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Stress-energy tensor near a charged, rotating, evaporating black hole*

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The recently developed two-dimensional stress-energy regularization techniques are applied to the two-dimensional analog of the Reissner-Nordström family of black-hole metrics. The calculated stress-energy tensor in all cases contains the thermal radiation discovered by Hawking. Implications for the evolution of the interior of a charged black hole are considered. The calculated stress-energy tensor is found to diverge on the inner, Cauchy, horizon. Thus the effect of quantum mechanics is to cause the Cauchy horizon to become singular. The stress-energy tensor is also calculated for the "most reasonable" two-dimensional analog of the Kerr-Newman family of black-hole metrics. Although the analysis is not as rigorous as in the Reissner-Nordström case, it appears that the correct value for the Hawking radiation also appears in this model.

Since the discovery of Hawking[1] that the gravitional field of a collapsing object will induce the emission of thermal radiation, much work has been done studying the properties of the radiation far from the collapsing object.[2,3,4] An outstanding problem is understanding the physics of the emission process in the neighborhood of the collapse. Owing to ambiguities in the definition of "particle" in regions where the curvature of spacetime is large, current efforts are aimed at calculating the effective stress energy of the radiation. The stress-energy tensor is also needed for any reasonable calculation of the back reaction on the gravitational field due to the emission of the thermal radiation.

Recently, Davies, Fulling, and Unruh[15] have succeeded in regularizing the stress-energy tensor of a massless scalar field in an arbitrary two-dimensional spacetime, using the method of geodesic point separation. When applied to the two-dimensional equivalent of a Schwarzchild black hole, their stress-energy tensor correctly reproduced the radiation flux at infinity.

In this paper I shall show that the two-dimensional formalism developed in Ref. 5 can be applied to a wider class of spherically symmetric spacetime metrics, the Reissner-Nordström metrics, and will in all cases reproduce the correct radiation flux at infinity.[6] I also find that the stress-energy tensor diverges on that segment of the inner horizon which is a Cauchy horizon. Since the most general black-hole metric, the Kerr-Newman metric, is only axisymmetric, no two-dimensional spacetime metric can correctly represent the full four-dimensional spacetime. Choosing what seems to be the "best" two-dimensional analog to the Kerr-Newman spacetime, I find that the calculated stress-energy tensor predicts the correct temperature for a Kerr-Newman black hole.

The metric of a two-dimensional spacetime can in general be written in the double null form

$$ds^2 = C(u, v)du\,dv\,, \tag{1}$$

where $C(u, v)$ is a conformal factor. Since all two-dimensional spacetimes are conformally flat, the massless scalar wave equation will be

$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}\phi = 0\,. \tag{2}$$

Null coordinates $\bar{u}, \bar{v}$ have always been found to exist in which the solutions to Eq. (2) take the form of the flat-space normal-mode solutions,

$$\begin{aligned} &\exp(-i\omega\bar{u})/(4\pi|\omega|)^{1/2}\,,\\ &\exp(-i\omega\bar{v})/(4\pi|\omega|)^{1/2}\,. \end{aligned} \tag{3}$$

The in-vacuum state on which the calculation is based is the state annihilated by the field operators with $\omega > 0$. Imposing the condition that the geometry be asymptotically flat, the in-vacuum state is uniquely defined by requiring the modes to be plane waves near past null infinity.

In Ref. 5, Davies, Fulling, and Unruh have calculated the expectation value of the stress-energy tensor in the in-vacuum state, using geodesic point separation. The expectation value is

$$T_{\mu\nu} = \theta_{\mu\nu} + \frac{R}{48\pi}g_{\mu\nu}\,, \tag{4}$$

where

$$\begin{aligned} &\theta_{\bar{u}\bar{u}} = -(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{u}\bar{u}}\,,\\ &\theta_{\bar{v}\bar{v}} = -(12\pi)^{-1}C^{1/2}(C^{-1/2})_{,\bar{v}\bar{v}}\,,\\ &\theta_{\bar{u}\bar{v}} = \theta_{\bar{v}\bar{u}} = 0\,, \end{aligned} \tag{5}$$

C is the conformal factor of Eq. (1), and R is the scalar curvature. This expression for $T_{\mu\nu}$ is conserved ($\nabla^{\mu}T_{\mu\nu} = 0$), but is not conformally invariant.

Now consider the rapid gravitational collapse of a thin charged shell of matter, as described in

Ref. 4, with appropriate modifications to allow for the presence of charge. In order to treat a two-dimensional problem, ignore the spherical angle coordinates θ and ϕ. Let R be the radius of the shell. The exterior spacetime metric is given by

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \tag{6}$$

or, in double null form,

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) du\, dv\,, \tag{7}$$

where $u = t - r^*$, $v = t + r^*$, r^* is the usual tortoise coordinate defined by

$$\frac{dr^*}{dr} = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}. \tag{8}$$

The interior metric is flat:

$$ds^2 = d\tau^2 - dr^2$$

or (9)

$$ds^2 = dU\, dV$$

in double null form.

The remaining problem is simply to relate the three sets of null coordinates, (u, v), (U, V), and $(\bar{u}, \bar{v})$. By arguments similar to those in Ref. 5, one obtains for the advanced-time coordinates

$$v = \bar{v} = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{-1/2} V \tag{10}$$

for all relevant times provided the collapse is sufficiently rapid. For the retarded-time coordinates, one has $u = \bar{u}$ for retarded times before the collapse begins, and

$$u = \frac{r_+^2}{(M^2 - Q^2)^{1/2}} \ln(A_+ - \bar{u}) - \frac{r_-^2}{(M^2 - Q^2)^{1/2}} \ln(A_- - \bar{u}) + B(\bar{u}) \tag{11}$$

for retarded times long after the collapse has begun. Here $\bar{u} = A_\pm$ are the equations of the future outer and inner horizons, $r_\pm$ is defined by $r_\pm = M \pm (M^2 - Q^2)^{1/2}$, and $B(\bar{u})$ is a slowly varying function of $\bar{u}$, dependent on the exact nature of the collapse, which does not affect the final results. One also obtains

$$\bar{u} = \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right)^{-1/2} U \tag{12}$$

for all times relevant to the calculation.

The conformal factor needed for Eq. (5) can now be found by combining Eqs. (7), (10), (11), and (12). It is given by

$$C(\bar{u}, \bar{v}) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{13}$$

before collapse, with r an implicit function of $\bar{u}$ and $\bar{v}$. Long after collapse has begun, it is given by

$$C(\bar{u}, \bar{v}) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left(\frac{r_+^2}{(M^2 - Q^2)^{1/2}} \frac{1}{(A_+ - \bar{u})} - \frac{r_-^2}{(M^2 - Q^2)^{1/2}} \frac{1}{(A_- - \bar{u})} + O(1)\right), \tag{14}$$

with $O(1)$ representing terms of order unity.

Evaluating $T_{\mu\nu}$ using Eq. (13) for the conformal factor, and then transforming back to the more usual u, v and r, t coordinates, one finds

$$\begin{aligned}
T_{uu} &= (24\pi)^{-1}\left(-\frac{M}{r^3} + \frac{3(M^2 + Q^2)}{2r^4} - \frac{3MQ^2}{r^5} + \frac{Q^4}{r^6}\right), \\
T_{vv} &= (24\pi)^{-1}\left(-\frac{M}{r^3} + \frac{3(M^2 + Q^2)}{2r^4} - \frac{3MQ^2}{r^5} + \frac{Q^4}{r^6}\right), \\
T_{uv} = T_{vu} &= (24\pi)^{-1}\left(-\frac{M}{r^3} + \frac{2M^2}{r^4} + \frac{3Q^2}{2r^4} - \frac{4MQ^2}{r^5} + \frac{3Q^4}{2r^6}\right),
\end{aligned} \tag{15}$$

$$\begin{aligned}
T_{tt} &= (24\pi)^{-1}\left(-\frac{4M}{r^3} + \frac{7M^2}{r^4} + \frac{6Q^2}{r^4} - \frac{14MQ^2}{r^5} + \frac{5Q^4}{r^6}\right), \\
T_{rr} &= (24\pi)^{-1}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-2}\left(-\frac{M^2}{r^4} + \frac{2MQ^2}{r^5} - \frac{Q^4}{r^6}\right), \\
T_{rt} &= T_{tr} = 0\,.
\end{aligned}$$

Recalculating $T_{\mu\nu}$ using Eq. (14), one finds that only the value of T_{uu} has changed, as compared to $T_{\mu\nu}$ in the u, v coordinate system in Eq. (15). Discarding terms which die off for large values of u, T_{uu} is

$$T_{uu} = (24\pi)^{-1}\left(\frac{M^2 - Q^2}{2r_+^4} - \frac{M}{r^3} + \frac{3(M^2 + Q^2)}{2r^4} - \frac{3MQ^2}{r^5} + \frac{Q^4}{r^6}\right). \tag{16}$$

The effect of the collapse is to add a constant term, $(M^2 - Q^2)/48\pi r_+^4$, to the expression for T_{uu}. It will appear at large r as an outward-going flux of radiation, whose magnitude precisely matches the equivalent result from the analysis of Hawking,

$$\frac{1}{2\pi}\int_0^\infty \frac{\omega\, d\omega}{\exp(\omega/kT) - 1} = \frac{\pi}{12}(kT)^2, \tag{17}$$

where, for a Reissner-Nordström black hole, the temperature is given by

$$T = \frac{(M^2 - Q^2)^{1/2}}{2\pi k r_+^2}. \tag{18}$$

Furthermore, if one takes the limit of T_{vv} as $r\to r_+$, one finds it approaches $-(M^2-Q^2)/48\pi r_+^4$, a flux of negative energy through the horizon which precisely balances the flux at infinity. Transforming $T_{\mu\nu}$ to a Kruskal-type coordinate system regular across the outer horizon, one finds that $T_{\mu\nu}$ is finite and well behaved there.

Thus, the two-dimensional stress-energy calculation reproduces the Hawking result not just for the Schwarzchild solution, but for the entire Reissner-Nordström class of black-hole metrics. This result increases one's confidence that the two-dimensional calculation accurately depicts some of the physics of the four-dimensional real world.

If the collapse is sufficiently rapid, the calculation of $T_{\mu\nu}$ can be extended inwards to the inner horizon at $r=r_-$. The stability of such horizons has long been in question, but little progress has been made in studying them.[7]

Evaluating $T_{\mu\nu}$ near $r=r_-$, one finds that $T_{\mu\nu}$ is as given in the u,v coordinate system in Eqs. (15) and (16), except that the term $(M^2-Q^2)/48\pi r_+^4$ in Eq. (16) has been replaced by a similar constant term, $(M^2-Q^2)/48\pi r_-^4$. Evidently, there is a flux of energy, similar to the Hawking radiation, traveling along the $u=-\infty$ segment of the inner horizon. Again evaluating $T_{\mu\nu}$ in a Kruskal-type coordinate system, this time one which is regular across the inner horizon, one finds that the stress energy is finite on the $u=-\infty$ segment of the inner horizon, but diverges on the $v=\infty$ segment of the inner horizon, the segment which is a Cauchy horizon. This singularity on the Cauchy horizon both closes off the throat of the Reissner-Nordström interior (exterior to the shell of matter), and in a sense "clothes" the naked singularity of the analytically extended Reissner-Nordström interior.

While the inclusion of the emission of charged particles and the back reaction of the metric is needed to obtain a realistic evolutionary scenario (e.g., where the black hole will quickly discharge), the above result is still interesting, as it shows the quantum mechanics of matter fields can drastically influence the internal structure of a black hole.

Now consider the most general black-hole metric, the Kerr-Newman spacetime. The metric is given in Boyer-Lindquist coordinates by

$$ds^2=\left(1-\frac{2Mr}{\Sigma}+\frac{Q^2}{\Sigma}\right)dt^2+\frac{4Mar\sin^2\theta}{\Sigma}d\phi dt-\frac{A}{\Sigma}\sin^2\theta\, d\phi^2-\frac{\Sigma}{\Delta}dr^2-\Sigma\, d\theta^2, \tag{19}$$

where $\Sigma=r^2+a^2\cos^2\theta$, $\Delta=r^2-2Mr+a^2+Q^2$, and $A=(r^2+a^2)^2-a^2\Delta\sin^2\theta$.

Since the Kerr-Newman metric possesses only axial, not spherical, symmetry, no two-dimensional spacetime can fully represent the four-dimensional geometry. An obvious choice for a "best possible" two-dimensional approximation of the Kerr-Newman metric is to take the metric on the symmetry axis: $\theta=0,\pi$ and $d\theta=d\phi=0$. The two-dimensional spacetime is then described by the metric

$$ds^2=\left(\frac{\Delta}{\Sigma}\right)dt^2-\left(\frac{\Sigma}{\Delta}\right)dr^2, \tag{20}$$

where now $\Sigma=r^2+a^2$. On the symmetry axis the metric also has the desirable property that $g_{tt}=0$ defines the horizon, whereas in general it defines the ergosurface.

Since no two-dimensional model can properly describe angular momentum, I shall make no attempt to analyze a rotating collapse, but rather simply calculate the stress-energy tensor for the static metric of Eq. (20), transformed to the double null form

$$ds^2=\left(\frac{\Delta}{\Sigma}\right)du\,dv. \tag{21}$$

As usual, $u=t-r^*$, $v=t+r^*$, and now r^* is defined by $dr^*/dr=\Sigma/\Delta$.

Calculating the stress-energy tensor, one obtains

$$\begin{aligned}T_{uu}=T_{vv}&=-\frac{1}{192\pi}\left(\frac{\Delta_{,r}{}^2-4\Delta}{\Sigma^2}+\frac{2(\Delta\Delta_{,r}\Sigma_{,r}+2\Delta^2)}{\Sigma^3}-\frac{3\Delta^2\Sigma_{,r}{}^2}{\Sigma^4}\right),\\ T_{uv}&=\frac{1}{48\pi}\left(\frac{\Delta}{\Sigma^2}-\frac{\Delta\Delta_{,r}\Sigma_{,r}+\Delta^2}{\Sigma^3}+\frac{\Delta^2\Sigma_{,r}{}^2}{\Sigma^4}\right).\end{aligned} \tag{22}$$

Since the effects of gravitational collapse have not been included, there is no constant term in T_{uu} representing the Hawking radiation, as there was in Eq. (16). To find the Hawking flux, we again consider the value of T_{vv} at $r=r_+$, the energy flux through the horizon, where now $r_+=M+(M^2-a^2-Q^2)^{1/2}$. Since $\Delta=0$ at $r=r_+$, T_{vv} is simply

$$\begin{aligned}T_{vv}\Big|_{r=r_+}&=-\frac{1}{192\pi}\left(\frac{\Delta_{,r}{}^2}{\Sigma^2}\right)\Bigg|_{r=r_+}\\ &=-\frac{1}{48\pi}\frac{M^2-Q^2-a^2}{(r_+{}^2+a^2)^2}.\end{aligned} \tag{23}$$

This is exactly the negative of the Hawking flux at infinity as predicted by Eq. (17), since the temperature of a Kerr-Newman black hole is given by

$$T=\frac{1}{2\pi k}\frac{r_+-M}{r_+{}^2+a^2}. \tag{24}$$

Thus, the thermal radiation discovered by Hawking is present even in this two-dimensional model of the Kerr-Newman netric. As this model certainly possesses extremely limited validity, it is then perhaps even more interesting that the Hawking radiation appears so persistently in the stress-energy tensor.

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[1]S. W. Hawking, Commun. Math. Phys. 43, 199 (1975).
[2]R. M. Wald, Commun. Math. Phys. 43, 9 (1975).
[3]J. B. Hartle and S. W. Hawking, Phys. Rev. D 13, 2188 (1976).
[4]W. G. Unruh, Phys. Rev. D 14, 870 (1976).
[5]P. C. W. Davies, S. A. Fulling, and W. G. Unruh, Phys. Rev. D 13, 2720 (1976).
[6]I have recently learned that the Reissner-Nordström calculation has been done independently by P. C. W. Davies, Proc. R. Soc. London (to be published), with particular application to the $Q > M$ naked-singularity case.
[7]See, for example, S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge Univ. Press, London, 1973), Chap. 5, Sec. 5.