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Quark-loop anomalies and the SU(4) mixing angles*

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Three mixing angles are used to describe the mixing of the three isoscalar mesons assigned to the $\underline{15}\oplus\underline{1}$ representation of SU(4). The formalism is applied to the pseudoscalar (P), vector (V), and tensor (T) multiplets, the mixing angles being determined for both linear and quadratic mass formulas. The mixing **angles** can also be independently determined from decay rates. To reduce the number of parameters involved, the effective *PVV*, *PVγ*, and *Pγγ* interactions are derived from the *VPP* interaction using **anomalies** in the partial conservation of the axial-vector current. The *VVV* vertex is also incorporated without introducing an additional parameter since a U(4) Yang-Mills effective Lagrangian is used. The mixing angles so determined in the case of the vectors and the pseudoscalars are used to reduce from five to two the number of parameters in the mass matrix. By fitting the remaining parameters to the observed masses it is found that the quadratic mass formula is strongly favored in both cases.

I. INTRODUCTION

Some time ago, one of us[1] discussed the derivation of the overall normalization of the pseudoscalar-vector-vector (PVV) interaction from that of the vector-pseudoscalar-pseudoscalar (VPP) interaction. This was applied to the pseudoscalar and vector SU(3) nonets by use of strong non-Abelian anomalies in the partial conservation of the axial-vector current (PCAC) due to divergent quark-loop graphs.[2] Since that time, several new mesons have been discovered[3] which are frequently interpreted as bound states of quarks, predominantly a charmed quark and its antiquark [in addition to the usual SU(3) quarks].[4] We discuss here the generalization of the methods of Ref. 1 to include this new quark in SU(4).

The usual discussion of the mixing of the $|8\rangle$, $|15\rangle$ and $|0\rangle$ states of the $\underline{15}\oplus\underline{1}$ representations of SU(4) to form three physical isoscalar mesons consists of either diagonalizing the mass matrix or assuming ideal mixing. The results often give poor predictions of decay rates.[5]

A compact method to describe the mixing of the basis states to form physical particle states is to use a rotation involving three Euler angles.[6] These mixing angles can be determined by mass-matrix diagonalization or, independently and more directly, by means of the decay rates of the particles involved.

In Sec. II, we introduce our mixing-angle formalism and determine the P, V, and T angles by mass-matrix diagonalization. In Sec. III, we use a U(4) version of the Yang-Mills theory as a model for VVV and VPP interactions. Using anomalies, we calculate the VVP, $PV\gamma$, and $P\gamma\gamma$ interactions, deriving hexadecimet symmetry [analogous to nonet symmetry in SU(3)] for the VVP coupling constants. In Secs. IV and V we determine the mixing angles for the vector and pseudoscalar multiplets in SU(4) from the available decay rates. The masses predicted by these angles are calculated as an overall consistency check.

Our conclusions are to be found in Sec. VI. Appendix A contains the mathematical details of our mixing-angle formalism, while Appendix B contains the $V_1 \to V_2 + V_3$ rate formula.

II. MIXING-ANGLE FORMALISM

Following Borchardt, Mathur, and Okubo,[7] we write the symmetry-breaking mass operator in the form

$$\mathfrak{M}' = T_8 + aT_{15}\,, \tag{1}$$

where a is a parameter to be determined. The elements of the mass matrix are

$$\begin{aligned} M_{ij} &= \overline{M}\delta_{ij} + A(d_{i8j} + ad_{i15j})\,,\\ M_{0i} &= B(\delta_{8i} + a\delta_{15i})\,,\\ M_{00} &= \overline{M}_0\,. \end{aligned} \tag{2}$$

Here we have used M to describe the elements of a mass matrix which may be linear, quadratic, inverse square, etc. in the masses as required. $\overline{M}$ and $\overline{M}_0$ are the SU(4)-invariant masses (or squared masses, as the case may be) of the $\underline{15}$ and $\underline{1}$ representations, respectively, while A and B are reduced matrix elements.

In the case of the vector mesons there are 5 known masses from which the 5 parameters can be determined. Nevertheless, it should be noted that Eq. (1) is not the most general possibility. Since[8]

$$\underline{15}\otimes\underline{15} = \underline{84}\oplus\underline{45}\oplus\underline{45}^*\oplus\underline{20}''\oplus\underline{15}_D\oplus\underline{15}_F\oplus\underline{1} \tag{3}$$

one may include additional $C=Y=I=0$ contributions to the mass matrix using $\underline{84}$ and $\underline{20}''$ representations. This possibility will not be pursued here, since it introduces at least two extra parameters.[9] However, it should remind us that a failure of the parameterization given by Eq. (2) does not vitiate SU(4). Generally speaking, decay rates are probably a more reliable way of determining mixing angles than mass-matrix diagonalization. This, of course, is the usual attitude taken in SU(3) meson decay phenomenology.

Because of the presence of off-diagonal elements in the mass matrix, the $|8\rangle$, $|15\rangle$, and $|0\rangle$ basis vectors will be mixed together to form the physical isoscalar mesons. We can parameterize the mixing by three Euler angles α, β, and θ in a rotation matrix $U(\alpha,\beta,\theta)$ by[10]

$$U(\alpha,\beta,\theta)=\begin{pmatrix}\cos\alpha\sin\theta+\sin\alpha\cos\beta\cos\theta & -\sin\alpha\sin\theta+\cos\alpha\cos\beta\cos\theta & \sin\beta\cos\theta\\ -\cos\alpha\cos\theta+\sin\alpha\cos\beta\sin\theta & \sin\alpha\cos\theta+\cos\alpha\cos\beta\sin\theta & \sin\beta\sin\theta\\ -\sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta\end{pmatrix}, \tag{4}$$

where

$$\begin{pmatrix}\omega\\ \phi\\ \psi\end{pmatrix}=U(\alpha_V,\beta_V,\theta_V)\begin{pmatrix}8\\ 15\\ 0\end{pmatrix} \tag{5}$$

in the vector case, and

$$\begin{pmatrix}X^0\\ \eta\\ \eta_c\end{pmatrix}=U(\alpha_P,\beta_P,\theta_P)\begin{pmatrix}8\\ 15\\ 0\end{pmatrix} \tag{6}$$

in the pseudoscalar case. This choice of Euler angles allows convenient comparison with results containing the SU(3) mixing angle θ, which is simply the limiting case $\alpha=0°$, $\beta=60°$ here, since $|0 \text{ of } SU(3)\rangle=\frac{1}{2}|15\rangle+\frac{1}{2}\sqrt{3}|0\rangle$.

Ideal mixing is given by $\alpha=0°$, $\beta=60°$, and $\theta=\tan^{-1}/\sqrt{2}=35.26°$. Since α is a measure of the octet contribution to the ψ particle, and hence governs $\psi\to PP$ decays, we expect $\alpha\approx 0°$ for all the narrow resonances.

In Table I we show the masses used for determining the mixing angles of the pseudoscalar, vector, and tensor particles. While the assignment of the vector particles is quite straightforward, some comments must be made about the pseudoscalar and the tensor particles. In the case of the pseudoscalars, there may be two candidates for the η_c, the $(c\bar{c})$ analog of the ψ. One has been reported by DESY,[11] where a state at 2800 MeV has been observed in the radiative decay of the ψ. However, this particle has not yet been observed at SLAC. Further, because its mass is 300 MeV below that of the ψ, earlier quark model estimates[12] put the decay width $\psi\to\eta_c+\gamma$ at greater than 100 keV, which has most emphatically not been observed. This would indicate the naive SU(4) quark model results are grossly in error, although it must be said that the corresponding SU(3) results have a discrepancy of about a factor of two.[13] The other candidate is a particle with mass 3455 MeV observed at SLAC[14] as an intermediate state in the decay $\psi'\to\psi\gamma\gamma$. Presuming this particle has approximately the same SU(3)-nonet content as the ψ, it would decay mainly via photon emission, as is observed. Thus, we include both cases as possible

TABLE I. Masses used in the calculation of the mixing angles. The masses in parentheses are those which are predicted, the first entry being obtained from the linear mass formula and the second from the quadratic. There are two solutions given for the pseudoscalars corresponding to different masses of the η_c. The masses are given in MeV. The masses are from Particle Data Group, Phys. Lett. 50B, 1 (1974), except where otherwise noted.

Pseudoscalar I	Pseudoscalar II	Vector	Tensor
π 138.0	π 138.0	ρ 770	A_2 1310
K 495.7	K 495.7	K^* 892.2	K^* 1421
η 548.8	η 548.8	ω 782.7	f 1270
X^0 957.6	X^0 957.6	ϕ 1019.7	f' 1516
D (1785) (2206)	D (2201) (2724)	D^* (1936) (2184)	D^{**} (2438) (2642)
F (2143) (2257)	F (2559) (2765)	F^* (2059) (2230)	F^{**} (2549) (2699)
η_c 2800 [a]	η_c 3455 [b]	ψ 3095 [c]	χ 3545 [d]

[a] See Ref. 11 [b] See Ref. 14. [c] See Ref. 3. [d] See Ref. 16.

TABLE II. Mass-matrix parameters and mixing angles corresponding to masses in Table I. The reduced matrix elements and the SU(4)-invariant masses have units GeV and GeV^2 for the linear and quadratic mass formulas, respectively. The angles are quoted in degrees.

J^P		$\bar{M}$	$\bar{M}_0$	A	B	a	α	β	θ
0^- (I)	Lin.	1.141	1.023	−0.413	−0.103	4.53	−3.719	75.53	−21.09
	Quad.	2.556	1.409	−0.262	−0.081	22.33	−0.640	74.27	−10.00
0^- (II)	Lin.	1.349	1.054	−0.413	−0.106	5.77	−2.860	75.86	−21.72
	Quad.	3.833	1.677	−0.262	−0.082	34.27	−0.415	74.60	−10.15
1^-	Lin.	1.414	1.425	−0.141	−0.099	9.77	−0.021	60.04	37.24
	Quad.	2.782	3.478	−0.235	−0.189	21.45	0.195	56.42	39.84
2^+	Lin.	1.929	1.853	−0.128	−0.090	10.42	−0.055	60.90	29.08
	Quad.	4.499	4.697	−0.350	−0.265	18.06	0.099	58.71	30.59

choices[15] for η_c, and label them I and II, respectively, in Tables I and II.

The other three new particles,[14,16] $\chi(3415)$, $\chi(3505)$, and $\chi(3545)$, have different decay properties than the η_c. All decay into hadrons, and since the $\chi(3415)$ has been reconstructed from $\chi\to\pi\pi$, $K\bar{K}$ events, it must have $J^{PG}=0^{++},2^{++},\ldots$. DESY has reconstructed[17] a $\psi'\to\chi\gamma$, $\chi\to\pi\pi$ decay with the χ mass at 3515 MeV, so that $\chi(3505)$ or $\chi(3545)$ might also have $J^{PG}=0^{++},2^{++}\ldots$. In that this would be in agreement with the charmonium prediction[12] of three states with $J^{PG}=0^{++},1^{++},2^{++}$ in the 3.4–3.6 GeV mass region, we choose to assign $\chi(3545)$ as a tensor meson.

Using these masses shown in Table I as inputs, the mass matrix can be diagonalized and the parameters and mixing angles obtained. This has been done for both the linear and quadratic mass formulas and the results are given in Table II. The masses predicted for the $D^*(u\bar{c}$ and $d\bar{c})$ and $F^*(s\bar{c})$ particles and their pseudoscalar and tensor analog are given in Table I. When these particles are observed, they will provide a good means of discriminating between the linear and quadratic mass formulas for bosons. Arguments based on SU(8) symmetry indicate that the parameter a should have the same value for both the vectors and pseudoscalars. Our results show that the ratio $a(\text{vector})/a(\text{pseudoscalar})$ has the value 0.6–1.0 for the quadratic case and 1.7–2.2 for the linear case, which would tend to favor the quadratic mass formula.

Finally, we note that β is reasonably close to the ideal value for vectors and tensors, but is off by ~15° for the pseudoscalars. The value of α is of order of 1° for the solutions considered. While this is small, it clearly does not yield proper decay widths. Taking the decay $\psi\to K^0\bar{K}^0$ as an example, $\alpha=0.5°$ and $\beta=60°$ predicts a width of ~30 keV compared to the experimental upper bound of 14 eV. More accurate methods of determining the mixing angles will be explored in the following sections.

III. EFFECTIVE LAGRANGIAN MODEL INCORPORATING NON-ABELIAN ANOMALIES

We wish to extend the SU(3) discussion[1] of the effective interaction inferred from non-Abelian PCAC anomalies to the SU(4) case. The effective PVV interaction may be written

$$\mathcal{L}^I=-\frac{N}{4\pi^2}\epsilon_{\mu\nu\sigma\tau}\,\mathrm{Tr}(\Pi\partial^\mu V^\nu\partial^\sigma V^\tau)\,, \tag{7}$$

where the couplings to the N (=3) color-degenerate fundamental quark quartets are absorbed in the definitions of the pseudoscalar quantity Π and the vector quantity V. In our case

$$\Pi=\sum_{i=0}^{15}\frac{\lambda_i}{2}P_i\frac{1}{F_i}\,, \tag{8}$$

where λ_i are the SU(4) generalizations[18] of the SU(3) λ_i's. The P_i's represent 16 pseudoscalar mesons and F_i their leptonic decay constants. For simplicity we assume that all of them are equal to the observed value for the pion,

$$F_i=F_\pi=94\text{ MeV}\,. \tag{9}$$

We write the vector V in terms of a 16-plet of vector mesons ϕ_i and the electromagnetic field as

$$V^\mu=g\sum_{i=0}^{15}\frac{\lambda_i}{2}\phi_i^\mu+eQA^\mu\,, \tag{10}$$

where g is the U(4)-invariant coupling of ϕ's to the quarks, e is the electric charge, and Q the GIM[19] (Glashow-Iliopoulos-Maiani) charge matrix $Q=\frac{1}{3}\mathrm{diag}(2,-1,-1,2)$. Duality ideas predict repetition of hadronic multiplets so one may add terms corresponding to them on the right-hand side as needed. Indeed, the $\rho'(1600)$ and $\psi'=\psi(3684)$ are prime candidates for vector-meson repetition. We shall be concerned here mainly with the lowest-mass SU(4) multiplets. Equations (7)–(10) yield

$$\mathcal{L}^{PVV} = -\frac{Ng^2}{16\pi^2 F_\pi} d_{ijk}\epsilon_{\mu\nu\sigma\tau} P^i \partial^\mu \phi_j^\nu \partial^\sigma \phi_k^\tau , \quad (11)$$

$$\mathcal{L}^{PV\gamma} = -\frac{Nge}{8\pi^2 F_\pi} \mathrm{Tr}(\lambda_i \lambda_j Q)\epsilon_{\mu\nu\sigma\tau} P^i \partial^\mu \phi_j^\nu \partial^\sigma A^\tau , \quad (12)$$

$$\mathcal{L}^{P\gamma\gamma} = -\frac{Ne^2}{8\pi^2 F_\pi} \mathrm{Tr}(\lambda_i Q^2)\epsilon_{\mu\nu\sigma\tau} P^i \partial^\mu A^\nu \partial^\sigma A^\tau . \quad (13)$$

We now consider the possibility that the 16 vector mesons ϕ_i^μ form a U(4) Yang-Mills–type field. This implies universal coupling of the ϕ's to the U(4) vector current. Consequently,

$$\mathcal{L}^{VPP} = -g f_{ijk} \partial_\mu P_i \phi_j^\mu P_k \quad (14)$$

and

$$\mathcal{L}^{VVV} = -g f_{ijk} \partial^\mu \phi_i^\nu \phi_\mu^j \phi_\nu^k . \quad (15)$$

We shall apply Eq. (15) to the decay process $\psi \to K^*\bar{K}^*$. The rate formula for the decay of a vector to two vectors can be found in Appendix B.

This defines our model. Note that (12) and (13) are derived directly from anomalies *without* the use of vector-meson dominance (VMD) or generalized VMD.[20] However, there is no incompatibility; Eqs. (12) and (13) may also be derived from (11), simple VMD, and hexadecimet symmetry.

It is intuitively plausible that generalized VMD should be compatible with Eqs. (12) and (13) as well. With every 16-plet of vector mesons, V_m, we associate a Yang-Mills universal coupling constant g_m. The original vector 16-plet will have $m=1$, so that $g=g_1$. Considering only the lowest-lying pseudoscalar 16-plet Eq. (11) generalizes to

$$\mathcal{L}^{PV_m V_n} = -\frac{N g_m g_n}{16\pi^2 F_\pi} d_{ijk}\epsilon_{\mu\nu\sigma\tau} P^j \partial^\mu \phi_{m,j}^\nu \partial^\sigma \phi_{n,k}^\tau . \quad (16)$$

Define $g_{\gamma m}$ to be the reciprocal of the coupling between γ and the ρ meson in the mth 16-plet, at zero photon mass. The normalization of the $g_{\gamma m}$'s may be inferred from the sum rule

$$1 = \sum \frac{g_m}{g_{\gamma m}} \quad (17)$$

which may be found by use of generalized VMD of the pion form factor at zero momentum transfer. (This shows that the simple VMD result $g=g_1=g_{\gamma 1}$ need not apply.) Using Eqs. (16) and (17), hexadecimet symmetry and generalized VMD, one may again derive[21] Eqs. (12) and (13). Thus, our model does not need generalized VMD but is compatible with it.

IV. VECTOR-MESON MIXING ANGLES

In order to use the results derived in the previous section, we must first find a value for the coupling constant g. By fitting to the rates for the decays $\rho\to\pi\pi$, $K^*\to K\pi$, $\phi\to K^+K^-$, $\phi\to K_L K_S$ one finds[1] $g^2/4\pi = 3.27$ and the mixing angle $\theta_V = 37.3°$. The VPP and VVV amplitudes $\psi\to K_L K_S$ and $\psi\to K^*(892) + \bar{K}^*(892)$ involve only the octet component of ψ (i.e., $-\sin\alpha \sin\beta$). Using Eqs. (14), (15), and (B1), we find

$$|\sin\alpha_V \sin\beta_V| \le 1.397\times 10^{-4} , \qquad |\sin\alpha_V \sin\beta_V| \le 0.518\times 10^{-4} \quad (18)$$

for the $K_L K_S$ and $K^*\bar{K}^*$ decays, respectively. The PVV decays $\psi\to\rho\pi$ and $\psi\to K\bar{K}^*(892)$ contain the 0, 8, and 15 components of the ψ, but as we see from Eq. (4) only α_V and β_V are involved; thus, they can be solved for explicitly. Ignoring the overall signs for the moment, we have an ambiguity in the relative signs of the amplitudes which yields two solutions. One of these solutions violates the bound in Eq. (18); the other solution is

$$\alpha_V = (-0.003\,60 \pm 0.001\,50)° , \qquad \beta_V = (59.994\,05 \pm 0.000\,55)° . \quad (19)$$

The overall sign of the amplitudes, and hence the angles themselves, was found by solving the quadratic Eq. (A2) for the parameter a using only the mixing angles. One solution was complex, while the other led to $a=19.73$. Using Eqs. (A3) and (A4), B and $\bar{M}_0$ can be eliminated in favor of A and $\bar{M}$, which can then be found by a two-parameter fit to the five vector masses. The result is given in Table III for the quadratic mass formula. The predicted masses differ from the observed masses by at most 2%. If a search is carried out by allowing the angles to vary within one standard deviation of the values found above, then the maximum discrepancy can be brought to well under 1%.

A fit was also tried for the linear mass formula, but the best fit involved discrepancies of about 10%. This is to be expected, since the parameter a,

TABLE III. Masses for vector mesons obtained from the mixing angles and a two-parameter fit to the observed masses. The parameters have the values $\bar{M} = 2.849\ \mathrm{GeV}^2$, $\bar{M}_0 = 2.872\ \mathrm{GeV}^2$, $A=-0.2633\ \mathrm{GeV}^2$, $B=-0.1853\ \mathrm{GeV}^2$, and $a=19.73$. Observed masses are from Table I.

Particle	Theoretical mass (MeV)	Observed mass (MeV)
ρ	759	770 ± 10
K^*	896	892.2 ± 0.5
ω	773	782.7 ± 0.6
ϕ	1021	1019.7 ± 0.3
ψ	3034	3095 ± 4
D^*	(2212)	···
F^*	(2263)	···

which is a function of the mixing angles alone, is predicted to be 19.7, close to the 21.5 obtained from the quadratic mass fit, but a factor of 2 different from the 9.8 of the linear mass fit (see Table I). Thus, the vector mixing angles favor a quadratic mass formula.

V. PSEUDOSCALAR MIXING ANGLES

The pseudoscalar mixing angles are more difficult to determine owing to the lack of definitive data on the η_c. As input to determine the angles, we use the ratios of the decay rates $\Gamma(\eta\to\gamma\gamma)/\Gamma(\pi^0\to\gamma\gamma)$ and $\Gamma(X^0\to\gamma\gamma)/\Gamma(X^0\to\rho\gamma)$, which effectively determine θ_P. The known rate for $\psi\to\eta\gamma$ and the bound on $\psi\to X^0\gamma$ can be used to obtain upper bounds on the deviation of α_P and β_P from ideality. As with the vector angles, there are two possible solutions. One of these leads to complex values for the mass-matrix parameters and is discarded. The other solution has $\alpha_P=-0.100\,31°$, $\beta_P=60.119\,22°$, and $\theta_P=-12.934°$. We do not quote an error for the angles since our results for α_P and β_P must be interpreted as bounds on the deviation from ideal mixing.

The fit to the masses obtained from these angles is shown in Table IV. Because of the ambiguity in the η_c mass, we did not include it in the χ^2 minimization routine. We find it predicted at 2922 MeV, although judging by the discrepancy of the η and X^0 masses, the error is likely to be 200 MeV. The value for the parameter a, as determined by the mixing angles, is 21.6. This agrees very well with the quadratic mass formula, and the SU(8) prediction that the parameter should have the same value for both the vector and pseudoscalar multiplets. The best fit for the linear mass formula is abysmal. The decay rates used and predicted in this model are given in Table V.

It is of interest to note that the bound obtained on α_P is some thirty times greater than the value of α_V. We ask if this is enough to comprehend the recent result[22] $\Gamma(\psi'\to\psi\eta)=9.8$ keV. It is conventional to assign the ψ' and $\rho'(1600)$ to a second vector-meson 16-plet V_2. We may get a rough idea of the universal Yang-Mills coupling g_2 from the

TABLE IV. Masses for pseudoscalar mesons obtained from the mixing angles and a two-parameter fit to the observed masses using a quadratic mass formula. The parameters have the values $\bar{M}=2.334$ GeV^2, $\bar{M}_0=2.904$ GeV^2, $A=-0.2462$ GeV^2, $B=-0.1499$ GeV^2, and $a=21.61$. Observed masses are from Table I.

Particle	Predicted mass (MeV)	Observed mass (MeV)
π	138.05	138.03 ± 0.005
K	482.0	495.7 ± 0.08
η	511.7	548.8 ± 0.6
X^0	1041	957.6 ± 0.3
η_c	2922	...
D	2107	...
F	2156	...

TABLE V. Decay rates found from the mixing angles $\alpha=-0.003\,60°$, $\beta=59.994\,05°$, $\theta=37.3°$ for the vectors, and $\alpha=-0.100\,31°$, $\beta=60.119\,22°$, $\theta=-12.934°$ for the pseudoscalars. Data are from Particle Data Group, Phys. Lett. 50B, 1 (1974), except where indicated. [mass (η_c) = 2922 MeV.]

Decay	Theoretical rate (keV)	Observed rate (keV)
$\psi\to K_LK_S$	0.0016	< 0.014 [a]
$\psi\to K^*(892)\bar{K}^*(892)$	0.18	0.18 ± 0.04 [a]
$\psi\to\rho\pi$	0.90	0.90 ± 0.28 [b]
$\psi\to K^*(892)\bar{K}$	0.034	< 0.041 [a]
$\rho\to\pi\pi$	171×10^3	$(150\pm10)\times10^3$
$K^*\to K\pi$	48.9×10^3	$(49.8\pm1.1)\times10^3$
$\phi\to K^+K^-$	2.05×10^3	$(1.96\pm0.14)\times10^3$
$\phi\to K_LK_S$	1.35×10^3	$(1.45\pm0.12)\times10^3$
$\pi^0\to\gamma\gamma$	7.47×10^{-3}	$(7.89\pm0.38)\times10^{-3}$ [c]
$\eta\to\gamma\gamma$	0.432	0.352 ± 0.135 [c]
$X^0\to\gamma\gamma$	5.73	< 19
$\eta_c\to\gamma\gamma$	269	...
$\rho\to\pi\gamma$	96.4	35 ± 10 [d]
$K^{*0}\to K^0\gamma$	218	75 ± 35 [e]
$\rho\to\eta\gamma$	66.5	< 160
$\phi\to\pi\gamma$	2.59	6.50 ± 1.94 [f]
$\omega\to\pi\gamma$	893	870 ± 61
$\psi\to\pi\gamma$	0.81×10^{-3}	< 0.38 [a]
$X^0\to\rho\gamma$	96.1	< 270
$\eta_c\to\rho\gamma$	82.9×10^{-3}	...
$\phi\to\eta\gamma$	159	81 ± 32 [f]
$\omega\to\eta\gamma$	6.97	< 50
$\psi\to\eta\gamma$	94.8×10^{-3}	$(95\pm29)\times10^{-3}$ [a]
$\phi\to X^0\gamma$	0.852	...
$X^0\to\omega\gamma$	11.3	< 80
$\psi\to X^0\gamma$	0.451	< 0.450 [a]
$\eta_c\to\phi\gamma$	0.262	...
$\eta_c\to\omega\gamma$	28.8×10^{-3}	...
$\psi\to\eta_c\gamma$	138	< 1 [a]

[a] See G. S. Abrams, in *Proceedings of the 1975 International Symposium on Lepton and Photon Interactions at High Energies, Stanford* (Ref. 17), p. 25, and G. J. Feldman, *ibid.*, p. 39, for recent reviews.

[b] B. Jean-Marie *et al.*, Phys. Rev. Lett. 36, 291 (1976).

[c] Weighted average of the results of Particle Data Group and those of A. Browman *et al.*, Phys. Rev. Lett. 32, 1067 (1974), and A. Browman *et al.*, *ibid.* 33, 1400 (1974).

[d] B. Gobbi *et al.*, Phys. Rev. Lett. 33, 1450 (1974).

[e] W. C. Carrithers *et al.*, Phys. Rev. Lett. 35, 349 (1975).

[f] Weighted average of the results of Particle Data Group and those of C. Bemporad, in *Proceedings of the 1975 International Symposium on Lepton and Photon Interactions at High Energies, Stanford* (Ref. 17), p. 113.

above rate by assuming ideal mixing for V_1 and V_2 and using the pseudoscalar angles that we have found. Using the anomaly Eq. (16) we obtain the enormous value $g_2=20g_1$ which implies the incredible result $\Gamma(\rho'(1600)\to\pi^+\pi^-)=160$ GeV. Thus, unless the leakage of the ψ' is considerably greater than the ψ, it would appear the ψ' and ρ' do not belong to the same multiplet. Apropos of these remarks we should mention the well-known fact that the assumption of a universal Regge slope also forbids putting $\rho'(1600)$ and $\psi'(3684)$ in the same multiplet.

The fact that the first term in Eq. (17) is empirically no more than 15% larger than unity indicates a large $g_{\gamma 2}$ to accompany the sizable g_2. However, the $g_{\gamma 2}$ inferred from $\psi'\to e^+e^-$ is of normal size. To interpret the ψ' as we have done then seems to imply that the series Eq. (17) consists of various large terms beyond the first canceling each other, an unlikely situation. Our conclusion is that $\psi'\to\psi\eta$ rate is far from being understood.

VI. CONCLUSION

We have described the mixing of the isoscalar mesons in SU(4) multiplets by means of three mixing angles. Cast in this form, the eigenvectors of the isoscalar mesons are readily obtained from the decay rates available and can be found more accurately than through the diagonalization of a mass matrix. Using a model for the VVV, VVP, and VPP interactions based on the strong anomalies in PCAC and on the assumption that the vector mesons form a U(4) Yang-Mills-type field, the decay rates can be parameterized by these mixing angles and just one independent coupling constant g. As can be seen from Table V, the theoretical rates are generally in good agreement with experiment, underlining the effectiveness of this simple parameterization. In particular, it is interesting to note that $\psi\to K^*\bar{K}^*$ affords the first opportunity to test a VVV interaction as could arise from a Yang-Mills model. In that we have assumed hexadecimet symmetry for the coupling constants, the $\rho\to\pi\gamma$ and $K^*\to K\gamma$ rates remain in poor agreement with the present theory, as in SU(3). Should these rates be confirmed by further experiment (or the mass of the η_c be found at much less than 3025 MeV, resulting in a large prediction for the $\psi\to\eta_c\gamma$ rate), then a theory with SU(4) or hexadecimet symmetry breaking would have to replace the approach taken here.

The angles obtained from this model were used to reduce the number of mass-matrix parameters from five to two. We find that linear mass formulas for the vector and pseudoscalar multiplets fit poorly when the rate information is taken into account. It is also found that the symmetry-breaking parameter a is the same for the vectors and pseudoscalars, as predicted by SU(8), only for the quadratic mass formula.

Note added. Since this paper was submitted, a possible candidate[23] for the D particle has been discovered at 1865 MeV. This mass is lower than expected from the quadratic mass formula and is in better agreement with the linear mass formula. It should be pointed out that if the $E(1420)$ were chosen in place of the $X^0(962)$ as the ninth member of the SU(3) pseudoscalar nonet, the mass predicted by the quadratic formula would be at least 200 MeV lower, and the discrepancy in the value for the mixing angle β (predicted by the mass formulas) would be resolved (~58° instead of ~75°, see Table II).

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APPENDIX A: DETAILS OF THE MIXING-ANGLE FORMALISM

In the discussion of the mixing problem we introduced an orthogonal matrix $U(\alpha,\beta,\theta)$ in Eq. (4). This matrix is used to diagonalize the mass matrix

$$M=\begin{bmatrix} \bar{M}+A\left(-\frac{1}{\sqrt{3}}+\frac{a}{\sqrt{6}}\right) & \frac{A}{\sqrt{6}} & B \\ \frac{A}{\sqrt{6}} & \bar{M}-(\frac{2}{3})^{1/2}Aa & Ba \\ B & Ba & \bar{M}_0 \end{bmatrix} \tag{A1}$$

via

$$\hat{M}=UMU^{-1}\,.$$

The three equations $\hat{M}_{ij}=0$ $(i\neq j)$ can be used to eliminate three of the mass-matrix parameters in favor of the three mixing angles. In particular, a is determined from the mixing angles via the quadratic equation

$$C_1a^2+C_2a+C_3=0\,, \tag{A2}$$

where

$$C_1=3U_{11}U_{21}U_{31}\,,$$

$$C_2=-\sqrt{2}\,U_{11}U_{21}U_{31}+3U_{12}U_{22}U_{32}+U_{21}U_{31}U_{12} - U_{23}U_{21}U_{32}{}^2+U_{33}U_{31}U_{22}{}^2\,,$$

$$C_3 = -\sqrt{2}\, U_{12}U_{22}U_{32} - U_{32}U_{22}U_{11} - U_{22}U_{23}U_{31}^2 + U_{33}U_{32}U_{21}^2\,,$$

where we have made use of the properties of orthogonal matrices. The other two conditions allow us to find B in terms of A [with a determined from (A2)],

$$B = \frac{A}{\sqrt{6}}\,\frac{-\sqrt{2}\,U_{31}U_{32} - U_{31}^2 + U_{32}^2 + 3aU_{31}U_{32}}{aU_{33}U_{31} - U_{33}U_{32}} \tag{A3}$$

and $\bar{M}_0$ in terms of $\bar{M}$ and A,

$$\bar{M}_0 = -\frac{1}{U_{33}U_{13}}\Big\{\bar{M}(U_{31}U_{11} + U_{32}U_{12}) + \frac{A}{\sqrt{6}}\,[-\sqrt{2}\;U_{31}U_{11} + U_{31}U_{12} + U_{32}U_{11} + a(U_{31}U_{11} - 2U_{32}U_{12})] + B[U_{31}U_{13} + U_{33}U_{11} + a(U_{32}U_{13} + U_{33}U_{12})]\Big\}. \tag{A4}$$

Given fixed values for the mixing angles, use of (A2) to (A4) determines the diagonal elements of $\hat{M}$ as a linear function of $\bar{M}$ and A,

$$\hat{M}_{ii} = U_{i1}^2\Big[\bar{M} + A\Big(-\frac{1}{\sqrt{3}} + \frac{a}{\sqrt{6}}\Big)\Big] + U_{i2}^2[\bar{M} - (\tfrac{2}{3})^{1/2}Aa] + U_{i3}^2\bar{M}_0 + 2U_{i1}U_{i2}\frac{A}{\sqrt{6}} + 2U_{i1}U_{i3}B + 2U_{i2}U_{i3}Ba\,. \tag{A5}$$

In the special case with $\alpha = 0$, the equations reduce to

$$B = \frac{A}{\sqrt{6}}\tan\beta \tag{A6}$$

and

$$\bar{M}_0 = \bar{M} - \frac{Aa}{\sqrt{6}}\,(3 - \tan^2\beta)\,. \tag{A7}$$

Setting $\alpha = 0$ and $\beta = 60^\circ$ implies $A = \sqrt{2}\,B$ and $\bar{M}_0 = \bar{M}$, and forces θ to be 35.3°, the ideal case.

APPENDIX B: VECTOR → VECTOR + VECTOR RATE FORMULA

Except for the process $V_1 \to V_2V_3$ all the rate formulas we use are well known so we do not state them. If the coupling constant for $V_1 \to V_2V_3$ is f, then we find

$$\Gamma = \frac{f^2}{4\pi}\,\frac{2}{3}\,\frac{p_{\text{c.m.}}^3}{m_1^2}\,S\,, \tag{B1}$$

where $p_{\text{c.m.}}$ is the center-of-mass momentum and

$$S = 1 + m_1^2\Big(\frac{1}{m_2^2} + \frac{1}{m_3^2}\Big) + \frac{1}{3}\Big(\frac{m_1 p_{\text{c.m.}}}{m_2 m_3}\Big)^2\,, \tag{B2}$$

where m_i is the mass of V_i. Apart from the factor S (B1) is the same as the $V \to PP$ rate formula. The factor S arises owing to the relativistic spin summation in the final state. Nonrelativistically S is two.

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