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Motion in the Schwarzschild field: A reply*

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Rebuttal is given to the criticisms of Cavalleri and Spinelli. It is shown that their point of view could lead to the conclusion that a test particle in Minkowski space moves with a speed approaching that of light. It is reaffirmed that a test particle in the Schwarzschild field crosses the Schwarzschild radius with a speed less than that of light.

Cavalleri and Spinelli[1] (hereafter referred to as CS) have taken exception to the point of view expressed in my earlier treatment[2] (hereafter referred to as I) of the motion of a freely falling test particle in the Schwarzschild field. A number of previous authors[3] had concluded that the speed of such a particle approaches the speed of light as the particle approaches the Schwarzschild radius, $r=2m$. The point of I was to establish that the test particle does indeed cross $r=2m$ with a speed less than the speed of light.

The point of view taken by CS is that the particle's motion is best understood by referring it to the coordinate system, S, in which the line element takes the form

$$ds^2=(1-2m/r)dt^2-(1-2m/r)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2)\,. \tag{1}$$

The system S is preferred, according to CS, because it is at rest with respect to the source, as evidenced by the fact that it is a static, rigid frame of reference, in which a certain curvature invariant is independent of the time coordinate. The velocity of the test particle may be defined by the Landau-Lifshitz prescription,[4] which leads to the expression

$$V^2=({g_{01}}^2-g_{00}g_{11})\left(g_{00}+g_{01}\frac{dx^1}{dx^0}\right)^{-2}\left(\frac{dx^1}{dx^0}\right)^2 \tag{2}$$

for the square of the velocity of a particle moving in the x^1 direction, where x^0 is the timelike coordinate. (Although other velocities have been discussed, this is the one upon which the present controversy centers.) It was shown in I that Eq. (2), when applied to a radially inward, timelike geodesic in the coordinate system S, yields

$$V^2=[a^2-(1-2m/r)]/a^2\,, \tag{3}$$

where a is a positive constant that is determined by the initial conditions. It is clear from Eq. (3) that $V^2\to 1$ as $r\to 2m$.

As an illustration of why the CS point of view can be misleading, let me supplement the arguments presented in I with an example in Minkowski space, which has the line element

$$ds^2=dt^2-dx^2-dy^2-dz^2\,. \tag{4}$$

New coordinates, u and v, may be introduced in the region $x>|t|>0$, as shown by Bergmann,[5] by means of the equations

$$u=\tanh^{-1}(t/x)\,,\quad v=(x^2-t^2)^{1/2}\,. \tag{5}$$

As indicated in Fig. 1, the curves $v=$ constant are hyperbolas in the (x,t) plane whose asymptotes are the null lines through the origin, and those for $u=$ constant are straight lines through the origin. The null lines that bound the region may be described as the limit of the hyperbolas $v=$ constant as $v\to 0$, or equivalently as the pair of lines obtained in the two limits of $u=$ constant as $u\to\pm\infty$. In terms of these new coordinates, the line ele-

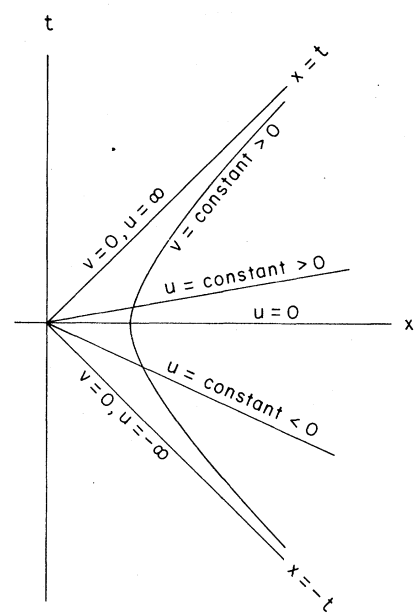

FIG. 1. The relation between two sets of coordinates in Minkowski space. The y and z coordinates are not shown.

ment (4) takes the form

$$ds^2 = v^2du^2 - dv^2 - dy^2 - dz^2 . \qquad (5)$$

It is clear from Eq. (5) that u is a timelike coordinate and v is a spacelike coordinate. Furthermore, the u direction is that of a hypersurface-orthogonal, timelike Killing vector. Both this coordinate system in Minkowski space and the system S in Schwarzschild space are static, rigid frames of reference, in which curvature invariants are independent of the time. In both cases the region under consideration has a null boundary that is approached either as the spatial coordinate approaches its lower limit or as the temporal coordinate approaches either positive or negative infinity.[6] The line element (5) could, in fact, be made to look even more like the Schwarzschild form (1) by a further coordinate transformation, as shown by Bergmann.[5]

If a timelike geodesic x = constant is expressed in terms of the u and v coordinates, it is easily seen that the velocity expression (2) yields $V^2 = \tanh^2 u$. As a test particle approaches the boundary of the region along such a geodesic (reaching that boundary, as in the Schwarzschild case, in a finite proper time), it is clear that $V^2 \to 1$.

It seems to me, then, that if one were to accept the CS argument for the Schwarzschild case, one would likewise have to conclude that the particle's speed approaches that of light in this Minkowski-space example. In both cases it seems to me to be more reasonable to take the point of view that since the limiting trajectory of the "rest" trajectories is null, the reason that the particle's speed appears to be approaching that of light is that it is measured relative to a reference system whose "stationary" clocks are moving with speeds approaching that of light.

Let me now comment briefly on certain other statements in the CS paper.

In the discussion of Eqs. (4) and (5) of their paper, CS attribute to me the claim that the test particle's speed cannot *approach* that of light as $r \to 2m$. My point was that, although in certain coordinate systems $V^2 \to 1$ as $r \to 2m$, this result can lead to a mistaken notion of the particle's actual behavior at $r = 2m$ since, as shown in I, the particle's actual speed there is less than that of light.

In their abstract, CS say that I found $V \leq \frac{1}{3}$ if $V = 0$ at $r = \infty$. This statement is presumably based on Eq. (10) of I. This equation is valid, as CS indicate, only in the coordinate system S^*, which is locally Minkowskian at a point on the Schwarzschild surface. As noted in I, however, every other locally Minkowskian frame at that point would likewise lead to a speed less than that of light. Since at any point of an arbitrary reference frame it is always possible to introduce a comoving locally Minkowskian frame, it seems clear that in any frame the particle's speed at $r = 2m$ is less than that of light.

To summarize my point of view, let me first agree with CS that in a number of interesting coordinate systems, including some that are nonsingular at $r = 2m$, certain expressions for a test particle's speed approach unity as the Schwarzschild radius is approached. One should not thereby be led to suppose, however, that the particle actually crosses that radius with the speed of light. To reiterate the concluding statement of I, it seems to me to be clear that freely falling test particles in the Schwarzschild field cross $r = 2m$ with a speed less than the speed of light.

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[1] G. Cavalleri and G. Spinelli, preceding paper, Phys. Rev. D 15, 3065 (1977).

[2] A. I. Janis, Phys. Rev. D 8, 2360 (1973).

[3] See, for example, Refs. 1–5 of I.

[4] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, 3rd ed. (Addison-Wesley, Reading, Mass., 1971), p. 250.

[5] P. G. Bergmann, Phys. Rev. Lett. 12, 139 (1964).

[6] In the Schwarzschild case, this is easily seen in the representation given by M. D. Kruskal, Phys. Rev. 119, 1743 (1960).