Spectral-function sum rules and the pion electromagnetic mass difference at finite temperature*

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Spectral representations are derived for current correlation functions in relativistic many-body theory. Equal-time current commutation relations allow the derivation of spectral-function sum rules at finite temperature. It is shown that the spectral representation for the Schwinger term in the usual time-space current commutator has a finite temperature-dependent part. The mass relations between the vector and axial-vector meson masses derivable from Weinberg sum rules remain unaltered. These results are used to evaluate the temperature dependence of the pion electromagnetic mass difference $m_{\pi^+} - m_{\pi^0}$ in the soft-pion limit.

I. INTRODUCTION

The nature of the dependence of symmetry on temperature and density in relativistic many-particle systems has been investigated recently.[1-5] The general approach has been to determine the minima of the effective potential in order to see whether a given symmetry is spontaneously broken or not, and further, to obtain the critical temperature and critical density at which symmetry changes may be expected.[2] This is done by extending the usual field-theory techniques to the many-particle system by defining the thermodynamic Green's functions, or the temperature Green's functions, to be the ground-state expectation value of the time-ordered products of fields averaged over the thermodynamic ensemble.[6,7] The particle spectrum becomes a function of the temperature and the density in a many-particle system, and the spectrum reflects the symmetry changes. It is of interest to investigate further consequences of the temperature-dependent field theory.

In the present paper the formalism of temperature-dependent field theory is extended to define thermodynamic current correlation functions. Spectral representations are derived for the thermodynamic current correlation functions in Sec. II, and it is shown that equal-time current commutation relations allow the derivation of spectral-function sum rules at finite temperature. The spectral representation for the Schwinger term[8,9] in the time-space current commutator at equal times has a finite temperature-dependent part. If chiral $SU(2)\times SU(2)$ current algebra is assumed the new thermodynamic spectral functions satisfy Weinberg sum rules.[10,11] The usual input of single-particle saturation of the spectral functions used to obtain mass and coupling-constant relations from these sum rules yields results obtained earlier by Weinberg.[10] This is to be expected because the position of the pole in the single-particle thermodynamic Green's function remains unaltered in the many-body formulation[5,7] when quantum corrections are ignored. An application of the sum rules to multiparticle production is under investigation.[12]

In Sec. III the pion electromagnetic mass difference at finite temperature is evaluated.[13] On using current algebra in the soft-pion limit the pion mass difference is related[14] to current correlation functions and to the photon propagator which is now temperature dependent. To order e^2 the mass difference is found to be

$$\Delta m_\pi{}^2(T)=\left(\frac{e^2}{4\pi}\right)\left(m_\rho{}^2\frac{9}{2\pi}\ln 2-1.76 m_\rho T\right).$$

At extremely high temperatures the electromagnetic mass difference is seen to vanish.

II. SPECTRAL-FUNCTION SUM RULES

A. The structure factor

The thermodynamic average of the product of two currents $J_\mu^a(x)$ and $J_\nu^b(y)$, where a, b are internal-symmetry indices and μ, ν are Lorentz indices, is defined as

$$\begin{aligned}\Delta^{ab}_{\mu\nu>}(x,y)&\equiv\langle\langle J_\mu^a(x)J_\nu^b(y)\rangle\rangle\\&=\mathrm{Tr}[e^{-\beta(H-\mu N)}J_\mu^a(x)J_\nu^b(y)]/\mathrm{Tr}(e^{-\beta(H-\mu N)}).\end{aligned}\tag{1}$$

In Eq. (1) $\beta=1/k_B T$ is the inverse temperature, where k_B is the Boltzmann constant set equal to unity, and μ is the chemical potential. Also, H is the Hamiltonian and N is the number operator. The double angular brackets after the first equality sign in Eq. (1) denote thermal averaging. We shall ignore the effects of a nonzero chemical potential in the following and consider only thermal excitations of bosons above the vacuum.

The cyclic property of the trace[6] allows us to write the Hermitian conjugate of (1) as

$$[\Delta^{ab}_{\mu\nu>}(x,y)]^* = \langle\langle J^b_\nu(y) J^a_\mu(x)\rangle\rangle = \Delta^{ab}_{\mu\nu<}(x,y)\,. \tag{2}$$

As a matrix in the indices (μ, a, x) and (ν, b, y) the above functions $\Delta^{ab}_{\mu\nu>,<}(x,y)$ are Hermitian so that

$$\Delta^{ab}_{\mu\nu>,<}(x,y) = [\Delta^{ba}_{\nu\mu>,<}(y,x)]^*\,, \tag{3}$$

and the functions are positive semidefinite,

$$\Delta^{ab}_{\mu\nu>,<}(x,y) \geq 0\,. \tag{4}$$

The two functions $\Delta^{ab}_{\mu\nu>}(x,y)$ and $\Delta^{ab}_{\mu\nu<}(x,y)$ are the same owing to the Hermiticity of J^a_μ, except that they are evaluated at different values of the arguments (μ, a, x) and (ν, b, y). We therefore have

$$\Delta^{ab}_{\mu\nu>,<}(x,y) = \Delta^{ba}_{\nu\mu<,>}(y,x) = [\Delta^{ab}_{\mu\nu<,>}(x,y)]^*\,. \tag{5}$$

If the medium is homogeneous the functions depend only on the coordinate difference $(x_0 - y_0, \vec{x} - \vec{y})$.

Owing to the cyclic nature of the trace, and the time translations being generated by H, the functions satisfy the relations

$$\Delta^{ab}_{\mu\nu<}(x,y) \equiv \Delta^{ab}_{\mu\nu<}(x_0 - y_0, \vec{x} - \vec{y}) = \Delta^{ab}_{\mu\nu>}(x_0 - y_0 - i\beta, \vec{x} - \vec{y})\,. \tag{6}$$

It can be shown that the Fourier transforms of $\Delta^{ab}_{\mu\nu>}$ and $\Delta^{ab}_{\mu\nu<}$ defined by

$$\Delta^{ab}_{\mu\nu>,<}(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \Delta^{ab}_{\mu\nu>,<}(\omega, \vec{k}) \tag{7}$$

satisfy the relations

$$\Delta^{ab}_{\mu\nu>,<}(\omega, \vec{k}) = \Delta^{ba}_{\nu\mu>,<}(\omega, \vec{k})^*\,, \tag{8}$$

$$\Delta^{ab}_{\mu\nu>,<}(\omega, \vec{k}) \geq 0\,, \tag{9}$$

$$\Delta^{ab}_{\mu\nu>}(\omega, \vec{k}) = \Delta^{ab}_{\mu\nu<}(-\omega, -\vec{k})^* = \Delta^{ba}_{\nu\mu<}(-\omega, -\vec{k})\,, \tag{10}$$

and

$$\Delta^{ab}_{\mu\nu<}(\omega, \vec{k}) = e^{-\beta\omega} \Delta^{ab}_{\mu\nu>}(\omega, \vec{k})\,. \tag{11}$$

Following the practice of nonrelativistic many-body theory[6,7] it is convenient to define the structure factor

$$S^{ab}_{\mu\nu}(x,y) = \langle\langle \{J^a_\mu(x), J^b_\nu(y)\}\rangle\rangle = \Delta^{ab}_{\mu\nu>}(x,y) + \Delta^{ab}_{\mu\nu<}(x,y)\,. \tag{12}$$

Its Fourier transform defined by

$$S^{ab}_{\mu\nu}(x,y) = \frac{1}{(2\pi)^4} \int d^4k\, e^{-ik(x-y)} S^{ab}_{\mu\nu}(\omega, \vec{k}) \tag{13}$$

satisfies the relations

$$S^{ab}_{\mu\nu}(\omega, \vec{k}) = S^{ab}_{\mu\nu}(-\omega, -\vec{k}) = S^{ab}_{\mu\nu}(\omega, \vec{k})^* \geq 0 \tag{14}$$

and

$$\begin{aligned} S^{ab}_{\mu\nu}(\omega, \vec{k}) &= (1 + e^{-\omega\beta}) \Delta^{ab}_{\mu\nu>}(\omega, \vec{k}) \\ &= (1 + e^{\omega\beta}) \Delta^{ba}_{\nu\mu<}(\omega, \vec{k}) \\ &= (1 + e^{\omega\beta}) \Delta^{ab}_{\mu\nu>}(-\omega, -\vec{k})\,. \end{aligned} \tag{15}$$

In internal-symmetry space $S^{ab}_{\mu\nu}(\omega, \vec{k})$ is a symmetric tensor. We shall assume that the ground state is invariant under internal-symmetry transformations so that

$$S^{ab}_{\mu\nu}(\omega, \vec{k}) = \delta_{ab} S_{\mu\nu}(\omega, \vec{k})\,. \tag{16}$$

Furthermore, symmetry in the Lorentz indices μ and ν, and positivity dictate that

$$S_{\mu\nu}(\omega, \vec{k}) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) S_1(k^2) + \frac{k_\mu k_\nu}{k^2} S_2(k^2) \tag{17}$$

with

$$S_1(k^2) \geq 0 \tag{18}$$

and

$$S_2(k^2) \geq 0\,. \tag{19}$$

Finally, we note that for conserved currents

$$k^\mu S_{\mu\nu}(k^2) = 0\,,$$

so that in this case

$$S_2(k^2) = 0\,. \tag{20}$$

B. The current commutator

The thermodynamic expectation value of the current commutator is defined by

$$\begin{aligned} \langle\langle [J^a_\mu(x), J^b_\nu(y)]\rangle\rangle &= \mathrm{Tr} e^{-\beta H} [J^a_\mu(x), J^b_\nu(y)] / \mathrm{Tr} e^{-\beta H} \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} [\Delta^{ab}_{\mu\nu>}(\omega, \vec{k}) - \Delta^{ab}_{\mu\nu<}(\omega, \vec{k})] \\ &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tanh\left(\frac{\beta\omega}{2}\right) S^{ab}_{\mu\nu}(\omega, \vec{k})\,. \end{aligned} \tag{21}$$

Using Eqs. (15) and (17) we have

$$\langle\langle [J^a_\mu(x), J^b_\nu(y)]\rangle\rangle = \delta_{ab} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tanh\left(\frac{\beta\omega}{2}\right) \left[-\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) S_1(k^2) + \frac{k_\mu k_\nu}{k^2} S_2(k^2)\right]\,. \tag{22}$$

For the commutator of time components of the currents at equal times we set $\mu = \nu = 0$ in (22) and obtain

$$\langle\langle [J_0^a(x), J_0^b(y)]\rangle\rangle_{x_0=y_0} = 0 \tag{23}$$

for any value of β.

The commutator of the time component of the current with the space component of the current is

$$\langle\langle [J^{0a}(x), J^{ib}(y)]\rangle\rangle = \delta_{ab}\int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \tanh\left(\frac{\beta\omega}{2}\right)\left(\frac{\omega k^i}{k^2}\right)[S_1(k^2)+S_2(k^2)]\,. \tag{24}$$

At equal times Eq. (24) becomes

$$\langle\langle [J^{0a}(x), J^{ib}(y)]\rangle\rangle_{x^0=y^0} = \delta_{ab}(-i\partial_i)\int \frac{d^4k}{(2\pi)^4} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\omega \tanh\left(\frac{\beta\omega}{2}\right)[S_1(k^2)+S_2(k^2)]\,. \tag{25}$$

Using the identity

$$\int_0^\infty dm^2\delta(m^2-k^2) = 1\,,$$

we express the structure factors in terms of the usual spectral variable m^2 to obtain

$$\langle\langle [J^{0a}(x), J^{ib}(y)]\rangle\rangle_{x^0=y^0} = \delta_{ab}(-i\partial_i)\int_0^\infty \frac{dm^2}{2\pi}\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\frac{S_1(m^2)+S_2(m^2)}{m^2}\right)\tanh[\tfrac{1}{2}\beta(\vec{k}^2+m^2)^{1/2}]\,. \tag{26}$$

In the limit of zero temperature we have

$$\lim_{\beta\to\infty}\tanh\left(\frac{\beta\omega}{2}\right) = \begin{cases} +1, & \text{for } \omega>0 \\ -1, & \text{for } \omega<0 \end{cases} = \epsilon(\omega)\,, \tag{27}$$

and Eq. (26) reduces to

$$\langle\langle [J^{0a}(x), J^{ib}(y)]\rangle\rangle_{x^0=y^0,\beta\to\infty} = -i\partial_i\delta(\vec{x}-\vec{y})\delta_{ab}\int_0^\infty \frac{dm^2}{2\pi}\left(\frac{S_1(m^2)+S_2(m^2)}{m^2}\right)\,. \tag{28}$$

The commutator appearing on the left side of (28) is known to be nonzero on general grounds[8] and we may write

$$[J^{0a}(x), J^{ib}(y)]_{x^0=y^0} = iC^{abc}J^{ic}(x)\delta(\vec{x}-\vec{y}) - i\mathcal{S}_{ab}\partial_i\delta(\vec{x}-\vec{y})\,, \tag{29}$$

where C^{abc} is the group structure constant of the symmetry group of the current algebra, and $\mathcal{S}_{ab}$ is the Schwinger term.

Substituting (29) into (28) we obtain

$$\langle\langle \mathcal{S}_{ab}\rangle\rangle = \delta_{ab}\langle\langle \mathcal{S}\rangle\rangle = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\omega\tanh\frac{\beta\omega}{2}\left(\frac{S_1(k^2)+S_2(k^2)}{k^2}\right) \tag{30}$$

as the spectral representation for the Schwinger term at finite temperature. We now show that in canonical field theory $\langle\langle \mathcal{S}\rangle\rangle$ has a finite many-body contribution besides the usual zero-temperature term obtained by taking the vacuum expectation value of the commutator in (29). Consider for simplicity electrodynamics of complex scalar fields Φ and the photon field A_μ, in which the currents are

$$J^0(x) = ie(\Phi^\dagger(x)\dot{\Phi}(x) - \dot{\Phi}^\dagger(x)\Phi(x)) + 2e^2\Phi^\dagger(x)\Phi(x)A^0(x)\,, \tag{31}$$
$$J^i(x) = e(\Phi^\dagger\partial^i\Phi - \partial^i\Phi^\dagger\Phi) + 2e^2\Phi^\dagger\Phi A^i\,.$$

Then canonical commutation relations for the fields lead to

$$\langle\langle [J^0(x), J^i(y)]\rangle\rangle = -ie^2[\partial_i\delta(\vec{x}-\vec{y})] \times\langle\langle(\Phi^\dagger(x)\Phi(y)+\Phi^\dagger(y)\Phi(x))\rangle\rangle\,. \tag{32}$$

The Fourier decomposition of the fields then leads to

$$\langle\langle [J^0(x), J^i(y)]\rangle\rangle = -i\partial_i\delta(\vec{x}-\vec{y})\frac{e^2}{(2\pi)^3} \times\int\frac{d^3k}{\omega}[1+n_B(\omega)+\bar{n}_B(\omega)]\,, \tag{33}$$

where

$$n_B(\omega) = 1/(e^{\beta\omega}-1)$$

is the Bose-Einstein distribution function.

We thus have additional finite contributions to the Schwinger term in many-body theory. At zero temperature n_B and $\bar{n}_B$ vanish and the usual field-theory term remains on the right side of (33). It may be noted that in nonrelativistic theory the Schwinger term in (33) reduces to

$$\frac{e^2}{(2\pi)^3}\int d^3k\,\frac{n_B(\omega)}{m},$$

leading to the longitudinal sum rule

$$\frac{Ne^2}{m}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tanh\left(\frac{\beta\omega}{2}\right)\left(\frac{S_1(k^2)}{\omega}\right). \tag{34}$$

The absence of the term $S_2(k^2)$ is due to current conservation.

In the algebra of field models the Schwinger term is a c number and the current commutator is given by

$$[J^{0a}(x),J^{ib}(y)]_{x0=y0}=iC^{abc}J^{ic}(x)\delta(\vec{x}-\vec{y})-i\delta^{ab}\frac{m_\rho^{\,2}}{f_\rho^{\,2}}\partial_i\delta(\vec{x}-\vec{y}), \tag{35}$$

and the corresponding sum rule is

$$\frac{m_\rho^{\,2}}{f_\rho}=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\omega\tanh\left(\frac{\beta\omega}{2}\right)[S_1(k^2)/k^2]. \tag{36}$$

In spinor field theory again similar conclusions can be drawn regarding the presence of a finite, temperature-dependent contribution to the Schwinger term.

C. The time-ordered product of currents

The thermodynamic average of the time-ordered product of currents is defined by

$$i\Delta^{ab}_{\mu\nu F}(x,y)=\theta(x_0-y_0)\Delta^{ab}_{\mu\nu>}(x,y)+\theta(x_0-y_0)\Delta^{ab}_{\mu\nu<}(x,y) \tag{37}$$

with $\Delta^{ab}_{\mu\nu>,<}$ given by (1). In terms of the structure factors the Fourier transform of (37) is

$$i\Delta^{ab}_{\mu\nu F}(k^2)=i\delta_{ab}\int\frac{d\omega'}{2\pi}\left[\frac{1}{\omega-\omega'+i\epsilon}\left(\frac{1}{1+e^{-\omega'\beta}}\right)-\frac{1}{\omega-\omega'-i\epsilon}\left(\frac{1}{1+e^{\omega'\beta}}\right)\right]\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)S_1(\omega',\vec{k})+\frac{k_\mu k_\nu}{k^2}S_2(\omega',\vec{k})\right]$$
$$-i\delta_{ab}\frac{g_{\mu0}g_{\nu0}}{k^2}\int\frac{d\omega'}{2\pi}\,\omega'\tanh\left(\frac{\beta\omega'}{2}\right)[S_1(\omega',\vec{k})+S_2(\omega',\vec{k})]. \tag{38}$$

The structure factor $S_{\mu\nu}$ is related to the temperature-dependent spectral function $\rho_{\mu\nu}(\omega,\vec{k})$ in the following way. The cyclic property (11) can be implemented by writing

$$\Delta^{ab}_{\mu\nu>}(\omega,\vec{k})=\rho_{\mu\nu}(\omega,\vec{k})[1+n_B(\omega)],$$
$$\Delta^{ab}_{\mu\nu<}(\omega,\vec{k})=\rho_{\mu\nu}(\omega,\vec{k})n_B(\omega),$$

so that

$$\rho^{ab}_{\mu\nu}(\omega,\vec{k})=\Delta^{ab}_{\mu\nu>}(\omega,\vec{k})-\Delta^{ab}_{\mu\nu<}(\omega,\vec{k})=\tanh\left(\frac{\beta\omega}{2}\right)S^{ab}_{\mu\nu}. \tag{39}$$

In the limit $\beta\to\infty$ we have

$$\rho^{ab}_{\mu\nu}(\omega,\vec{k})=\epsilon(\omega)S^{ab}_{\mu\nu}(\omega,\vec{k}).$$

In terms of $\rho_{\mu\nu}$ the causal function is

$$\Delta^{ab}_{\mu\nu F}(\omega,\vec{k})=\delta_{ab}\int\frac{d\omega'}{2\pi}\left\{\frac{\rho_{\mu\nu}(\omega',\vec{k})[1+n_B(\omega')]}{\omega-\omega'+i\epsilon}-\frac{\rho_{\mu\nu}(\omega',\vec{k})n_B(\omega')}{\omega-\omega'-i\epsilon}\right\}, \tag{40}$$

which, for $\beta\to\infty$, reduces to the causal function of the usual field theory:

$$\Delta^{ab}_{\mu\nu F}(\omega,\vec{k})=\delta_{ab}\int\frac{d\omega'}{2\pi}\frac{\rho_{\mu\nu}(\omega',\vec{k})}{\omega-\omega'+i\epsilon}. \tag{41}$$

For a single particle the spectral function $\rho_{\mu\nu}$ is

$$\rho_{\mu\nu}(\omega',\vec{k})=\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)2\pi\delta(\omega^2-\omega'^2)\,\epsilon(\omega-\omega'). \tag{42}$$

From Eqs. (39) and (40) it is clear that $\rho_{\mu\nu}$ represents the discontinuity in $\Delta_{\mu\nu F}$ across the real axis in the complex frequency plane.

D. Weinberg sum rules at finite temperature

Consider spectral functions for vector and axial-vector currents. In SU(2) × SU(2) current algebra the equality of the Schwinger terms in the vector and the axial-vector current commutators gives

$$\int_0^\infty\frac{d\omega^2}{2\pi}\tanh\left(\frac{\beta\omega}{2}\right)\left[\frac{S_1^V(k^2)}{k^2}-\left(\frac{S_1^A(k^2)+S_2^A(k^2)}{k^2}\right)\right]=0. \tag{43}$$

The structure factor S_2^V is absent because of current conservation. In terms of spectral functions we have

$$\int_0^\infty dm^2\left[\frac{\rho_1^V(m^2,\beta)}{m^2}-\left(\frac{\rho_1^A(m^2,\beta)+\rho_2^A(m^2,\beta)}{m^2}\right)\right]=0. \tag{44}$$

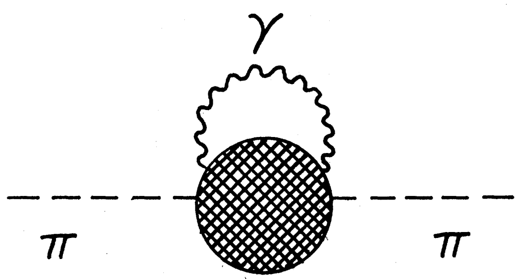

FIG. 1. Feynman diagram for the electromagnetic self-energy of the pion.

This is Weinberg's first sum rule.[10] The spectral functions here are of course temperature dependent and the states appearing in the matrix elements defining the spectral functions are weighted by Boltzmann factors.

As a direct consequence of the field-current identity we have for vector currents V_μ and axial-vector currents A_μ

$$\langle\langle[\partial_0 V_i^a(x) - \partial_i V_0^a(x), V_j^b(y)]\rangle\rangle_{x^0=y^0} = \langle\langle[\partial_0 A_i^a(x) - \partial_i A_0^a(x), A_j^b(y)]\rangle\rangle_{x^0=y^0} \tag{45}$$

leading to the second Weinberg sum rule[10,11]

$$\int_0^\infty \frac{d\omega^2}{2\pi}\tanh\left(\frac{\beta\omega}{2}\right)[S_1^V(k^2) - S_1^A(k^2)] = 0\,, \tag{46}$$

or

$$\int_0^\infty dm^2[\rho_1^V(m^2,\beta) - \rho_1^A(m^2,\beta)] = 0\,. \tag{47}$$

In the specific case of single-particle saturation the spectral functions in (44) and (47) reduce to those of the zero-temperature theory. It can be shown[5,7] that the propagators of free field theory have their singularity structure unaltered on going over to temperature-dependent field theory. On saturating the specral functions by the ρ meson, the axial vector A_1, and the pion, using the field-current identity[15]

$$V^\mu = (m_\rho^2/f_\rho)\rho^\mu\,, \tag{48}$$

and the corresponding relation for the axial-vector mesons

$$\langle 0|A^\mu(x)|A_1\rangle = \frac{m_A^2}{f_A}\frac{\epsilon^\mu}{\sqrt{2\omega}}\,, \tag{49}$$

and the partial conservation of the axial-vector current[15]

$$\partial_\mu A^\mu(x) = F_\pi m_\pi^2 \varphi_\pi(x)\,, \tag{50}$$

we obtain from (44) and (47) the well-known relations

$$m_\rho^2/f_\rho^2 = m_A^2/f_A^2 + F_\pi^2 \tag{51}$$

and

$$(m_\rho^2/f_\rho)^2 = (m_A^2/f_A)^2\,. \tag{52}$$

Use of the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation[16]

$$F_\pi^2 m_\rho^2/2f_\rho \tag{53}$$

then leads to the Weinberg relation

$$m_{A_1}^2 = 2m_\rho^2\,. \tag{54}$$

The relations (51), (52), and (54) are consequences of the theory at the single particle or the tree approximation. At this level of the theory the couplings and masses remain unchanged from the zero-temperature theory even though decay widths will show thermal broadening due to modifications in the phase space brought about by the many-body aspects of the theory at finite temperature.[5]

III. THE PION ELECTROMAGNETIC MASS DIFFERENCE AT FINITE TEMPERATURE

Let us apply the results of Sec. II in the evaluation of the electromagnetic mass difference of pions

$$\Delta m_\pi^2(\beta) = m_{\pi^+}^2 - m_{\pi^0}^2 \tag{55}$$

at finite temperature. To order e^2 it is given by the Feynman diagram of Fig. 1. The photon propagator at finite temperature is

$$\begin{aligned} iD_{\mu\nu}(q^2) &= i(-g_{\mu\nu} + q_\mu q_\nu/q^2)\left[\frac{1}{q^2+i\epsilon} - 2\pi i\delta(q^2)n_B(q_0)\right] \\ &= i(-g_{\mu\nu} + q_\mu q_\nu/q^2)D(q^2)\,. \end{aligned} \tag{56}$$

The mass difference is then given by[13,14]

$$\Delta m_\pi^2(\beta) = \frac{-ie^2}{2}\int\frac{d^4q}{(2\pi)^4}D^{\mu\nu}(q^2)M_{\mu\nu}(p,q)\,, \tag{57}$$

where

$$M_{\mu\nu}(p,q) = i\int e^{iqz}d^4z[\langle\langle\pi^+(p)|T^*(V_\mu^{\rm em}(z)V_\nu^{\rm em}(0))|\pi^+(p)\rangle\rangle - \langle\langle\pi^0(p)|T^*(V_\mu^{\rm em}(z)V_\nu^{\rm em}(0))|\pi^0(p)\rangle\rangle]\,. \tag{58}$$

The matrix elements in (58) can be evaluated in the soft-pion limit. On reducing out the pions we have

$$M_{\mu\nu}(p,q)=\frac{-i(p^2-m_\pi^2)}{(F_\pi m_\pi^2)^2}\int e^{iqz}dz\int dxdy\, e^{ip(x-y)}[\langle\langle T^*(\partial_\rho A^{\rho(-)}(x)\partial_\lambda A^{\lambda(+)}(y)V_\mu^{\text{em}}(z)V_\nu^{\text{em}}(0))\rangle\rangle$$
$$-\langle\langle T^*(\partial_\rho A^{\rho(3)}(x)\partial_\lambda A^{\lambda(3)}(y)V_\mu^{\text{em}}(z)V_\nu^{\text{em}}(0))\rangle\rangle] .$$

Using current algebra in the soft-pion limit $M_{\mu\nu}(0,q)$ can be written as

$$M_{\mu\nu}(0,q)=\frac{2i}{F_\pi^2}\int dz\, e^{iqz}[\langle\langle T^*(V_\mu^{(3)}(z)V_\nu^{(3)}(0))\rangle\rangle-\langle\langle T^*(A_\mu^{(3)}(z)A_\nu^{(3)}(0))\rangle\rangle]$$
$$=\frac{2i}{F_\pi^2}[i\Delta_{\mu\nu F}^V(q^2,\beta)-i\Delta_{\mu\nu}^A(q^2,\beta)], \tag{59}$$

where the temperature dependence of the causal functions of currents is explicitly shown.

Now the mass difference is given by

$$\Delta m_\pi^2=\frac{3ie^2}{F_\pi^2}\int\frac{d^4q}{(2\pi)^4}D(q^2)\int\frac{dq_0'}{2\pi}[S_1^V(q_0',\vec{q})-S_1^A(q_0',\vec{q})]\left[\frac{1-n_B(q_0')}{q_0-q_0'+i\epsilon}-\frac{n_B(q_0')}{q_0-q_0'-i\epsilon}\right]. \tag{60}$$

We use Weinberg sum rules (51) and (52), obtained by single-particle saturation of the structure functions by ρ, A_1, and π mesons, to write

$$\Delta m_\pi^2=\frac{3e^2i}{F_\pi^2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{q^2+i\epsilon}-\frac{\pi i}{q_0}n_B(q_0)\delta(q_0-|\vec{q}|)+\frac{\pi i}{q_0}n_B(-q_0)\delta(q_0+|\vec{q}|)\right]\left(\frac{m_\rho^4}{f_\rho^2}\right)$$
$$\times\left\{\frac{1}{q^2-m_\rho^2}-\frac{1}{q^2-m_A^2}-\frac{\pi i}{q_0}n_B(q_0)[\delta(q_0-\mathcal{E}_\rho)-\delta(q_0-\mathcal{E}_A)]+\frac{\pi i}{q_0}n_B(-q_0)[\delta(q_0+\mathcal{E}_\rho)-\delta(q_0+\mathcal{E}_A)]\right\}.$$

Hence

$$\Delta m_\pi^2(\beta)=\frac{i6e^2m_\rho^2}{(2\pi)^4}\int\frac{d^4q}{q^2}\left(\frac{1}{q^2-m_\rho^2}-\frac{1}{q^2-m_A^2}\right)-3e^2\int\frac{d^3q}{(2\pi)^3}\left[\frac{n_B(|\vec{q}|)}{|\vec{q}|}\right]+6e^2\int\frac{d^3q}{(2\pi)^3}\left[\frac{n_B(\mathcal{E}_\rho)}{\mathcal{E}_\rho}-\frac{n_B(\mathcal{E}_A)}{2\mathcal{E}_A}\right]. \tag{61}$$

The first term in (61) is the usual zero-temperature expression of Das *et al.*[14] A Wick rotation of the k_0 integration gives

$$\Delta m_\pi^2(\beta^{-1}=0)=\frac{3e^2}{4\pi}\frac{m_\rho^2}{2\pi}\ln 2 . \tag{62}$$

The second term in (61) represents the correction to the pion self-energy due to the thermal background of photons at finite temperature, and we have

$$-3e^2\int\frac{d^3q}{(2\pi)^3}\frac{n_B(|\vec{q}|)}{(|\vec{q}|)}=-\frac{e^2}{4\pi}\frac{\pi}{\beta^2}$$
$$=-\left(\frac{e^2}{4\pi}\right)\pi T^2 . \tag{63}$$

The last term in (61) is evaluated in the high-temperature limit using the result

$$\int\frac{d^3q}{(2\pi)^3}\frac{1}{(q^2+m^2)^{1/2}}\left(e^{\beta(q^2+m^2)^{1/2}}-1\right)^{-1}\underset{\beta\to 0}{\sim}\left[\frac{1}{12\beta^2}-\frac{m}{4\pi\beta}+\frac{m^2}{4\pi}\left(\tfrac{1}{2}\ln 4\pi+\tfrac{1}{4}-\tfrac{1}{2}\gamma-\ln\beta m\right)+\cdots\right].$$

Thus the pion mass difference is

$$\Delta m_\pi^2=\frac{3e^2}{4\pi}\frac{m_\rho^2}{2\pi}\ln 2-\pi\left(\frac{e^2}{4\pi}\right)T^2+\left[\frac{e^2}{4\pi}\pi T^2-\frac{e^2}{4\pi}(6-3\sqrt{2})m_\rho T+\frac{e^2}{4\pi}\frac{3m_\rho^2}{\pi}\ln 2\right]+\cdots$$
$$=\frac{e^2}{4\pi}\left(\frac{9}{2\pi}m_\rho^2\ln 2-1.76\,m_\rho T\right). \tag{64}$$

At high enough temperatures the mass difference vanishes. As a rough estimate of the temperature at which Δm_π^2 is zero we retain the leading term in the temperature to obtain

$$T=\frac{9}{2\pi}\frac{m_\rho\ln 2}{1.76}\simeq 430\text{ MeV}.$$

Higher-order effects and refinements in the theory could naturally change this estimate.

In this paper we have extended the temperature formalism to relativistic current correlation functions and have shown how the current-algebra sum rules are modified in many-body theory. We have presented an application of these sum rules to the evaluation of the pion electromagnetic mass difference at finite temperature. An application of the sum rules to multiparticle production[12] and relativistic transport theory is under investigation.

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