Comments and Addenda

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Comment on N/D generation of the ρ resonance

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Several calculations have found that the ρ meson can be generated in an apparently natural way by N/D methods without recourse to inelastic channels or Castillejo-Dalitz-Dyson poles. In the context of the N/D framework, this means that the ρ is a dynamical resonance of the pion-pion system. Tryon has challenged these results, asserting that the method used to generate the ρ was biased and that the solutions severely violate rigorous sum rules. This paper argues that both claims made by Tryon are unsubstantiated.

Controversy as to whether the ρ meson can be considered to be a dynamical pion-pion resonance has a long and confused history.[1] In the context of the N/D method, the question is in principle a simple one: Given the proper crossed-channel processes, does the ρ emerge automatically in a single-channel calculation, or must it be inserted by hand into the direct channel by means of a CDD pole?[1,2] Unfortunately, in practice, the problem cannot be disentangled from the effects of approximations, both in inputs and in the N/D method itself.

In 1969,[3] I introduced a technique which allowed a simple treatment of the nearby left-hand cut, thereby eliminating many of the stability problems of earlier calculations.[4-6] Following this, a series of papers was published describing further improvements, tests of the method, and applications to pion-pion scattering.[7-10] In these, several coworkers and I found that the ρ did emerge without recourse to CDD poles. In one of these,[8] I also discussed a set of sum rules which had been used by Tryon, and concluded that these rules, though rigorous, were not useful in practice because of systematic cancellations in the low-energy inputs. Subsequent to this, Tryon published another paper,[11] which attempted to show that the ρ was not dynamically produced, but the method which he used turned out to be invalid.[12,13] Finally, Tryon[14] has criticized my own 1973 paper[8] on two counts: first, that the gap-matching method which I used was biased, and second, that results obtained there showed severe violations of a rewritten version of the sum rules. I argue in what follows that these claims are unsubstantiated.

The first claim is that the gap-matching method, as used in my calculations, is inherently biased and, in effect, forces the appearance of a ρ meson by generating an incorrect left-hand cut.

We will assume in what follows that the physical amplitude $A(\nu)$ exists in the gap $-1 \leq \nu < 0$, and that its analytic continuation into the physical region contains the ρ meson. Also, let us assume, as Tryon does at the beginning of his Sec. V, that the Froissart-Gribov amplitude A^{FG} also exists in the gap, and that it can be continued into the physical region. Obviously, if

$$A(\nu) = A^{FG}(\nu), \quad -1 \leq \nu < 0 \tag{1}$$

then the continuation of A^{FG} will contain the ρ meson, and in fact will be everywhere identical to the physical amplitude. In particular, it will generate the correct left-hand cut and will presumably satisfy the sum rules exactly. This is of course true of any continuation, be it by the N/D method or by some other technique.

I am uncertain what Tryon's viewpoint is here. His viewpoint may be that (1) is correct, but that to actually calculate A^{FG} in the gap and then continue it into the physical region by the N/D method is biased; if this is what my N/D calculations are doing, why should the distant left-hand cut be incorrect, and why should the sum rules be violated? Alternatively, Tryon's viewpoint may be that (1) is incorrect; if so, why would I get a ρ in my N/D calculations? It is perfectly possible to argue that (1) is wrong, but in this case the incorrect assumptions which go into (1) should be clearly specified.

Actually, all of this is not closely related to real calculations where approximations are made, because then A^{FG} most certainly does not agree with the physical amplitude, and the difference between them are quite crucial. For example, suppose that we use a ρ meson in "zero-width" approximation as input in the crossed channels. The corresponding Froissart-Gribov expression is

$$A^{\mathrm{FG}}(\nu)=\frac{3}{4\nu}\; m_\rho\,\Gamma_\rho\,P_1\left(1+\frac{2(\nu+1)}{\nu_\rho}\right)Q_1\left(1+\frac{\nu_\rho+1}{\nu}\right)\,, \tag{2}$$

which is certainly not the same as the physical amplitude, since it does not contain a resonance of any kind when continued to the physical region, does not satisfy unitarity, and does not even contain a right-hand cut. Obviously, if one uses this as the basis of an N/D calculation and requires $A^{N/D}\simeq A^{\mathrm{FG}}$ in the gap, then N/D can produce an output ρ only by generating the proper corrections to A^{FG}. Of course, one could improve A^{FG} considerably by removing the zero-width approximation for the input ρ and by including more crossed-channel partial waves, say $l\leq 5$, but the computed gap amplitude would still have all the deficiencies which are present in (2). Getting an output ρ in this situation is far from trivial, and the N/D method can succeed only by making further requirements that the gap amplitude be consistent with the nearby left-hand cut and with unitarity.

Obviously, in discussing real calculations, it is extremely important to take proper account of approximations, and any discussion that does not do this is inadequate. In his actual discussion (Sec. V), Tryon works almost entirely with amplitudes computed by direct-channel dispersion relations, and does not take into account the fact that there will be crucial differences between these amplitudes and Froissart-Gribov amplitudes when approximations are made. For example, using the same input as was used in (2), a dispersion relation gives

$$A(\nu)=\frac{m_\rho\,\Gamma_\rho}{4}\;\frac{\nu}{\nu_\rho(\nu-\nu_\rho)}\;\;,\;\;-1\leq\nu<0 \tag{3}$$

[Tryon's equation (16b)], which is quite different from (2). Thus, if in an N/D calculation, one matched the N/D amplitude to (3), one would indeed expect a ρ to emerge since (3) itself has a pole at $\nu=\nu_\rho$. However, in my own calculations, $A^{N/D}$ was always matched to A^{FG}, not to A, and the fact that (3) and (2) agree approximately in the gap (the disagreement as $\nu\to 0$ is less than 3%) is not relevant.

A further point of confusion is that Tryon uses the same symbols to represent the physical cuts and the N/D cuts, although *a priori*, these could be quite different. Thus, not only should Tryon's Eq. (15) be given as an approximate equality, but it is not clear whether A_{DL}, etc., refer to the N/D or the physical amplitudes. [In the discussion immediately preceding (15), they refer to N/D cuts, but immediately afterwards, they refer to physical cuts.] In any event, if one assumes that the *nearby* physical and N/D cuts are equal, as was done in my own calculations, then the condition $A\simeq A^{\mathrm{FG}}\simeq A^{N/D}$ implies only

$$A_{\mathrm{DL}}^{N/D}+A_{\mathrm{R}}^{N/D}\simeq A_{\mathrm{DL}}+A_{\mathrm{R}}\,,\;\;-1\leq\nu<0\,. \tag{4}$$

From (4), it is certainly not possible to conclude that because $A_{\mathrm{R}}\gg A_{\mathrm{DL}}$ for $-1\leq\nu<0$, $A_{\mathrm{R}}^{N/D}\gg A_{\mathrm{DL}}^{N/D}$ for $-1\simeq\nu<0$, or that $A_{\mathrm{R}}^{N/D}\simeq A_{\mathrm{R}}$ in the physical region $\nu>0$. The fact is that it is simply impossible to predict what the N/D method will do on the basis of comparisons with a direct-channel dispersion relation: To predict the N/D output, one must take into account the actual differences between A^{FG} and A, the nature of the nearby left-hand cut, and unitarity.

Almost all of Tryon's assertions about bias (Sec. V) are based on conjectures of the type which I have discussed above, and only one actual calculation appears to substantiate his claims. This is the N/D calculation based on (4), or his Eq. (16b), which gives a ρ meson, but unphysical results at very high energy; from this, he concludes that the appearance of the ρ is itself evidence of bias in the N/D method. Actually, the results reported by Tryon are very easy to understand. In the first place, since the analytic continuation of (16b) from the gap contains a pole at $\nu=\nu_\rho$, it should not be very surprising that the N/D method gives a resonance near there. On the other hand, the assumed asymptotic properties of the N/D method force the phase shifts to go to zero eventually, so that if they once rise through 90°, they must inevitably fall through 90° also. All of the observed behavior is therefore built into the calculation from the outset and the output is actually just what should be expected. In real calculations, the behavior at very high energies which Tryon discusses (>100 GeV) is completely irrelevant, since this is well into the inelastic region where the results are not expected to be correct. If one really hopes to get correct results at such energies, a realistic treatment of unitarity would of course be necessary.

In the case discussed above, a ρ was assumed to be in the direct channel, and so its appearance in an N/D calculation can tell nothing about the presence or absence of bias. On the other hand, if there is really some bias in the gap-matching method, it should be easy to construct test cases where a ρ is *not* assumed in the direct channel, but is still produced by N/D. No such examples have been produced. Another way to show bias would be to start with a unitary resonance form whose analytic properties seemingly require a Castillejo-Dalitz-Dyson (CDD) pole [Tryon's example is unsuitable because the analytic properties of (16b) are inappropriate for the purpose], and show that the N/D method generates the resonance without the CDD pole. The fact is, however, that in all test cases reported to date,[7,9,15] the gap-matching method, as used in my pion-pion calculations, behaves propertly.

The second allegation made by Tryon is that my N/D calculations show severe violations of sum rules. As was mentioned before, the question of agreement with the sum rules was discussed in my 1973 paper,[8] and the reader is referred there for comments made at that time. In his current paper, Tryon has rewritten the sum rules in a slightly different form and has given the corresponding results in his Table II. Unfortunately, he has not discussed the question of what constitutes agreement or disagreement in a real calculation where approximations are made. Consider the following factors:

1. Elastic unitarity was used in the N/D calculation, whereas in reality, there is considerably inelasticity above the ρ. The approximation gives no apparent trouble if one is interested only in low energies, as in the ρ meson calculation of Ref. 8, but of course the sum rules extend over all energies.

2. The entire distant left-hand cut was replaced by two poles. This is surely a gross approximation and the parametrization obviously represents the cut in only an average way. The parametrization was never intended to represent the actual cut in any detail and can hardly be expected to satisfy high-order sum rules.

3. Approximations were made in the inputs, and these affect not only the N/D results, which are of course only approximate, but also the sum rules themselves. The problem can be simply illustrated by looking at Tryon's own inputs, given in his Table I. Thus, in the R_2 sum rule, the $l=0$, 1, and 2 partial waves give large individual contributions, but the sum $(S_0+S_2+P_1+D_0+D_2)$ is only 0.07. That is, the low-energy contributions cancel almost completely, and 1.61 of the 1.68 total comes from the g-meson and high-energy terms. Similarly, for R_3, the $l=0$, 1, and 2 terms contribute only 0.07 of the 0.89 total. What this means is that all of the best experimental data are being thrown out, and that the right-hand sides of the R_2 and R_3 sum rules are being determined almost completely by high-energy terms which are poorly known and which have large inelasticity. This difficulty was pointed out in Ref. 8 and the cancellations for the inputs used in that paper can be inferred from Table V, where individual contributions to the old form of the sum rules are given. Unfortunately, Tryon gives only the total in his Table II, so the cancellations are not apparent.

In view of these difficulties, the numbers that the N/D calculations produce for the left-hand side of the sum rules seem to me to be quite reasonable. There is no automatic reason for the N/D solution to satisfy the sum rules, yet all of the numbers it produces are of the right order of magnitude, and this seems to me to be all that can be expected.

The only conclusion I can draw is that there is no reason at present to believe that something is "wrong" with the N/D calculations. A ρ meson is generated using elastic unitarity and no CDD pole, and in the N/D framework, such a particle is considered to be a dynamical resonance of the pion-pion system. I should hasten to add, however, that I have no idea what this means in the context of other approaches such as current-algebra or quark models.

[1]For an interesting problem in the early days, see F. Zachariasen, in *Recent Developments in Particle Physics*, edited by M. J. Moravcsik (Gordon and Breach, New York, 1966).
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