\pi{}^2 - f_\kappa{}^2}{2 f_K f_\pi}(m_K{}^2 - m_\pi{}^2) + t\,\frac{(f_K{}^2 - f_\pi{}^2 + f_\kappa{}^2)}{2 f_K f_\pi}\,\frac{(m_\kappa{}^2 - m_K{}^2)\, m_\kappa{}^2}{(m_\kappa{}^2 - t)^2}\,.$$

The constant term differs from the original Dashen-Weinstein[1] result in order ϵ^3 (about 1%). However, the derivative of $f_0(t)$ evaluated at the unphysical point $t=m_\pi{}^2+m_K{}^2$ differs from the modified[2] Dashen-Weinstein result by about 40%.

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[8]In this calculation, we may adopt the usual naive power-counting approach in the parameter ϵ. The reason for this is that our assumption of a zero width π-$K(\kappa)$ resonance forces us to neglect threshold effects. See footnote 7 in Ref. 5 cited above.

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