PHYSICAL REVIEW D VOLUME 15, NUMBER 11 1 JUNE 1977

Meson spectrum in the quark model with a phenomenological potential*

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(Received 9 July 1976; revised manuscript received 22 November 1976)

The masses of mesons are calculated under the assumption that a meson is a bound state of a quark-antiquark pair interacting via a phenomenological potential. The form of the potential, which contains a Coulomb-type term, a linear confining term, plus spin-orbit, tensor, spin-spin, and other terms, is suggested by gauge field theory, but the magnitudes of the terms are determined by a phenomenological fit to the data. Predictions are given for the masses of as yet unseen mesons.

I. INTRODUCTION

Recently, a number of papers have appeared in which meson masses are calculated under the assumption that mesons are bound states of quark-antiquark pairs. A sample of such calculations is given in Refs. 1–9. At least two features distinguish this recent work from the older literature on the subject.[10] First, the new work benefits from additional experimental information, including information about the ψ family of mesons.[11,12] Second, in recent work the interaction between a quark and an antiquark is often taken to vary as $1/r$ at small distances[13] and as r at large distances.[14,15] Some of the recent treatments, however, use other interactions, such as a harmonic-oscillator potential,[16] or a $1/r$ potential with an infinite boundary.[17]

The primary motivation for taking a $1/r$ plus linear potential between quarks has come primarily from non-Abelian gauge field theory. In particular De Rújula, Georgi, and Glashow[5] have argued at length that such an interaction naturally arises in a theory in which the forces are mediated by an octet of massless vector bosons (color gluons). We refer the reader to the paper of De Rújula, Georgi, and Glashow[5] for details and additional references.

The first of these new calculations of the meson spectrum were made within the framework of the nonrelativistic Schrödinger-equation model, but subsequent calculations have incorporated some relativistic effects in a variety of ways. The Dirac equation,[18] the Klein-Gordon[19] equation, the Bethe-Salpeter equation,[16] and other equations[20] have been used, and the potential between quarks has been inserted either as the fourth component of a four-vector[8] or as a true scalar.[18,19] The wide variety of wave equations that have been used to treat this problem only serves to point out the fact that at present we do not have a generally accepted theory of how to treat the relativistic two-body problem.

Unfortunately, if a given potential is used in different wave equations, different mass spectra result. This fact has been explicitly demonstrated by Goldman and Yankielowicz,[18] who calculated the charmonium spectrum by using the potentials of Harrington *et al.*[2] and Eichten *et al.*[3] in a reduced-mass Dirac equation. Goldman and Yankielowicz obtained energy-level spacings which differed by as much as 40% from those found by Eichten *et al.* and Harrington *et al.* using the nonrelativistic Schrödinger equation. The differences are likely to be even greater in the case of light quarks, whose motion is more inherently relativistic.

In view of the fact that the calculated energy levels are so strongly dependent on the form of the equation used, and the fact that no one equation can be demonstrated to be the correct one, we have adopted a phenomenological approach to the problem. We continue to be guided by gauge theory as to the kinds of terms that can appear in a potential between quarks. However, we regard the details of the potential as phenomenological. In particular, our potential contains a number of free parameters which we adjust to obtain qualitative agreement with the observed meson mass spectrum.

This paper is basically a generalization of our former work on bound states of strange quarks and their antiquarks[6] to include ordinary quarks and charmed quarks (charmonium) as well. As in our previous work, we keep the same kinds of terms in the potential that are present in positronium,[21] as these terms are likely to arise from the exchange of massless vector gluons. Specifically, we include a $1/r$ term in the potential plus spin-spin and spin-orbit terms, a tensor term,[22] and terms which go like the spin or orbital angular momentum squared. These terms all result from the nonrelativistic reduction of the Breit equation.[21] Lastly, we include a linear term in the potential to confine the quarks.[14,15]

In our calculation we omit the nondiagonal part of the tensor force, because if we were to include it, we would have to solve simultaneous differential equations, and the calculations would become too long. This approximation is a good one only if the tensor interaction is small. To anticipate our results, it turns out that in some cases the tensor force is small, but in others it is not. We shall discuss the effects of the tensor force again after we give our results.

For the wave equation, we take the ordinary Schrödinger equation, but use relativistic kinematics. We mean by this that our procedure is to find the eigenvalue of the Schrödinger equation which corresponds to the square of the relative momentum. We then use the relativistic connection between energy and momentum to obtain the meson mass. In the equal-mass case, this procedure is equivalent to solving a Klein-Gordon equation with a potential inserted as a true scalar.

We have not included all the terms that arise in the reduction of the Breit equation. We have omitted a term which goes like ∇^4 and another term which goes like $(1/r)\nabla^2 + \nabla^2(1/r)$. If we had included these terms, our wave equation would not be of a Schrödinger-type. We partly justify omitting these terms by our use of relativistic kinematics. The ∇^4 term, for example, arises from an expansion of the momentum term in the Breit equation. To include it in the wave equation at the same time that we use relativistic kinematics would amount to double-counting. We need to keep the other terms that arise in the reduction of the Breit equation because they give rise to spin- and angular-momentum-dependent terms in the potential. We take care of possible double-counting in this case by multiplying the magnitudes of these terms by phenomenological parameters.

Our reason for using relativistic kinematics is that the momenta of the bound quark-antiquark pairs are not small compared to their masses. Even in the case of charmonium, where the quark masses are large, we have determined that the use of relativistic kinematics makes a substantial difference in the spectrum. Indeed, we have verified by explicit calculation that if we use relativistic kinematics with the potential of Harrington *et al.*,[2] we obtain a meson spectrum closer to that found by Goldman and Yankielowicz[18] (who used the Dirac equation) than that found by the original authors. Thus, our use of relativistic kinematics with the Schrödinger equation enables us to include a major effect of relativity while at the same time keeping a tractable wave equation. Again, we stress that if we were to use a different wave equation the parameters of our potential would undoubtedly have to be changed somewhat for us to obtain a comparable fit to the observed meson spectrum.

Our procedure is to solve the Schrödinger equation numerically with the assumed form of the interaction. In addition to our omitting the nondiagonal part of the tensor force, we make one other approximation in obtaining this solution. We omit all δ-function or contact terms in the interaction from the potential in obtaining the numerical solution, and later evaluate the contribution from the contact terms in perturbation theory. Our reason for so doing is that in some cases the δ-function interaction is attractive, and there is no acceptable s-wave exact solution to the Schrödinger equation in this case. (The δ-function interaction arises as a result of the reduction of the Breit equation, which does not have this singularity.) Despite our approximations, our work contains the most detailed numerical calculations of which we are aware. In no other calculation of the meson spectrum, to our knowledge, is the Schrödinger equation solved numerically with spin-orbit, quadratic orbital angular momentum, and tensor terms in the potential, in addition to $1/r$ and linear terms.

In Sec. II we give the details of the interaction and the method for obtaining the meson masses. In Sec. III we present our results and discuss them.

II. DETAILS OF THE CALCULATION

As discussed in the Introduction, our prescription is to solve the bound-state Schrödinger equation

$$-\nabla^2\psi + 2\mu U\psi = k^2\psi \ , \tag{1}$$

where μ is the reduced mass of the quark and antiquark, U is the potential, and k^2 is an eigenvalue corresponding to the square of the momentum of the bound quark or antiquark. The potential U contains, in addition to $1/r$ and linear terms, a spin-orbit term proportional to $\vec{L}\cdot\vec{S}$, where $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the total spin of the quarks, a quadratic orbital angular momentum term proportional to L^2, and a tensor term proportional to S_{12}, which is the usual tensor operator. In the case of positronium, these last three terms go like $1/r^3$ or like $(1/r)dV/dr$, where V is the Coulomb potential. In our case, the potential analogous to the Coulomb potential is V_s given by

$$V_s = -\alpha_s/r + \beta r \ , \tag{2}$$

where α_s is the square of the strong coupling constant and β is a parameter which measures the strength of the confining term. If we let the functions multiplying the $\vec{L}\cdot\vec{S}$, L^2, and S_{12} terms go like $(1/r)(dV_s/dr)$, these terms will contain a $1/r^3$

singularity at the origin. As is well known, there are no acceptable bound-state solutions with such a singular (attractive) potential. We therefore introduce a cutoff parameter a into the potential by making the replacement

$$(1/r)dV_s/dr \to (1/r)[\alpha_s/(r^2+a^2)+\beta]. \qquad (3)$$

If $a\to 0$, the expression on the right-hand side of (3) becomes $(1/r)dV_s/dr$. We also introduce three parameters γ_{LS}, γ_L, and γ_T describing the strength of the $\vec{L}\cdot\vec{S}$, L^2, and S_{12} terms. We thus write for U

$$U=-\alpha_s/r+\beta r+(8\mu^2 r)^{-1}[\alpha_s/(r^2+a^2)+\beta] \times(3\gamma_{LS}\vec{L}\cdot\vec{S}+\gamma_L L^2+\tfrac{1}{2}\gamma_T S_{12}). \qquad (4)$$

As we have previously remarked, when solving the Schrödinger equation with this potential, we include only the diagonal elements of S_{12}. We have introduced the quantity $(8\mu^2)^{-1}$ into our expression for U to help us interpret the parameters γ_{LS}, γ_L, and γ_T. For positronium, α_s becomes the usual fine-structure constant, γ_{LS}, γ_L, and γ_T are all unity, and $a=\beta=0$. In our case, we let γ_{LS}, γ_L, and γ_T, as well as α_s and β depend on isospin and on the kind of quarks which are interacting.

There is in addition a contact term U_c in the interaction. As in our previous work, we write it as

$$U_c=\lambda\pi\alpha_s(2\mu)^{-2}\delta(\vec{r})(1+S^2+\tfrac{2}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2), \qquad (5)$$

where λ is a parameter. The form of Eq. (5) is chosen as an aid to interpreting the parameter λ, since $\lambda=1$ for positronium.

We solve the Schrödinger equation with the potential U to obtain the eigenvalue k_{nLJ}^2 in a particular state, which depends on the orbital angular momentum L, the total angular momentum J, and the quantum number n, which numbers the levels of given L, J, and parity P in order of increasing mass. We then evaluate the expectation value of the contact term U_c in perturbation theory, using the wave function ψ_{nLJ} of this state. Then the new momentum-squared eigenvalue K_{nLJ}^2 is given by

$$K_{nLJ}^2=k_{nLJ}^2+2\mu\langle\psi_{nLJ}|U_c|\psi_{nLJ}\rangle. \qquad (6)$$

The expectation value of U_c in Eq. (6) vanishes unless $L=0$.

TABLE I. Effective quark masses in MeV according to some recent papers. Most of the authors made no attempt to distinguish between the mass of the u and d quarks.

m_u	m_d	m_s	m_c	Reference
336	338	540	1500	Wu[a]
260	260	475	2000	Kang[b]
163	163	286	1228	Gunion[c]
275	275	400	1400	Cheng[d]
336	336?	540	1660	De Rújula[e]

[a]Reference 23.
[b]Reference 8.
[c]Reference 4.
[d]Reference 17.
[e]Reference 5.

From K_{nLJ} we determine the mass of the meson M_{nLJ} as

$$M_{nLJ}=(m_1^2+K_{nLJ}^2)^{1/2}+(m_2^2+K_{nLJ}^2)^{1/2}, \qquad (7)$$

where m_1 and m_2 are the masses of the bound quark and antiquark.

The Schrödinger equation, the potential U, and M_{nLJ} all contain expressions for the quark masses. We regard these masses as effective masses of constituent quarks which are parameters of the model, and will be somewhat different in other models. However, in most calculations using a constituent-quark picture, the effective masses turn out to be similar; they are a few hundred MeV for the ordinary u and d quarks, about half a GeV for the strange s quark, and about 1.5 GeV for the charmed c quark. We give in Table I some values of effective quark masses used in previous papers.[4,5,8,17,23] In the present work, to avoid having still further adjustable parameters we use the masses found by Cheng and James.[17] Our results would not be qualitatively different if we had used some other quark masses from Table I.

We have solved the Schrödinger equation numerically and have found a set of parameters which gives a good qualitative fit to the data. These parameters are given in Table II. We defer a dis-

TABLE II. Values of the parameters appearing in the potential of Eqs. (4) and (5).

		α_s	β (fm^{-2})	γ_{LS}	γ_L	γ_T	a (fm)	λ
$I=1$	$u\bar{u}, d\bar{d}$	3.00	3.93	0.183	0.343	0.06	0.412	0.0246
$I=0$	$u\bar{u}, d\bar{d}$	2.73	4.55	0.032	0.285	0.88	0.412	0.0167
$I=\frac{1}{2}$	$s\bar{u}, s\bar{d}, \bar{s}u, \bar{s}d$	2.73	4.96	0.124	0.096	−0.08	0.412	0.0225
$I=0$	$s\bar{s}$	1.37	2.44	0.856	0.953	0.14	0.437	0.0360
$I=0$	$c\bar{c}$	0.98	5.78	0.353	2.074	1.69	0.437	0.212

cussion of the values of these parameters to the next section.

III. RESULTS AND DISCUSSION

We present in Tables III through VII our fits to the observed meson spectra and our predictions of as yet undiscovered mesons. We give in the tables only a small number of predicted mesons out of the infinite number that exist for our potential, and that would exist for other infinite confining potentials. The experimental masses listed in these tables are taken from the 1976 edition of the Review of Particle Properties of the Particle Data Group,[12] except for some new results on charmonium states reported by Goldhaber.[24]

Most of the calculations of meson masses, other than those involving charmed quarks, were completed before the 1976 edition of the Review of Particle Properties became available and so are based on fits to the masses given in the 1974 edition.[25] For the most part, the differences between the older values of the masses and the more recent values are small. However, the mass of the ϵ meson is given as less than 700 MeV in the 1974 tables and as 1200 MeV in the 1976 tables. In our calculation we used the earlier smaller value in our fit to the data, and some of our parameters in the second row of Table II might change considerably if we redid the fit with the newer value. We believe that if the reported mass of the ϵ can change by more than 500 MeV in two years, the last word has not yet been said on this subject. One possibility is that the observed ϵ at 1200 MeV should be identified with a radially excited $^3P_0$ state, which in Table IV is calculated to be at 1328 MeV. The state at 600 MeV would then correspond to the lowest state with $J^{PC}=0^{++}$. There is evidence[12] that the $I=0$ $\pi\pi$ phase shift passes through 90° near 800 MeV, and this might reflect the existence of a 0^{++} state near that energy.

We shall now discuss some general features of

TABLE III. Comparison of experimental values of the masses of the isospin $I=1$ mesons (from the Particle Data Group, Ref. 12) with the calculated masses. Also given are predicted values of the mesons of some mesons which have not been observed.

Name	J^{PG}	$n\,^{2S+1}L_J$	Experimental mass (MeV)	Calculated mass (MeV)
π	0^{--}	$1\,^1S_0$	138	138
ρ	1^{-+}	$1\,^3S_1$	773 ± 3	770
δ	0^{+-}	$1\,^3P_0$	976 ± 3	977
A_1	1^{+-}	$1\,^3P_1$	~1100	1170
predicted	0^{--}	$2\,^1S_0$		1214
B	1^{++}	$1\,^1P_1$	1228 ± 10	1254
ρ'	1^{++}	$2\,^3S_1$	~1250[a]	1318
A_2	2^{+-}	$1\,^3P_2$	1310 ± 5	1327
predicted	0^{+-}	$2\,^3P_0$		1395
predicted	1^{+-}	$2\,^3P_1$		1493
A_3	2^{--}	$1\,^1D_2$	~1640	1531
predicted	0^{--}	$3\,^1S_0$		1532
predicted	1^{++}	$2\,^1P_1$		1545
predicted	2^{+-}	$2\,^3P_2$		1592
g	3^{-+}	$1\,^3D_3$	1690 ± 20	1595
ρ'	1^{-+}	$3\,^3S_1$	~1600	1598

[a] Evidence for the $\rho'(1250)$ is not compelling, and this meson has been omitted from the main meson table of the Particle Data Group, Ref. 12.

TABLE IV. Comparison of experimental values of the masses of $I=0$ mesons (from the Particle Data Group, Ref. 12) with the values calculated assuming that the mesons are bound states of ordinary-quark-antiquark pairs. Also given are predicted values of the masses of some mesons which have not been observed.

Name	J^{PC}	$n\,^{2S+1}L_J$	Experimental mass (MeV)	Calculated mass (MeV)
η	0^{-+}	$1\,^1S_0$	549	548
ϵ	0^{++}	$1\,^3P_0$	~1200	600
ω	1^{--}	$1\,^3S_1$	783	782
f	2^{++}	$1\,^3P_2$	1271 ± 5	1311
predicted	0^{-+}	$2\,^1S_0$		1321
predicted	1^{+-}	$1\,^1P_1$		1324
predicted	0^{++}	$2\,^3P_0$		1328
predicted	1^{--}	$2\,^3S_1$		1379
predicted	1^{++}	$1\,^3P_1$		1433
predicted	1^{--}	$1\,^3D_1$		1509
predicted	3^{--}	$1\,^3D_3$		1598
predicted	2^{-+}	$1\,^1D_2$		1602
predicted	2^{++}	$2\,^3P_2$		1617
predicted	2^{+-}	$2\,^1P_1$		1625
ω'	1^{--}	$3\,^3S_1$	1667 ± 10	1673
h	4^{++}	$1\,^3F_4$	2020 ± 30	1806

TABLE V. Comparison of experimental values of the masses of $I=\frac{1}{2}$ strange mesons (from the Particle Data Group, Ref. 12) with the calculated values. Also given are predicted values of the masses of some mesons which have not been observed.

Name	J^P	$n\,^{2S+1}L_J$	Experimental mass (MeV)	Calculated mass (MeV)
K	0^-	$1\,^1S_0$	496	496
K^*	1^-	$1\,^3S_1$	894	895
κ	0^+	$1\,^3P_0$	1250 ± 100	1246
Q_1	1^+	$1\,^3P_1$	~1300	1279
Q_2	1^+	$1\,^1P_1$	~1400	1377
predicted	0^-	$2\,^1S_0$		1443
K^*	2^+	$1\,^3P_2$	1421 ± 3	1445
predicted		$2\,^3S_1$		1529
predicted	1^-	$1\,^3D_1$		1559
predicted	2^-	$1\,^3D_2$		1617
L	2^-	$1\,^1D_2$	1765 ± 10	1657
predicted	0^+	$2\,^3P_0$		1666
predicted	1^+	$2\,^3P_1$		1683

TABLE VI. Comparison of experimental values of the masses of $I=0$ mesons (from Ref. 12) with values calculated assuming the meson are bound states of strange quarks and their antiquarks. Also given are predicted values of some mesons which have not been observed.

Name	J^{PC}	$n^{2S+1}L_J$	Experimental mass (MeV)	Calculated mass (MeV)
η'	0^{-+}	$1\,^1S_0$	958	958
S^*	0^{++}	$1\,^3P_0$	~993 ± 5	993
	1^{--}	$1\,^3S_1$	1020	1020
D	1^{++}	$1\,^3P_1$	1286 ± 10	1291
E	0^{-+}	$2\,^1S_0$	1416 ± 10	1416
predicted	0^{++}	$2\,^3P_0$		1426
predicted	1^{--}	$1\,^3D_1$		1434
predicted	1^{--}	$2\,^3S_1$		1442
f'	2^{++}	$1\,^3P_2$	1516 ± 3	1516
predicted	1^{++}	$2\,^3P_1$		1571
predicted	2^{--}	$1\,^3D_2$		1593
predicted	2^{-+}	$1\,^1D_2$		1648
predicted	1^{+-}	$2\,^1P_1$		1654
predicted	0^{-+}	$3\,^1S_0$		1660

TABLE VII. Comparison of experimental values of the masses of heavy mesons (from Refs. 12 and 24) with values calculated assuming the mesons are states of charmonium. Also given are predicted values of some mesons which have not been observed.

Name	J^{PC}	$n^{2S+1}L_J$	Experimental mass (MeV)	Calculated mass (MeV)
χ	0^{-+}	$1\,^1S_0$	~2750	2794
J/ψ	1^{--}	$1\,^3S_1$	3098 ± 3	3089
χ	0^{++}	$1\,^3P_0$	3415 ± 10	3421
χ	0^{-+}	$2\,^1S_0$	3455 ± 10	3580
χ	1^{++}	$1\,^3P_1$	3500 ± 10	3584
χ	2^{++}	$1\,^3P_2$	3550 ± 10	3587
ψ	1^{--}	$2\,^3S_1$	3684 ± 4	3705
ψ	1^{--}	$1\,^3D_1$	3950 ± 20	3882
predicted	0^{++}	$2\,^3P_0$		3882
predicted	1^{++}	$2\,^3P_1$		3989
predicted	2^{++}	$2\,^3P_2$		3990
predicted	0^{-+}	$3\,^1S_0$		4004
ψ	1^{--}	$3\,^3S_1$	~4100	4094
ψ	1^{--}	$4\,^3S_1$	4414 ± 7	4396

the meson mass spectrum we have calculated. Despite the rather large number of adjustable parameters in our model, we were not able to obtain quantitative agreement with experiment for all meson masses. The largest qualitative discrepancy is that the calculated high-mass levels for ordinary quark-antiquark pairs tend to be lower in energy than the experimental masses. The principal reason for this is the combination of a linear confining potential and relativistic kinematics. Relativistic wave equations, such as the Dirac equation and the Klein-Gordon equation, have this same qualitative feature. We have verified this fact explicitly for the Klein-Gordon equation, and Goldman and Yankielowicz[18] have verified it for the Dirac equation, in both cases for scalar potentials and quarks of equal masses.

We would not want to argue that this result implies that the use of relativistic kinematics is not good. Another alternative, within the general framework of our model, is to say that a linear confining potential is not quite correct. In fact, to obtain better agreement with experiment, it appears that we would need a confining potential which rises somewhat more steeply with distance than a linear potential.

It is a further feature of our calculation that there are many predicted mesons with masses below 2 GeV which have not been observed. Allowing for the qualitative feature that our model gives masses which tend to be too low, we would guess that a considerable number of mesons ought to exist below 2 GeV with the quantum numbers given in our tables but with somewhat higher masses. If these mesons should turn out not to exist at all, our model would be in deep trouble, as would other models with absolutely confined quarks.

There is a particular feature of our calculated $I=1$ spectrum in Table III which is worth pointing out. This is that the first radial excitation of the ρ meson is calculated to be at 1318 MeV, rather than at the position of the ρ', which experimentally is at 1600 MeV. But our calculated second radial excitation of the ρ is at 1598 MeV, very close to the ρ'. We identify the 1318-MeV level with a ρ' state at 1250 MeV, but, unfortunately, the present experimental evidence for the existence of this ρ' state is not compelling. The spectrum found by Gunion and Willey[4] is also of this character.

The same qualitative feature occurs in the $I=0$ spectrum of Table IV. We have to identify the known ω' meson of mass 1667 MeV as the second radial excitation of the ω, rather than the first. Our predicted first radial excitation is at 1379 MeV, and there is as yet no experimental evidence in favor of such a state. We encounter a similar situation with respect to the h meson. In Table IV we have identified the h with the lowest $^3F_4$ bound state of a quark and antiquark. The calculated value of this state lies 200 MeV below the experimental value of the h, but the first radial excitation of the $^3F_4$ state lies fairly close to the h mass. Thus, an alternative interpretation to Table IV is that there is an $^3F_4$ state near 1800 MeV which has not yet been seen experimentally, and the h is a radial excitation of that unobserved state. It is clear from these considerations that considerably more experimental work in meson spectroscopy needs to be done in order to provide a good test of our phenomenological potential.

We next discuss the parameters of our model. It can be seen from Table II that the dimensionless parameters α_s, which multiply the $1/r$ term in the potential, range from about 1 for charmonium to 3 for the $I=1$ mesons. According to the idea of asymptotic freedom,[13] α_s should decrease as the quark mass increases, in qualitative agreement with the variation of α_s which we have found. But our values of α_s are considerably larger than those expected, especially for charmonium. Eichten *et al.*,[3] for example, give $\alpha_s = 0.3$ for charmonium. Barbieri *et al.*[7] give $\alpha_s = 0.27$ for $c\bar{c}$, $\alpha_s = 0.36$ for $s\bar{s}$, and $\alpha_s = 0.42$ for $u\bar{u}$ and $d\bar{d}$, using our notation that α_s is the coefficient of the $1/r$ term in the potential. (Barbieri *et al.* define α_s slightly differently from us. What we call α_s they call $4\alpha_s/3$.)

There is a qualitative reason why we obtain large values of α_s. With a Coulomb potential, as is well known, there is a degeneracy between levels of differing L (neglecting spin-orbit effects, etc.). Thus, for example, the $1D$, $2P$, and $3S$ levels are degenerate. For the linear potential, however, the $1D$ lies lower than the $2P$, which lies lower than the $3S$ level. The experimental level spacing, it turns out, corresponds to a situation which, for low-lying levels, is intermediate between the Coulomb and linear cases, and thus requires a substantial amount of Coulomb potential in the interaction. In particular, in the case of charmonium, the relatively large value of α_s comes in part from our identification of the $\psi(3950)$ as the 1^3D_1 state of charmonium. The mass of this state is only slightly below the mass the $\psi(4100)$, which we interpreted as a $3S_1$ radial excitation of the $\psi(3095)$ and $\psi(3685)$. With another interpretation of this state, we might be able to use a considerably lower value of α_s. We should point out that at the time previous calculations of the charmonium spectrum were carried out, the $\psi(4100)$ was not resolved from the $\psi(3950)$, and so this problem did not arise. In other words, the known ψ spectrum is now richer than it was, and leads to more restrictions on the potential, if all the observed states are interpreted as states of charmonium. Other interpretations are of course possible, for example, that still other heavy quarks exist, but we shall not consider such possibilities further.

Turning from α_s to the parameter β, which measures the strength of the confining potential, we see from Table II that β is comparable in magnitude in all isospin states and for all kinds of quarks. However, when we assumed that a single constant would suffice for β, we got considerably poorer agreement with the data. Thus, with our particular form of the interaction, we only approximately obtain the result suggested by De Rújula *et al.*[5] that the confining interaction is the same for all quarks.

The parameter γ_{LS} measures the strength of the spin-orbit interaction. The splitting between states of a given L and different J is governed both by this interaction and by the tensor interaction. For the moment, let us consider the effect of the $\vec{L}\cdot\vec{S}$ interaction alone. In perturbation theory, for $L=1$, the splitting between the $J=2$ and $J=1$ levels is twice as great as between the $J=1$ and $J=0$ levels. We obtain essentially the perturbation result when γ_{LS} is small, but, as can be seen from Table II, in the case of $s\bar{s}$, γ_{LS} is near unity. In this case the observed level splitting varies considerably from the perturbation-theory result, and our calculation, to our knowledge, is the only one which is able to account for this fact.

We next consider the parameter γ_T which governs the strength of the tensor interaction. Recall that we made the approximation that the nondiagonal part of the tensor force could be neglected. This approximation is a good one if γ_T is small. We see from Table II that *a posteriori* our approximation is good in three cases but not good in two cases. In the cases where the approximation is poor, there will be appreciable shifting of the energy levels arising from the nondiagonal terms. Despite this defect in our calculation, we have treated the tensor term at least as well as previous workers, who have used perturbation theory on the diagonal elements of S_{12} as well as neglecting the off-diagonal elements.

As we have already remarked, the parameter a is a cutoff to enable us to obtain acceptable bound-state solutions to the Schrödinger equation. We varied this parameter only in two cases, and kept it fixed in the other cases, as can be seen from Table II. The cutoff a is between 0.4 and 0.5 fm, a value which is similar (by coincidence?) to the radius of the repulsive core in the nucleon-nucleon interaction.

The last parameter we discuss is λ, which measures the strength of the contact interaction. From Table II we see that λ is small in all cases, even in the case of the $I=1$ mesons, where it is responsible for the large splitting between the π and ρ mesons. Thus, a small value of λ does not necessarily mean that our perturbation-theory treatment is good, but only that the splitting is smaller than would be expected for $\alpha_s \gtrsim 1$ using a positronium analogy. An alternative way to proceed, not using perturbation theory, would be to smear out the δ function in the interaction and use the smeared-out potential in the Schrödinger equation. We have not done this because it would require assuming a particular form for the interaction as well as introducing at least one other parameter into the calculation.

Because our values of α_s are considerably higher than values which are rather commonly accepted, we have investigated the consequences of constraining α_s to be small. In particular, we have required α_s to be below unity for all quark-antiquark interactions and around 0.4 for charmonium. We find that in order to obtain average level spacings which are approximately in agreement with experiment, the parameter β multiplying the linear confining potential must be rather large ($\beta \approx 4-6\ \mathrm{fm}^{-2}$). With a large value of β and a small value of α_s, the ground-state levels are generally too high. We can rectify this deficiency by adding another parameter to the potential. A particularly simple way to do this is to replace the linear term βr by $\beta(r-r_0)$. The need for the parameter r_0 is in effect an admission that we can obtain approximately correct level spacings, but cannot obtain the absolute values of the levels. A similar problem was encountered by Eichten *et al.*[3] in calculating the charmonium spectrum. These authors fit only the mass difference between the $\psi'(3684)$ and $\psi(3098)$, but not the individual masses.

With α_s very small for charmonium, the $1\,^3D_1$ state would normally lie considerably below the $3\,^3S_1$ state in energy. In this picture, therefore, we probably should not interpret the observed peak at 3950 MeV as the $1\,^3D_1$ state of charmonium. But this is not a real difficulty, because at present there is no generally accepted interpretation for this peak. If the peak at 3950 MeV is not the $1\,^3D_1$ state in question, then the $1\,^3D_1$ state should exist below this energy. As yet, there is no evidence for the existence of such a state.

Turning to the mesons composed of uncharmed-quark-antiquark pairs, and using the parameter r_0, we obtain spectra which differ in some details from the spectra given in Tables III–VI. The details depend in part on whether or not we regard the evidence for the $\rho'(1250)$ as compelling. If we omit the $\rho'(1250)$ and regard the $\rho'(1600)$ as the first radially excited state of the ρ, the level spacing is of course considerably increased. This affects our calculated higher-energy levels as well, as they are also shifted to higher energy. A solution of the same general character in the $I=0$ state would not include the predicted $\omega(1379)$ state of Table IV, and a number of other levels would be raised considerably in energy.

In conclusion, with no constraints on the values of the parameters of our model, we have been able to obtain good qualitative agreement with the observed meson mass spectrum. We have also predicted the existence of a number of mesons which have not been seen. The details of our parameters and the predicted levels depended on our particular guesses about the quantum numbers of some states. With other choices for the quantum numbers of these states, we would have obtained somewhat different parameters, and the predicted levels would have shifted somewhat in energy. For phenomenological work, it is therefore very important that further experimental effort be made to pin down the quantum numbers of the observed mesons.

We found that use of relativistic kinematics and a linear confining potential tended to reduce the level spacing of the higher-lying levels below that observed by experiment. We conjecture that this same result will be true for a wide class of relativistic wave equations. We found that the phenomenological constant α_s, which governs the strength of the $1/r$ potential, varies between about 1 and 3 for our best fits. These are values which are larger than obtained previously. However, if α_s is constrained to be small, we are still able to obtain qualitatively good fits to the data by introducing another parameter, r_0. Finally, we found that spin-orbit interactions, L^2 interactions, tensor interactions, and spin-spin interactions all appeared to be needed in our fit to the observed low-lying meson states.

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*Work supported in part by the National Science Foundation and in part by the U. S. Energy Research and Development Administration.

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