Note on motion in the Schwarzschild field

G. Cavalleri
CISE-Centro Informazioni Studi Esperienze, Segrate, Milano and Istituto di Fisica dell'Universitá di Milano, Italy

G. Spinelli
Istituto di Matematica del Politecnico di Milano, Italy
(Received 30 December 1975; revised manuscript received 14 June 1976)

We reassert, against Janis's criticism, that the speed V of a freely falling test particle in the Schwarzschild field approaches the speed of light as it approaches the Schwarzschild radius ($r_S = 2m$) if measured in a reference system S at rest with the source. Janis's result, i.e., $V < 1$ for a freely falling test particle even for $r \to 2m^+$ (in particular, $V \leq 1/3$ if $V = 0$ at $r \to \infty$), holds only with respect to a system S^* a general point of which moves with $V' \to 1^-$ when it approaches the Schwarzschild radius.

The radial motion of a test particle in a Schwarzschild field was treated by some authors[1,2] who reached the same conclusion that the particle velocity V approaches the light velocity when the distance r between the test particle and the source approaches the Schwarzschild radius $2m$. This result is obtained in a reference system S at rest with the source.

In an earlier paper[2] we considered not only the "renormalized" velocity (i.e., the velocity measured by local real clocks and meter sticks) but also the "semirenormalized" velocity (local meter sticks but far clocks or, equivalently, ideal clocks unaffected by the gravitational field) and completely "unrenormalized" velocity (which should be measured by ideal clocks and meter sticks unaffected by the gravitational field). When the test particle approaches the Schwarzschild radius, all the above three velocities approach the corresponding light velocity (either renormalized, semirenormalized, or unrenormalized, respectively).

In a recent paper[3] Janis first criticized all the preceding results, and then by some coordinate transformations he obtained in a particular reference system S^* a velocity which does not approach the light velocity when the test particle approaches the Schwarzschild radius. Obviously one can choose any system one wants and we acknowledge that S^* is more convenient for particular purposes. However, we do not agree with Janis's criticism against the results obtainable in S. In particular we assert that it is just S^* and not S that locally (for $r \to 2m$) approaches the light speed with respect to the source. Let us examine some details of the problem.

First Janis says, writing the line element as

$$ds^2 = (1-2m/r)dt^2 - (1-2m/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \qquad (1)$$

"it is clear that an observer at rest at the Schwarzschild radius, $r = 2m$, must move with the speed of light," and then "the reason a test particle's speed approaches the speed of light is that it is measured by a family of observers whose speeds approach the speed of light."

We are not discussing here the behavior of the particle for any value of $r > 0$. On the contrary, we treat only a much more simple and restricted problem in order to avoid doubts of interpretation. Precisely we limit our reasonings to the region

$$r > 2m\,, \qquad (2)$$

where the Killing vector field is timelike, and thus one has a stationary geometry.

Let us consider a radius issuing from the source and, for $r > 2m$, a succession of coordinate clocks (considered as pointlike particles) each at rest with the source. Obviously their trajectory in space-time has $dr = d\theta = d\phi = 0$ and then the proper time registered by them is given by

$$ds^2 = (1 - 2m/r)dt^2\,, \qquad (3)$$

where r gives the position of the clock.

The time rates of such coordinate clocks become slower and slower as the clocks approach the Schwarzschild radius, and the limit for $r \to 2m^+$ would be $ds^2 = 0$. But this does not mean, as Janis infers, that the clocks are moving with a velocity increasing from one to another and tending to the velocity of light. On the contrary, each clock is at rest (for $r > 2m$) and only their time rates change. Moreover, we do not have a family of observers (one for each clock) but a single observer (the single coordinate system which has all the above-mentioned clocks) *at rest* with the source.

Let us now consider the other objection on the test particle velocity. Janis[3] obtains for the observer S at rest with the source his Eq. (4)

$$V^2=[a^2-(1-2m/r)]/a^2\ , \qquad (4)$$

where a is a finite positive constant depending on the initial conditions. In the case of a particle coming from infinity with initial speed $dr/dt=-v_0$ [that is when $a=(1-v_0^2)^{-1/2}$], Eq. (3) gives our renormalized velocity [given by Eq. (4) of Ref. 2]. It is clear that $V\to 1^-$ as $r\to 2m^+$. Janis observes that $g_{\mu\nu}t^\mu t^\nu=1$ for $r>2m$ ($t^\mu=dx^\mu/ds$). Strangely enough, from this fact, which implies the trajectory to be timelike, he infers that the particle velocity cannot *approach* the light velocity (as r approaches $2m$) even in S. However, $g_{\mu\nu}t^\mu t^\nu$ is a discontinuous function of V; it is equal to 0 for a lightlike world line but it is

$$\lim_{V\to 1^-} g_{\mu\nu}t^\mu t^\nu=1\ . \qquad (5)$$

Consequently there is no contradiction between (4) and (5); i.e., the particle velocity can tend to the light velocity for $r\to 2m^+$.

On the contrary, Janis considers (4) and (5) in contradiction and as due to a nonconvenient choice of the coordinates. Consequently Janis performs two successive transformations.

First he transforms from the coordinate system (t, r, θ, ϕ) to the advanced Eddington-Finkelstein coordinates (v, r, θ, ϕ), where

$$v=t+r+2m\ln(r-2m)\ . \qquad (6)$$

Then he transforms from (v, r, θ, ϕ) to his coordinates (x^0, x^1, x^2, x^3) defined by[3]:

$$\begin{aligned} x^0&=\frac{1}{\sqrt{2}}\left[v\left(1+\frac{v}{8m}\right)-\left(r-2m\right)\left(1-\frac{v}{4m}\right)-m(\theta-\tfrac{1}{2}\pi)^2-m\phi^2\right],\\ x^1&=\frac{1}{\sqrt{2}}\left[v\left(1+\frac{v}{8m}\right)+\left(r-2m\right)\left(1-\frac{v}{4m}\right)-m(\theta-\tfrac{1}{2}\pi)^2-m\phi^2\right],\\ x^2&=r(\theta-\tfrac{1}{2}\pi)\ ,\\ x^3&=r\phi\ . \end{aligned} \qquad (7)$$

In such a reference frame S^*, the particle velocity is

$$V^2\to(2a^2-1)^2/(2a^2+1)^2\ , \qquad (8)$$

which is less than unity for every finite positive value of a (which depends on the initial conditions).

This fact is not due to the absence of singularities at $r=2m$ for S^*. Indeed, as noted by Janis himself, even the use of the advanced Eddington-Finkelstein coordinates (which do not present difficulties at $r=2m$) leads to $V\to 1^-$ for $r\to 2m^+$.

In our opinion the reason lies in the fact that the coordinate system of S^* is in motion with respect to the source, and its velocity approaches the light speed when approaching the Schwarzschild radius.

That the reference system S is at rest while it is S^* that moves with respect to the source, can be proved by the following reasoning. Let us calculate the curvature invariant[4]:

$$\mathfrak{I}=R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}=48m^2/r^6\ . \qquad (9)$$

Notice that for a point fixed with respect to S (i.e., $r=$const), the relevant $\mathfrak{I}$ is constant, i.e., it does not depend on t. This fact, for $r>2m$ (where one has a stationary geometry), is a confirmation that S is actually at rest with the source.

Since $\mathfrak{I}$ is an invariant in value, to find its expression as a function of the coordinates in S^* we only have to transform r in (6). The relationship between r and the coordinates x^0, x^1, x^2, x^3 of S^* can be obtained by Eq. (7) and it is

$$\begin{aligned} \frac{1}{\sqrt{2}}(x^0+x^1)&=4m\left(1+\frac{1}{\sqrt{2}}\,\frac{x^0+x^1}{r-m}\right)\\ &\quad\times\left(\frac{3}{2}+\frac{1}{\sqrt{2}}\,\frac{x^0-x^1}{r-m}\right)\\ &\quad-m\left(\frac{x^2}{r}\right)^2-m\left(\frac{x^3}{r}\right)^2\ . \end{aligned} \qquad (10)$$

Then, if we take a point at rest with respect to S^*, i.e., if we fix x^1, x^2, and x^3, in this case $\mathfrak{I}$ is no longer independent of the time coordinate since it depends on x^0. Since the source is assumed to be constant, the dependence of $\mathfrak{I}$ on x^0 means that S^* is moving in a real sense with respect to the source.

Now that it is proved that for $r>2m$, S is at rest while S^* is moving, let us calculate the relative velocities. One of the two systems (S^*) is not rigid, thus the relative velocities change from point to point. Let us take a point P at rest with S, i.e., whose coordinates r, θ, and ϕ are constant. For simplicity let it be $\theta=\frac{1}{2}\pi$ and $\phi=0$. The point P is observed by S^* as moving with a speed dx^1/dx^0. Taking r as a parameter we obtain by (6) and (7) that

$$\lim_{r\to 2m^+}\frac{dx^1}{dx^0}=1^-\ . \qquad (11)$$

This means that the points at rest with the coordinate system S^* are moving with an increasing speed as they approach the Schwarzschild radius and that such speed tends to the light speed.

Now that such results have been stated, it is not at all surprising that Janis finds for a freely falling test particle a speed with respect to S^* which does not approach the light speed. Let us make this simple example. A point particle A is moving with a speed approaching the light speed. From A we shoot a point particle B with a velocity <1 with respect to the first one. For the rest observer both A and B do approach the light speed. In our case, A represents a general point at rest with S^* while B is the test particle.

What is interesting in Janis's treatment (i.e., in chosing S^*) is the possibility of discriminating between the different approach to unity according to the different initial conditions.

However, in conclusion, we can just reverse Janis's statement after his Eq. (1) and say that the reason a test particle's speed in S^* does *not* approach the speed of light is that it is measured by an observer the points at rest of whom approach the speed of light with respect to the gravitational source.

We again emphasize that we have not studied the behavior of the particle in the region $r<2m$ (where the Killing vector field is spacelike and one cannot have a stationary geometry). In this paper we only considered[5] the region $r>2m$.

ACKNOWLEDGMENTS

One of the authors (G.S.) is glad to acknowledge that his contribution to the research was performed under the auspices of the C.N.R. Gruppo per la Fisica-Matematica.

[1] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, 3rd ed. (Addison-Wesley, Reading, Mass., 1971); Ya. Zel'dovich and I. Novikov, *Relativistic Astrophysics* (Univ. of Chicago Press, Chicago, 1971), Vol. 1; J. Jaffe and I. Shapiro, Phys. Rev. D 6, 405 (1972); F. Markley, Am. J. Phys. 41, 45 (1973).

[2] G. Cavalleri and G. Spinelli, Lett. Nuovo Cimento 6, 5 (1973).

[3] A. I. Janis, Phys. Rev. D 8, 2360 (1973).

[4] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 822.

[5] However, in passing we give a hint for solving a puzzling problem relevant to the crossing of $r=2m$. The paths of both a massive particle and a photon can be continued through the Schwarzschild radius and into the interior region. Geodesic motion preserves the character of the respective tangent vectors, timelike and null. In this invariant sense, the particle and the photon move differently not only outside the Schwarzschild radius but also at that radius and within it. Now, if the velocity as measured by S for $r>2m$ is such that

$$\lim_{r\to 2m^+} V=1^- ,$$

then any extension to (or across) $r=2m$ will necessarily be discontinuous. The explanation is simple in terms of S and S^*. The material framework S (with relevant clocks) cannot be at rest for $r<2m$ where the geometry is no longer stationary. There is a discontinuity for the material parts pertaining to S. On the contrary, there is no such discontinuity for S^* which always judges the massive particles to have a speed lower than the light speed. In this sense S^* is more convenient for studying the crossing of the Schwarzschild radius. However, this has nothing to do with the fact that, for $r>2m$, S (and not S^*) is at rest with the source and that, for S, a freely falling particle has a speed tending to unity when $r\to 2m^+$.