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Classical solutions for SU(4) gauge fields: Interacting monopoles

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The notion of local idempotents is introduced, and their relation to a class of solutions for SU(n) gauge fields is pointed out. This class includes the known monopole-type solutions for SU(2) and SU(3) gauge fields—coupled to scalars and spinors. Next, these ideas are used to study solutions for SU(4) gauge fields. The following classes of solutions are studied. Corresponding to two commuting SU(2) subgroups of SU(4) one has two monopole-type contributions from the space components, $\vec{W}(x)$, of the gauge field. They are directly coupled among themselves, the remaining SU(4) components providing a tensor-type interaction. They are also coupled to a scalar field $\Phi(x)$ and the time component $W_0(x)$. Two different possibilities for $\Phi(x)$ and $W_0(x)$ are considered in detail. An exact solution is given for a point monopole interacting with a particular system of finite mass. Simple variational calculations are used to obtain finite mass for the total system. Brief remarks are added concerning other possibilities; e.g., how pseudoparticles can be studied from our point of view.

I. INTRODUCTION: A GENERAL POINT OF VIEW CONCERNING A CLASS OF SOLUTIONS FOR SU(n) GAUGE FIELDS

In two previous papers[1] we have studied static classical solutions for SU(3) gauge fields coupled to scalar octets and to spinor octets and triplets. They generalize the results of the authors quoted in Ref. 1. In this paper we will study monopole-type solutions for SU(4) gauge fields.

Since SU(4) contains two commuting SU(2) subgroups, one can evidently simply add two SU(2) solutions (say of the monopole type) without any interference between them. This we consider to be a trivial generalization. Our aim in this paper is to show how additional terms may be introduced so that we no longer have two isolated systems. Such terms may be considered as providing interactions between the two monopole-type systems. *The terms, monopoles and dyons, have often been used for brevity, and are to be taken in a generalized sense—referring to components of field tensors having a point-charge-like asymptotic behavior* ($\simeq\vec{x}/r^3$). One need not even identify any one of them with the electromagnetic field. This aspect will be further discussed in Sec. IV.

However, before considering SU(4) we will make certain remarks, from a more general point of view, concerning certain classes of solutions for non-Abelian gauge fields. For this purpose we start by introducing the concept of "local idempotents." Let us explain this term.

Michel and Radicati have shown,[2] in their study of broken symmetries, that the solutions found in certain variational approaches can be studied systematically from a geometrical point of view—in terms of critical orbits and idempotents of the symmetric product, for the representation in question.[3]

We are not considering broken symmetries. However, the equations of motion of the gauge fields are variational ones, obtained from the Lagrangian. So one may try to see whether at least certain classes of solutions can be constructed in terms of idempotent vectors in the group space which must now be "local" (i.e., depending on x_μ) since we are dealing with local gauge symmetries. Let us consider SU(n) as the gauge group. For gauge fields we have only the adjoint representation. The vectors in the representation space are Hermitian, traceless, $n\times n$ matrices. Thus a global idempotent vector (independent of x_μ) is defined as satisfying the relation

$$\lambda^2=\alpha\lambda+\beta 1\ , \tag{1.1}$$

where α,β are constants and 1 is the $n\times n$ unit matrix.

Let

$$\phi(x)=V(x)\lambda V(x)^{-1}\ . \tag{1.2}$$

Then

$$\phi^2(x)=\alpha\phi(x)+\beta 1\ . \tag{1.3}$$

We call $\phi(x)$ a local idempotent. This is somewhat restrictive since α,β are constants. But this is the class of local idempotents we will utilize. (One may generalize by introducing x-dependent coefficients.) $\phi(x)$ thus obtained may have singularities in space-time, depending on our choice of $V(x)$. For the time being let us consider a generic point away from singularities.

We have

$$[2\phi(x)-\alpha 1]^2=(\alpha^2+4\beta)1\ . \tag{1.4}$$

Henceforth we will assume the normalization

$$(\alpha^2+4\beta)=1\ , \tag{1.5}$$

which is always possible since we consider $\phi(x)$ to be Hermitian and not proportional to 1. When $\alpha=0$, $\phi(x)$ may be called a local nilpotent, and we will choose $\beta=\frac{1}{4}$.

The essential new feature that arises is that now $\phi(x)$ varies with x_μ and we have to consider its derivatives. From (1.3) we obtain

$$\phi(x)(\partial_\mu\phi(x))=(\partial_\mu\phi(x))(\alpha-\phi(x)) \ . \tag{1.6}$$

and

$$[\phi(x),(\partial_\mu\phi(x))(\partial_\nu\phi(x))]=0 \ . \tag{1.7}$$

The generalization to products of arbitrary odd and even numbers of derivatives is evident. From there other relations can be obtained. Let us note a few which are particularly useful for the *static solutions* $[\partial_0\phi(x)=0]$ we will study:

$$[\phi(x),[\phi(x),\vec{\nabla}\phi(x)]]=\vec{\nabla}\phi(x) \ , \tag{1.8}$$

$$(\vec{\nabla}\phi(x))\times[\phi(x),\vec{\nabla}\phi(x)]+[\phi(x),\vec{\nabla}\phi(x)]\times(\vec{\nabla}\phi(x))=0 \ , \tag{1.9}$$

$$(i[\phi(x),\vec{\nabla}\phi(x)])\times(i[\phi(x),\vec{\nabla}\phi(x)])=(\vec{\nabla}\phi(x))\times(\vec{\nabla}\phi(x)) \ . \tag{1.10}$$

Let us now consider gauge transformations of the form

$$U(x)=e^{-i\theta(x)\phi(x)} \ . \tag{1.11}$$

Later on we will also consider several idempotents $[\phi_i(x)]$ simultaneously. But let us start with this simple example. Such transformations have been studied in Ref. 4, and we utilize some of their results in a more general context. Moreover, we do not want to *necessarily* identify each $\phi_i(x)$ thus introduced with a scalar field. This will be made more explicit through SU(4) examples.

In order to obtain a suitable ansatz for the gauge fields, let us start from the most trivial solution for the pure gauge fields, namely,

$$W_\mu(x)=0 \ , \tag{1.12}$$

and use the gauge transformation (1.11). We obtain, using (1.4) and other related properties,[5]

$$\begin{aligned}(W_\mu(x))_{\rm tr} &= U(x)W_\mu(x)U(x)^{-1}+(i\partial_\mu V(x))V(x)^{-1}\\ &= i(\partial_\mu U(x))U(x)^{-1}\\ &= \sin\theta(x)(\partial_\mu\phi(x))\\ &\quad+(\cos\theta(x)-1)(i[\phi(x),\partial_\mu\phi(x)])\\ &\quad+(\partial_\mu\theta(x))\phi(x)\end{aligned} \tag{1.13}$$

and

$$(F_{\mu\nu}(x))_{\rm tr}=U(x)F_{\mu\nu}(x)U(x)^{-1}=0 \ . \tag{1.14}$$

The result (1.14) is ensured partly because of the constraints on the coefficients involving $\theta(x)$ and partly because of the special properties of $\phi(x)$. Thus as an ansatz one may utilize the form (1.13), relaxing the constraints on the coefficients. The properties of $\phi(x)$ will still lead to relatively simple forms for the field tensor and equations of motion. Thus the search for solutions other than pure gauge ones will not be a hopeless task. In this way idempotents may lead to a useful ansatz.

As a simple example for the static case let us set to begin with [for any idempotent $\phi(x)$ in SU(n)],

$$\begin{aligned}&W_0(x)=0 \ ,\\ &\vec{W}(x)=a(x)(\vec{\nabla}\phi(x))\\ &\qquad+(b(x)-1)(i[\phi(x),\vec{\nabla}\phi(x)])\\ &\qquad+\vec{c}(x)\phi(x) \ .\end{aligned} \tag{1.15}$$

Using properties such as (1.8), (1.9), (1.10), and so on one obtains [setting $F_{ij}(x)=e_{ijk}F_k(x)$, and normalizing the gauge field coupling constant to 1]

$$\begin{aligned}\vec{F}(x)\equiv\vec{\nabla}\times\vec{W}(x)+i\vec{W}(x)\times\vec{W}(x)&=(\vec{\nabla}a(x)-b(x)\vec{c}(x))\times(\vec{\nabla}\phi(x))+(\vec{\nabla}b(x)+a(x)\vec{c}(x))\times(i[\phi(x),\vec{\nabla}\phi(x)])\\ &\quad+(\vec{\nabla}\times\vec{c}(x))\phi(x)+(a^2(x)+b^2(x)-1)(i\vec{\nabla}\phi(x)\times\vec{\nabla}\phi(x)).\end{aligned} \tag{1.16}$$

The pure gauge result is obtained on setting

$$\begin{aligned}&a(x)=\sin\theta(x) \ ,\\ &b(x)=\cos\theta(x) \ ,\\ &\vec{c}(x)=\vec{\nabla}\theta(x) \ .\end{aligned} \tag{1.17}$$

This serves as a check. But now one can search for other nontrivial solutions. It is known, for example, that for SU(2), setting

$$\begin{aligned}&a(x)=0,\quad \vec{c}(x)=0,\quad b(x)=b(r)\\ &\text{and}\\ &\phi(x)=\frac{\hat{\mathbf{x}}\cdot\vec{\tau}}{2r} \ ,\end{aligned} \tag{1.18}$$

one obtains the gauge field part of the famous monopole-type solutions.[6] For SU(2), of course, any direction is a nilpotent one. The SU(3) solutions given in Ref. 1 can all be obtained by starting with gauge transformations of the form

$$U(x)=\exp[-i\theta_1(r)\phi_+(x)-i\theta_2(r)\phi_-(x)]\ , \qquad (1.19)$$

where

$$\phi_\pm(x)=-V(x)\left(\lambda_3\pm\frac{1}{\sqrt{3}}\lambda_8\right)V(x)^{-1}$$

and

$$V(x)=e^{i\pi\lambda_1/4}e^{-i\theta\lambda_5}e^{i\varphi\lambda_2}\ ; \qquad (1.20)$$

θ and φ are the spherical angles. That is, we calculate

$$i(\vec{\nabla}U(x))U(x)^{-1}$$

corresponding to (1.19) and then generalize the coefficients as in (1.15). Then we may add a nonzero $W_0(x)$ component and a scalar octet by using linear combinations of $\phi_\pm(x)$. [This corresponds to the SO(3) embedding. The use of the SU(2) subgroup $(\frac{1}{2}(\lambda_1,\lambda_2,\lambda_3))$ seems to permit only the addition of an effectively Abelian λ_8 term.] SU(4) examples will be given in the following sections.

In this paper again (as in the examples quoted) we will consider only radially symmetric solutions—more precisely for the coefficients we take $a(x)=a(r)$ and so on, while the $\phi_i(x)$'s are constructed using spherical harmonics as in (1.18). Results such as (1.13) are evidently more general, with corresponding possibilities of applications.

Let us finally note that starting with (1.15) if we make an additional gauge transformation

$$U(x)=e^{-i\tilde{\theta}(x)\phi(x)}\ , \qquad (1.21)$$

then $(\vec{W}(x))_{tr}$ is obtained by transforming the coefficients $a(x), b(x), \vec{c}(x)$ such that the new ones are

$$\begin{aligned}\tilde{a}(x)&=a(x)\cos\tilde{\theta}(x)+b(x)\sin\tilde{\theta}(x)\ ,\\ \tilde{b}(x)&=b(x)\cos\tilde{\theta}(x)-a(x)\sin\tilde{\theta}(x)\ ,\\ \tilde{\vec{c}}(x)&=\vec{c}(x)+\vec{\nabla}\tilde{\theta}(x)\ .\end{aligned} \qquad (1.22)$$

Thus, depending on our choice of $a(x), b(x), \vec{c}(x)$ it may be possible to eliminate one of these coefficients in the new gauge. This may simplify the search for different types of solutions. However, when there are several $\phi(x)$'s the situation is more complicated. Let

$$\phi_i(x)=V(x)\lambda_iV(x)^{-1}\quad(i=1,\ldots,n-1)\ , \qquad (1.23)$$

where the λ_i's are suitable linear combinations of the generators of the Cartan subalgebra. The $\phi_i(x)$'s commute among themselves. But $\phi_i(x)$ does not commute, in general, with all $\vec{\nabla}\phi_j(x)$ $(i\neq j)$ though this may happen in some particular cases, such as (1.20). Thus even if we choose commuting $\phi_i(x)$'s [by using the same $V(x)$ for each λ_i] we are not led to a superposition of forms such as (1.13) and (1.15). Our SU(4) example will make this explicit. The effect of additional gauge transformations of the form

$$\exp\left[-i\sum_i\tilde{\theta}_i(x)\phi_i(x)\right]$$

can, of course, always be examined with profit. Another possible generalization would be to consider $[(n-1)$ or more$]$ noncommuting idempotents by introducing simultaneously different transformations $V_i(x)$.

II. SU(4) SOLUTIONS

We will use the following notations. They display in an explicit fashion the structure of what we will call the interaction terms. This is our reason for repeating certain well-known things. Let (lm) be the 4×4 matrix with only one nonzero element, 1, on the lth row and the mth column.

Let

$$J_i=-i\epsilon_{ijk}(jk),\quad K_i=-i((i4)-(4i)) \qquad (2.1)$$
$$(i,j,k=1,2,3)\ .$$

Let

$$\vec{L}_\epsilon=\tfrac{1}{2}(\vec{J}+\epsilon\vec{K})\quad(\epsilon=\pm1)\ . \qquad (2.2)$$

Then

$$[L_\epsilon^i, L_{-\epsilon}^j]=0,\quad [L_\epsilon^i, L_\epsilon^j]=i\epsilon_{ijk}L_\epsilon^k\ . \qquad (2.3)$$

These are the six antisymmetric SU(4) matrices generating two commuting SU(2) subgroups. The remaining nine symmetric generators of SU(4) can be taken to be

$$L_+^iL_-^j\quad(i,j=1,2,3)\ . \qquad (2.4)$$

Thus this basis is seen to be well adapted to display tensorial "spin-spin" type couplings between two commuting SU(2) groups. In this paper we will study solutions where this type of coupling is present. We have

$$L_\epsilon^iL_\epsilon^j=\tfrac{1}{2}i\epsilon_{ijk}L_\epsilon^k+\tfrac{1}{4}\delta_{ij}1\ . \qquad (2.5)$$

Hence L_ϵ^i are nilpotents with the normalization (1.5). We have two sets of 4×4 "Pauli matrices."

The following relations permit a ready conversion to the familiar λ basis:

$$-4L_+^iL_-^i=(ii)-(jj)-(kk)+(44) \qquad (2.6)$$
$$(\text{no sum; } i,j,k \text{ cyclic})\ ,$$
$$-4L_+^iL_-^j=(ij)+(ji)+\epsilon_{ijk}((k4)+(4k))\ . \qquad (2.7)$$

The three generators of the Cartan subalgebra can be chosen as

$$L_+^3,\ L_-^3,\ \text{and } 2L_+^3L_-^3\ . \qquad (2.8)$$

Before restricting ourselves to the case of radial symmetry, let us give some results in a more general form.

Let

$$\phi(x) = \vec{v}_\epsilon(x)\cdot\vec{L}_\epsilon \ , \tag{2.9}$$

where

$$\vec{v}^2(x) = 1 \ ,$$

when

$$\phi_\epsilon{}^2(x) = \tfrac{1}{4}1 \ .$$

$\phi_\epsilon(x)$ is obtained by suitably rotating L_ϵ^3, using an $SU_\epsilon(2)$ transformation.

Corresponding to (2.8) the three nilpotents are now [with $V(x)$ of (1.23) now of the form $V_+(x)V_-(x)$]

$$\phi_+(x),\ \phi_-(x),\ (2\phi_+(x)\phi_-(x)) \ . \tag{2.10}$$

Let us consider first the space components of the gauge field. The generalization of (1.13) to (1.15), used for the two commuting SU(2) subgroups, leads to a form

$$\begin{aligned}\vec{W}(x) &= \sum_\epsilon \{a_\epsilon(x)\vec{\nabla}\phi_\epsilon(x) + (b_\epsilon(x)-1)i[\phi_\epsilon(x), \vec{\nabla}\phi_\epsilon(x)] + \vec{c}_\epsilon(x)\phi_\epsilon(x)\}\\ &= \sum_\epsilon \{\vec{\nabla}\phi_\epsilon(x)[a_\epsilon(x) - (b_\epsilon(x)-1)(i2\phi_\epsilon(x))] + \vec{c}_\epsilon(x)\phi_\epsilon(x)\} \ .\end{aligned} \tag{2.11}$$

[We have utilized (1.6) with $\alpha = 0$.]

This may lead, for example, to two noninteracting monopoles. To obtain a suitable ansatz for a coupled interacting system, let us now utilize the third nilpotent, namely a gauge transformation

$$U(x) = e^{-i\theta(x)(2\phi_+(x)\phi_-(x))} \ . \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\begin{aligned}(\vec{W}(x))_{tr} &= U(x)\vec{W}(x)U(x)^{-1} + i(\vec{\nabla}U(x))U(x)^{-1}\\ &= \sum_\epsilon (\vec{\nabla}\phi_\epsilon(x))\{[a_\epsilon(x) - b_\epsilon(x)(i2\phi_\epsilon(x))]e^{i\theta(x)(4\phi_+(x)\phi_-(x))} + i2\phi_\epsilon(x)\}\\ &\quad + \sum_\epsilon \vec{c}_\epsilon(x)\phi_\epsilon(x) + (\vec{\nabla}\theta(x))(2\phi_+(x)\phi_-(x)) \ .\end{aligned} \tag{2.13}$$

Continuing to apply our technique we again choose a simple generalized form and write

$$\begin{aligned}\vec{W}(x) &= \sum_\epsilon (\vec{\nabla}\phi_\epsilon(x))\{[a_\epsilon(x) - b_\epsilon(x)(i2\phi_\epsilon(x))][d_\epsilon(x) + e_\epsilon(x)(i4\phi_+(x)\phi_-(x))] + i2\phi_\epsilon(x)\}\\ &\quad + \sum_\epsilon \vec{c}_\epsilon(x)\phi_\epsilon(x) + \vec{f}(x)(2\phi_+(x)\phi_-(x)) \ .\end{aligned} \tag{2.14}$$

For the space part of the field tensor this gives

$$\begin{aligned}\vec{F}(x) &= \vec{\nabla}\times\vec{W}(x) + i\vec{W}(x)\times\vec{W}(x)\\ &= \sum_\epsilon [i(\vec{\nabla}\phi_\epsilon)\times(\vec{\nabla}\phi_\epsilon)][(a_\epsilon{}^2 + b_\epsilon{}^2)(e_\epsilon{}^2 + d_\epsilon{}^2) - 1]\\ &\quad - \sum_\epsilon (\vec{\nabla}\phi_\epsilon)\times\{[(\vec{\nabla}a_\epsilon - b_\epsilon\vec{c}_\epsilon) - (\vec{\nabla}b_\epsilon + a_\epsilon\vec{c}_\epsilon)(i2\phi_\epsilon)](d_\epsilon + e_\epsilon(i4\phi_+\phi_-))\\ &\qquad + [a_\epsilon - b_\epsilon(i2\phi_\epsilon)][(\vec{\nabla}d_\epsilon + e_\epsilon\vec{f}) + (\vec{\nabla}e_\epsilon - d_\epsilon\vec{f})(i4\phi_+\phi_-)]\}\\ &\quad + \sum_\epsilon (\vec{\nabla}\times\vec{c}_\epsilon)\phi_\epsilon + (\vec{\nabla}\times\vec{f})(2\phi_+\phi_-)\\ &\quad - 2(d_+e_- - d_-e_+)[(\vec{\nabla}\phi_+)\times(\vec{\nabla}\phi_-)(4\phi_+\phi_-)(a_+ - b_+(i2\phi_+))(a_- - b_-(i2\phi_-))] \ .\end{aligned} \tag{2.15}$$

[We have suppressed the arguments (x) for brevity.]

Equations (2.14) and (2.15) may be used as a starting point for a search for solutions with different types of symmetries and boundary conditions. In what follows we will look for only relatively simple types of "spherically symmetric" solutions. But we will look for such solutions that include some nontrivial coupling between the two SU(2) subgroups.

Let us now write

$$\phi_\epsilon(x) = \frac{\vec{x}}{r}\cdot\vec{L}_\epsilon \equiv \hat{r}\cdot\vec{L}_\epsilon \quad (\hat{r} \equiv \vec{x}/r) \ , \tag{2.16}$$

put in (2.14) and (2.15)

$$a_\epsilon(x) = \vec{c}_\epsilon(x) = \vec{f}(x) = 0 \ ,$$

and write (2.17)

$$b_\epsilon(x) d_\epsilon(x) \equiv \eta_\epsilon(r), \quad b_\epsilon(x) e_\epsilon(x) \equiv \zeta_\epsilon(r) \ .$$

The coefficients $\eta_\epsilon(r), \zeta_\epsilon(r)$ are now functions of the radial distance r only. Henceforth we will often suppress the arguments (x) and (r). No confusion is likely.

In what follows we will use systematically the results given in Appendix A. They will not be referred to separately on each occasion. We obtain

$$\vec{W} = \sum_\epsilon (\vec{\nabla}\phi_\epsilon)[(-\eta_\epsilon + 1)(i2\phi_\epsilon) + \zeta_\epsilon(2\phi - \epsilon)] \tag{2.18}$$

and

$$\begin{aligned}\vec{F} = &- \sum_\epsilon (\hat{r}\phi_\epsilon) \frac{1}{r^2} (\eta_\epsilon{}^2 + \zeta_\epsilon{}^2 - 1) \\ &- \sum_\epsilon (\vec{\nabla}\phi_\epsilon)[\eta_\epsilon' + \zeta_\epsilon'(i4\phi_+\phi_-)] + 2(\vec{\nabla}\phi_+) \times (\vec{\nabla}\phi_-)(\eta_+\zeta_- - \eta_-\zeta_+) \qquad \left(\eta' \equiv \frac{d}{dr}\eta\right) \ .\end{aligned} \tag{2.19}$$

Unless $\zeta_+/\eta_+ = \zeta_-/\eta_- =$ constant, we have a nontrivial coupling between the two SU(2) subgroups. The first step toward the equations of motion gives

$$\begin{aligned}\vec{\nabla} \times \vec{F} + i(\vec{W} \times \vec{F} + \vec{F} \times \vec{W}) = \sum_\epsilon (\vec{\nabla}\phi_\epsilon) \Bigg\{ &\left(\frac{i2\phi_\epsilon}{r^2}\right) [r^2\eta_\epsilon'' - \eta_\epsilon(\eta_\epsilon{}^2 + \zeta_\epsilon{}^2 - 1) - \zeta_{-\epsilon}(\eta_\epsilon\zeta_{-\epsilon} - \eta_{-\epsilon}\zeta_\epsilon)] \\ &- \left(\frac{2\phi_{-\epsilon}}{r^2}\right) [r^2\zeta_\epsilon'' - \zeta_\epsilon(\eta_\epsilon{}^2 + \zeta_\epsilon{}^2 - 1) + \eta_{-\epsilon}(\eta_\epsilon\zeta_{-\epsilon} - \eta_{-\epsilon}\zeta_\epsilon)] \Bigg\} \\ &+ \frac{\hat{r}}{2r^2}(4\phi_+\phi_-)\left[\sum_\epsilon (\eta_\epsilon'\zeta_\epsilon - \eta_\epsilon\zeta_\epsilon')\right] + 2\hat{r}(\vec{\nabla}\phi_+ \cdot \vec{\nabla}\phi_-)(\eta_+\zeta_-' - \zeta_+\eta_-' - \zeta_-\eta_+' + \eta_-\zeta_+') \ .\end{aligned} \tag{2.20}$$

Now we will consider the effect of introducing a nonzero $W_0(x)$ component and a scalar field (all the fields are in the adjoint representation). It is known[1] that they can be made to play quite similar roles, so far as formal calculations are concerned, though their physical significance is quite different.

For SU(2) and SU(3) (Refs. 1, 4, and 6) the idempotents themselves were introduced as scalar fields or as $W_0(x)$. Here we have three candidates,

$$\phi_+(x), \ \phi_-(x), \ \text{and} \ 2\phi_+(x)\phi_-(x) \ .$$

Instead of superposing all three, we have found it interesting to consider separately the following two possibilities, for the scalar field $\phi(x)$, namely

$$\phi(x) = \sum_\epsilon \frac{c_\epsilon(r)}{r} \phi_\epsilon(x) \tag{2.21}$$

and

$$\phi(x) = \frac{c(r)}{r}(2\phi_+(x)\phi_-(x)) \ . \tag{2.22}$$

We find (2.22) to be a particularly interesting possibility. But it is also of interest to compare the two cases. We will give a parallel discussion of these two alternatives.

Let us first give the respective contributions to the equations of motion. For scalar fields we have to add the contribution from scalar potential $V(\phi)$, but we will come to that point later on.

Case I. Let

$$\phi = \sum_\epsilon \frac{c_\epsilon}{r} \phi_\epsilon \ .$$

Then

$$\begin{aligned}\vec{D}\phi &\equiv \vec{\nabla}\phi + i[\vec{W}, \phi] \\ &= \sum_\epsilon \left\{ \hat{r}\left(\frac{c_\epsilon}{r}\right)' \phi_\epsilon + (\vec{\nabla}\phi_\epsilon)\left(\frac{c_\epsilon}{r}\right)(\eta_\epsilon + \zeta_\epsilon(i4\phi_+\phi_-)) \right\},\end{aligned} \tag{2.23}$$

$$i[\phi,\vec{\mathrm{D}}\phi]=\sum_{\epsilon}(\vec{\nabla}\phi_\epsilon)\left(\frac{c_\epsilon}{r}\right)^2[-\eta_\epsilon(i2\phi_\epsilon)+\zeta_\epsilon(2\phi_{-\epsilon})]\ , \tag{2.24}$$

$$\vec{\mathrm{D}}\cdot(\vec{\mathrm{D}}\phi)\equiv\vec{\nabla}\cdot(\vec{\mathrm{D}}\phi)+i(\vec{\mathrm{W}}\cdot(\vec{\mathrm{D}}\phi)-(\vec{\mathrm{D}}\phi)\cdot\vec{\mathrm{W}})$$
$$=\sum_{\epsilon}\left[\frac{c_\epsilon''}{r}-\frac{2c_\epsilon}{r^3}(\eta_\epsilon{}^2+\zeta_\epsilon{}^2)\right]\phi_\epsilon+[(\vec{\nabla}\phi_+)\cdot(\vec{\nabla}\phi_-)](i2\phi_+)\frac{2}{r}[(c_++c_-)(\eta_-\zeta_+-\eta_+\zeta_-)]\ . \tag{2.25}$$

Case II. Let

$$\phi=\frac{c}{r}(2\phi_+\phi_-)\ .$$

Then

$$\vec{\mathrm{D}}\phi=\left(\frac{c}{r}\right)'\hat{r}(2\phi_+\phi_-)+\sum_{\epsilon}(\vec{\nabla}\phi_\epsilon)[\eta_\epsilon(2\phi_{-\epsilon})+\zeta_\epsilon(i2\phi_\epsilon)]\left(\frac{c}{r}\right)\ , \tag{2.26}$$

$$i[\phi,\vec{\mathrm{D}}\phi]=\sum_{\epsilon}(\vec{\nabla}\phi_\epsilon)[-\eta_\epsilon(i2\phi_\epsilon)+\zeta_\epsilon(2\phi_{-\epsilon})]\left(\frac{c}{r}\right)^2\ , \tag{2.27}$$

$$\vec{\mathrm{D}}\cdot(\vec{\mathrm{D}}\phi)=\left\{\frac{c''}{r}-\left(\frac{2c}{r^3}\right)\left[\sum_{\epsilon}(\eta_\epsilon{}^2+\zeta_\epsilon{}^2)\right]\right\}(2\phi_+\phi_-)+\frac{4c}{r}(\eta_+\eta_-+\zeta_+\zeta_-)(\vec{\nabla}\phi_+\cdot\vec{\nabla}\phi_-)\ . \tag{2.28}$$

For $W_0(x)$ we need (corresponding to $\vec{\mathrm{D}}\phi$)

$$\vec{\mathrm{F}}_{(0)}\equiv\vec{\nabla}W_0+i[\vec{\mathrm{W}},W_0]\ . \tag{2.29}$$

Assuming by turns

$$W_0(x)=\sum_{\epsilon}\frac{d_\epsilon(r)}{r}\phi_\epsilon(x) \tag{2.30}$$

and

$$W_0(x)=\frac{d(r)}{r}(2\phi_+(x)\phi_-(x))\ , \tag{2.31}$$

we have again exactly analogous expressions. For (2.30) the analogy with SU(2) dyons is evident. It will be convenient to discuss certain features of (2.31) after the equations of motion.

The equations of motion are

$$\vec{\nabla}\times\vec{\mathrm{F}}+i(\vec{\mathrm{W}}\times\vec{\mathrm{F}}+\vec{\mathrm{F}}\times\vec{\mathrm{W}})-i[W_0,\vec{\mathrm{F}}_{(0)}]+i[\phi,\vec{\mathrm{D}}\phi]=0\ , \tag{2.32}$$

$$\vec{\mathrm{D}}\cdot\vec{\mathrm{F}}_{(0)}=0\ , \tag{2.33}$$

$$\vec{\mathrm{D}}\cdot\vec{\mathrm{D}}\phi-\frac{\delta V(\phi)}{\delta\phi}=0\ , \tag{2.34}$$

where $V(\phi)$ is some suitable scalar potential. For example, we may take

$$V(\phi)=-\frac{\mu^2}{2}(\mathrm{Tr}\phi^2)+\tfrac{1}{4}\lambda(\mathrm{Tr}\phi^2)^2\quad(\mu^2>0)\ . \tag{2.35}$$

Let us first point out certain constraints that simplify the equations of motion. We will give the results for the cases where the same type of ansatz is made for $\phi(x)$ and $W_0(x)$. The corresponding equations for other possible combinations are obtained easily. We will not write down all the alternatives systematically.

Case I. Let us first consider case I, namely the system given by (2.18), (2.21), and (2.30). The following results may easily be verified.

Let us set

$$\zeta_\epsilon=K_\epsilon\eta_\epsilon\ , \tag{2.36}$$

where K_ϵ is a *constant*. For

$$K_+=K_-\ , \tag{2.37}$$

we obtain effectively a gauge transformation of two noninteracting SU(2) systems. This is the trivial generalization. We have only to substitute known analytic [Ref. 7, for $V(\phi)=0$] or numerical[6,8] solutions for SU(2) [or SO(3), since we are considering scalar triplets for each SU(2)].

However, there is another possibility for nonzero ζ_ϵ (Ref. 9). Let

$$(1+K_+K_-)=0\ ,$$

$$c_+=-c_-\equiv C\ , \tag{2.38}$$

$$d_+=-d_-\equiv D\ , \tag{2.39}$$

and

$$\eta_-=\alpha\eta_+\ , \tag{2.40}$$

where

$$\alpha^2=\frac{1+K_+{}^2}{1+K_-{}^2}=K_+{}^2\ . \tag{2.41}$$

Let us set

$$\sum_{\epsilon}(\eta_\epsilon{}^2+\zeta_\epsilon{}^2)=2\eta_+{}^2(1+K_+{}^2)\equiv\eta^2\ . \tag{2.42}$$

Now the whole system of equations of motions is reduced to [with $\eta_- = \pm\zeta_+, \zeta_- = \mp\eta_+$, due to (2.41)]

$$r^2\eta'' - \eta(\eta^2 + C^2 - D^2 - 1) = 0\ , \tag{2.43}$$

$$r^2D'' - D\eta^2 = 0\ , \tag{2.44}$$

$$r^2C'' - C\eta^2 - \tfrac{1}{2}\frac{\delta V(C)}{\delta C} = 0\ . \tag{2.45}$$

Thus we almost get back the equations for SU(2) but not quite (compare Ref. 7).

Case II. Now we consider the system given by (2.18), (2.22), and (2.31). Here for nonzero c and/or d the two SU(2)'s remain coupled and the possibility (2.37) does not arise.

But again setting

$$\zeta_\epsilon = K_\epsilon \eta_\epsilon\ , \tag{2.36$'$}$$

$$1 + K_+K_- = 0\ , \tag{2.38$'$}$$

$$\eta_- = \alpha\eta_+\ , \tag{2.40$'$}$$

and

$$\sum_\epsilon (\eta_\epsilon{}^2 + \zeta_\epsilon{}^2) \equiv \eta^2\ , \tag{2.42$'$}$$

we have finally

$$r^2\eta'' - \eta(\eta^2 + c^2 - d^2 - 1) = 0\ , \tag{2.46}$$

$$r^2d'' - 2d\eta^2 = 0\ , \tag{2.47}$$

and

$$r^2c'' - 2c\eta^2 - \frac{\delta V(c)}{\delta c} = 0\ . \tag{2.48}$$

This time we have exactly the same equations as for SU(2) (compare Ref. 7). Moreover, this time α is not fixed [we do not have a counterpart of (2.41)].

The SU(2) solutions [for $V(\Phi) = 0$], namely (see Ref. 7)

$$\eta = \frac{\beta r}{\sinh\beta r}\ ,$$
$$d = \sinh\gamma(\beta r \coth\beta r - 1)\ , \tag{2.49}$$
$$c = \cosh\gamma(\beta r \coth\beta r - 1)\ ,$$

evidently again provide a regular solution of the coupled nonlinear system (2.46)–(2.48).

But they do not lead to finite mass—as can be seen from (A15)–(A17). This is as far as we have been able to proceed with exact solutions. In a particular case, the source of the divergence can be exhibited in a very simple fashion.

Let

$$\eta_- = \zeta_- = 0 \tag{2.50}$$

[when (2.38$'$) is no longer necessary]. We now get a situation where a *point* monopole [for $SU_-(2)$] is interacting with a *finite* monopole [for $SU_+(2)$] and a symmetrical "electric" component or more precisely with $W_0 = (d/r)(2\Phi_+\Phi_-)$. The only divergence is that due to the point monopole at the origin. [The presence of the point monopole is necessary to cancel certain extra terms in $\vec{D}\Phi$ and $\vec{F}_{(0)}$.] The asymptotically point-charge-like fields are given by

$$\mathrm{Tr}(\phi_\epsilon \vec{F})$$
and
$$\mathrm{Tr}(2\phi_+\phi_-\vec{F}_{(0)})\ . \tag{2.51}$$

Also, $\mathrm{Tr}(2\Phi_+\Phi_-\vec{F}) = 0$. This aspect will be further discussed in Sec. IV.

III. VARIATIONAL ESTIMATES

Let us now indicate some simple variational calculations that lead to a finite mass for the coupled system. Let us note that setting (for case II) $\eta_\epsilon = \zeta_\epsilon = 0$, i.e.,

$$\vec{W} = \sum_\epsilon (\vec{\nabla}\phi_\epsilon)(i2\phi_\epsilon)\ , \tag{3.1}$$

$$W_0 = d_0(2\phi_+\phi_-)\ , \tag{3.2}$$

$$\phi = c_0(2\phi_+\phi_-)\ , \tag{3.3}$$

where d_0 and c_0 are now constants, such that

$$\frac{\delta V(\phi)}{\delta\phi} = 0\ , \tag{3.4}$$

the equations of motion are satisfied everywhere except at the origin (where there is a singularity). Moreover, we do not have two isolated systems, since the monopoles are symmetrically coupled to W_0 and Φ. Another way of looking at this situation is to note that only the matrices $\vec{J}$ survive in $\vec{W}$ in (3.1). One may add Φ_ϵ terms to (3.2) and (3.3). But the simpler example will suffice to illustrate our point.

Consider now the variational calculation indicated in Sec. IV of Ref. 8. A simple generalization consists in setting, for the case we are considering (maintaining $\zeta_\epsilon = 0$),

$$\eta_\epsilon = \frac{b_\epsilon}{r^2 + b_\epsilon}\ ,$$
$$d = \frac{d_0 r^2}{(r^2 + a_1)^{1/2}}\ , \tag{3.5}$$
$$c = \frac{c_0 r^2}{(r^2 + a_2)^{1/2}}\ .$$

Substituting these in expressions (A15), (A17) and a similar one for $\Phi(x)$, we obtain a finite mass integral and eventually a variational estimate. The difference with the corresponding result of Ref. 8 arises only from the sums on ϵ.

Setting for example

$$b_\epsilon = b \ , \tag{3.6}$$

one obtains the mass

$$M = (4\pi)\,\frac{\pi}{32}\left(\frac{21}{\sqrt{b}} + d_0^{\,2} f(a_1, b) + c_0^{\,2} f(a_2, b)\right) + \text{contributions from } V(\phi) \ , \tag{3.7}$$

where

$$f(a,b) \equiv \frac{1}{(a-b)^2}\left\{2\sqrt{a}\left[\frac{(a-b)^2}{2} - 16b^2\right] + 16b^{3/2}(b+a)\right\} . \tag{3.7$'$}$$

These expressions are very close to the corresponding ones in Ref. 8. Analogous solutions may now be obtained.

The crudest such estimate may be obtained by setting for example $a_1 = a_2 = 0$ and with $V(\phi)$ as in Ref. 8 for example, when

$$M = (4\pi)\,\frac{\pi}{32}\left[\frac{21}{\sqrt{b}} + (c_0^{\,2} + d_0^{\,2})16\sqrt{b}\right] . \tag{3.8}$$

Here c_0 is assumed to be fixed by (3.4), the stability condition. Now for any fixed d_0, the minimum is obtained, on varying b, for

$$b = \frac{21}{16(c_0^{\,2} + d_0^{\,2})} . \tag{3.9}$$

When $\zeta_\epsilon \neq 0$ (for both cases I and II), the mass integral can be made finite for the parametrization

$$\eta_\epsilon = \frac{a_\epsilon \cos\theta_\epsilon}{r^2 + a_\epsilon} \ , \quad \zeta_\epsilon = \frac{b_\epsilon \sin\theta_\epsilon}{r^2 + b_\epsilon} \ , \tag{3.9$'$}$$

where θ_ϵ is a constant angle and

$$\theta_{-\epsilon} = \theta_\epsilon + n\pi(\text{const}). \tag{3.10}$$

To this we have to add some suitable parametrization for W_0 and ϕ.

Here we are not interested in the numerical values obtained through such crude approximations. We merely indicate the simplest possibilities. Increasing the number of parameters or using integral equations,[8] such estimates can be improved.

IV. REMARKS

We have seen how generalizing gauge transformations along locally idempotent vectors one obtains valuable *Ansätze* for the gauge fields. One can try to place the idea on firmer ground by trying to show that under certain conditions these are the only types of solution possible. But if, in order to be able to make such a statement, one has to start by imposing too many severe restrictions on the nature of the solutions to be obtained, the undertaking loses much of its interest. Here we have tried to explore certain new possibilities that arise for SU(4). Even there we have chosen some simple cases without trying to be exhaustive. Other types of solutions are worth exploring—by starting with idempotents satisfying different types of asymptotic conditions. By relaxing the constraints of radial symmetry one may, for example, try to obtain two or more monopoles at different points. We intend to study elsewhere the possibilities of constructing axially symmetric solutions. Another entirely different class of solutions for SU(4) gauge fields has been given by Kaku.[10]

Let us now come back to the definition (2.51). For SU(2) 't Hooft defined[6] the electromagnetic field tensor as

$$\mathfrak{F}_{\mu\nu} = \tfrac{1}{2}\mathrm{Tr}(\hat{\phi}\tilde{F}_{\mu\nu}) \ , \tag{4.1}$$

where

$$\tilde{F}_{\mu\nu} \equiv F_{\mu\nu} + i[D_\mu\hat{\phi}, D_\nu\hat{\phi}] \ , \tag{4.2}$$

and we denote the normalized $\phi(x)$ as $\hat{\phi}$.

This definition gives a gauge-invariant way of eliminating the terms quadratic in the gauge field, and one has,[11] for this case,

$$\begin{aligned} \mathfrak{F}_{\mu\nu} &= \partial_\mu \boldsymbol{B}_\nu - \partial_\nu B_\mu + \tfrac{1}{2} i\,\mathrm{Tr}\{\hat{\phi}[\partial_\mu\hat{\phi}, \partial_\nu\hat{\phi}]\} \ , \\ B_\mu &\equiv \tfrac{1}{2}\mathrm{Tr}(\hat{\phi}W_\mu) \ , \end{aligned} \tag{4.3}$$

where B_μ behaves as a Maxwell field and the magnetic charge is carried by the $\hat{\phi}$ terms in (4.3). The role of Higgs fields concerning such topologically conserved quantum numbers has been much discussed recently.[11,12]

However, it will be noted that for all such solutions

$$D_\mu\hat{\phi} \to 0 \ , \tag{4.4}$$

asymptotically, and thus

$$\mathfrak{F}_{\mu\nu} \to \tfrac{1}{2}\mathrm{Tr}(\hat{\phi}F_{\mu\nu}) \ . \tag{4.5}$$

Indeed 't Hooft's asymptotic solution[6] gives

$$\vec{\mathrm{F}} = -\frac{\vec{\mathrm{x}}}{r^3}\hat{\phi} \ \text{with} \ \hat{\phi} = \frac{\vec{\tau}\cdot\vec{\mathrm{x}}}{r} \ , \tag{4.6}$$

and

$$\tfrac{1}{2}\mathrm{Tr}(\hat{\phi}\vec{\mathrm{F}}) = -\frac{\vec{\mathrm{x}}}{r^3} = \vec{\nabla}\left(\frac{1}{r}\right) \tag{4.7}$$

is the point monopole field. For finite solutions with this asymptotic behavior, the same definition can be chosen.

It is this aspect that we have generalized for our case, where there are several idempotents and different choices for $\phi(x)$ and $W_0(x)$ are possible. The reason is that for the same $F_{\mu\nu}$, we would like our "magnetic" (or other) fields and charges not to vary along with different choices of scalar fields introduced for the purpose of constructing solutions. From this point of view one first selects the gauge where $F_{\mu\nu}$ and the vectors $\phi_i(x)$ have suitable asymptotic properties and then (subject to proper normalizations and sign conventions) defines the projections

$$\mathrm{Tr}(\phi_i F_{\mu\nu}) \tag{4.8}$$

to be certain fields (e.g. electromagnetic).

In any other gauge one can take the projections along the covariantly transformed idempotent $(U(x)\phi_i(x)U(x)^{-1})$, whether or not each $\phi_i(x)$ is introduced as a scalar field. In case II, one thus obtains new types of charges. Moreover, for SU(n) ($n \geq 3$) scalars can be constructed out of symmetric products of $F_{\mu\nu}$'s with themselves. This gives formulas analogous to (4.2) with a more intrinsic significance. We will discuss this in detail elsewhere.

It is interesting to compare the consequences of definitions (4.1) and (4.2) for our two cases.

For case I,

$$\vec{\mathrm{F}} - i\vec{\mathrm{D}}\hat{\phi} \times \vec{\mathrm{D}}\hat{\phi} = \sum_{\epsilon}\left[\frac{\hat{r}}{r^2}\phi_\epsilon - (\vec{\nabla}\phi_\epsilon)(\eta_\epsilon' + \zeta_\epsilon'(i4\phi_+\phi_-))\right] \pm 2(\vec{\nabla}\phi_+ \times \vec{\nabla}\phi_-)(1 + 4\phi_+\phi_-)(\eta_+\zeta_- - \eta_-\zeta_+)$$
$$[\hat{\phi} \simeq (\phi_+ \pm \phi_-)]. \tag{4.9}$$

For case II,

$$\vec{\mathrm{F}} - i\vec{\mathrm{D}}\hat{\phi} \times \vec{\mathrm{D}}\hat{\phi} = \sum_{\epsilon}\left[\frac{\hat{r}}{r^2}\phi_\epsilon - (\vec{\nabla}\phi_\epsilon)(\eta_\epsilon' + \zeta_\epsilon'(i4\phi_+\phi_-))\right] \quad (\hat{\phi} \simeq 2\phi_+\phi_-)\,. \tag{4.10}$$

Thus for both cases (irrespective of the choice of ϕ)

$$\mathrm{Tr}(\phi_\epsilon \vec{\mathrm{F}}) = \frac{\hat{r}}{r^2}\,, \tag{4.11}$$

$$\mathrm{Tr}(2\phi_+\phi_-\vec{\mathrm{F}}) = 0\,. \tag{4.12}$$

We see that the monopoles are not strictly related to the Higgs scalars. (Some relevant comments are added in Appendix B.) The strengths of the poles are one in our units (or $1/g$ for coupling constant g).

Let us now look at another aspect of our solutions. For case I,

$$\mathrm{Tr}(\vec{\mathrm{F}}\cdot\vec{\mathrm{F}}_{(0)}) = \frac{1}{r^2}\frac{d}{dr}\left[\sum_{\epsilon}\left(\frac{d_\epsilon}{r}\right)(1 - \eta_\epsilon^{\,2} - \zeta_\epsilon)\right], \tag{4.13}$$

and for case II,

$$\mathrm{Tr}(\vec{\mathrm{F}}\cdot\vec{\mathrm{F}}_{(0)}) = 0\,. \tag{4.14}$$

Thus for case I we have (with $F^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\delta}F^{\rho\delta}$)

$$\frac{1}{8\pi^2}\int d^3x(\mathrm{Tr}F_{\mu\nu}F^*_{\mu\nu}) = \frac{1}{\pi}\left[\sum_{\epsilon}\left(\frac{d\epsilon}{r}\right)(1 - \eta_\epsilon^{\,2} - \zeta_\epsilon^{\,2})\right]_{r=0}^{\infty} \tag{4.15}$$

Marciano and Pagels[13] have already pointed out [using the SU(2) example] how dyon-type solutions can contribute to chirality nonconservation through nonzero anomaly terms. In our example, (4.15) may or may not be nonzero depending on the precise asymptotic behaviors of the d_ϵ. We have not explored all possibilities. For finite-energy solutions the $r = 0$ limit of (4.15) must be zero. For

$$\frac{d_+}{r} = -\frac{d_-}{r} \to \text{constant}$$
$$\to r \to \infty$$

the upper limit can again be zero. The solutions most studied in connection with the axial-vector current anomalies are the "pseudoparticle" solutions,[14] namely the regular solutions of four-dimensional Euclidean Yang-Mills equations.

The SU(2) example of Ref. 14 has a simple interpretation from our point of view. We again start with the pure gauge form [as in (1.13)] and introduce a very simple type of "deformation" of the coefficients—namely a space-time dependent overall multiplying factor.

Thus

$$W_\mu = f(\tau)((i\partial_\mu U)U^{-1})\,, \tag{4.16}$$

where (in the notation of Ref. 14)

$$\tau^2 = x_1^{\,2} + x_2^{\,2} + x_3^{\,2} + x_4^{\,2} = r^2 + x_4^{\,2}$$
$$U = e^{-i\theta(x)\vec{\sigma}\cdot\vec{x}/r}\,. \tag{4.17}$$

Here $(\vec{\sigma}\cdot\vec{x})/r$ is an idempotent with Pauli matrices $\vec{\sigma}$. The solution of Ref. 14 is obtained with

$$\cos\theta(x) = x_4/\tau\,,$$
$$\sin\theta(x) = r/\tau\,, \tag{4.18}$$
$$f(\tau) = \tau^2/(\tau^2 + x^2)\,.$$

We intend to study elsewhere pseudoparticle solutions for other groups.

We have illustrated our method by choosing one type of SU(4) example, which we consider to be particularly interesting. A fairly systematic construction of spherically symmetric pointlike solutions has been attempted by Brihaye and Nuyts,[15] without, however, including our example. Without attempting a complete analysis of their results, we have tried to indicate, in Appendix C, how their examples fit in with our point of view. Idempotents again seem to play a decisive role. Here we just note that the coupling terms in our example fall off as $r \to \infty$ and would not have been noticed had we looked at asymptotic forms only. However, since we are particularly interested in finite-energy solutions, we cannot afford to do that. Nor can we extremize different parts of the Lagrangian separately,[15] as is admissible for classification of pointlike solutions.

Let us repeat that we have exploited only the very simple symmetric product for adjoint representations [(1.1)]. However, for other representations, the definition of idempotents can be much more complicated.[2] Higher-order products can also be introduced from SU(4) onward for certain representations. It is difficult to say beforehand whether such cases will be tractable enough to be used profitably in the gauge field context—but it would be worth trying. In particular, chiral gauge field Lagrangians[16] with scalar fields transforming as $(n,\bar{n})$ representation of $SU(n)\otimes SU(n)$ should naturally bring into play the interesting idempotent structure of $(n,\bar{n})$ representations. However, at the present stage this is only a conjecture.

In this paper we have presented a point of view helpful in a systematic search for solutions. If one thinks that Yang-Mills fields are of interest in particle physics and then finds that they have classical solutions with remarkable properties, then a more thorough search is evidently desirable. The search for certain new types of quantization methods has its *raison d'être* in the existence of special types of classical solutions. In fact, from the path-integral point of view, all finite-energy solutions, providing extrema, are relevant. One may, of course, hope to exploit more directly the particlelike properties of some solutions in terms of suitable models.

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APPENDIX A

In the Introduction we have given certain useful consequences [(1.6) to (1.10)] of (1.3). These are quite general. When we postulate a certain explicit form for $\phi(x)$, we have naturally more relations. We collect here results particularly useful for the spherical symmetry case of Sec. II.

Let

$$\phi_\epsilon(x)=\frac{\vec{x}}{r}\cdot\vec{L}_\epsilon\equiv\hat{r}\cdot\vec{L}_\epsilon\ ; \tag{A1}$$

then

$$r(\vec{\nabla}\phi_\epsilon(x))=\vec{L}_\epsilon-\hat{r}(\hat{r}\cdot\vec{L}_\epsilon)\ . \tag{A2}$$

$\vec{\nabla}\phi_\epsilon(x)$ diverges at the origin and the spherical angles are not well defined on the z axis. Here we write the results in a form valid at nonsingular points.

One obtains [using (2.3)]

$$\hat{r}\cdot\vec{\nabla}\phi_\epsilon(x)=0\ , \tag{A3}$$

$$i[\phi_\epsilon(x),\vec{\nabla}\phi_\epsilon(x)]=\hat{r}\times(\vec{\nabla}\phi_\epsilon(x))\ , \tag{A4}$$

$$\hat{r}\times(i[\phi_\epsilon(x),\vec{\nabla}\phi_\epsilon(x)])=-\vec{\nabla}\phi_\epsilon(x)\ , \tag{A5}$$

$$i(\vec{\nabla}\phi_\epsilon(x))\times(\vec{\nabla}\phi_\epsilon(x))=-\frac{1}{r^2}\,\hat{r}\phi_\epsilon(x)\ . \tag{A6}$$

If we use also (2.4) we obtain (always for $r\neq 0$)

$$(\vec{\nabla}\phi_\epsilon(x))^2=\frac{1}{2r^2}\,1\ , \tag{A7}$$

$$\vec{\nabla}^2\phi_\epsilon(x)=-\frac{2}{r^2}\,\phi_\epsilon(x)\ , \tag{A8}$$

$$(\vec{\nabla}\phi_+(x))\cdot(\vec{\nabla}\phi_-(x))=\frac{1}{r^2}\,(\vec{L}_+\cdot\vec{L}_- -\phi_+(x)\phi_-(x))\ . \tag{A9}$$

The following relations are necessary for simplifying and regrouping coefficients

$$\{(\vec{\nabla}\phi_+(x))\times(\vec{\nabla}\phi_-(x))\}(i2\phi_+(x))=-\{(\vec{\nabla}\phi_+(x))\times(\vec{\nabla}\phi_-(x))\}(i2\phi_-(x))$$
$$=\frac{\hat{r}}{r^2}(\vec{L}_+\cdot\vec{L}_- -\phi_+(x)\phi_-(x))=\hat{r}(\vec{\nabla}\phi_+(x)\cdot\vec{\nabla}\phi_-(x))\ , \tag{A10}$$

$$\{(\vec{\nabla}\phi_+(x))\cdot(\vec{\nabla}\phi_-(x))\}(i2\phi_+(x))=-\{(\vec{\nabla}\phi_+(x))\cdot(\vec{\nabla}\phi_-(x))\}(i2\phi_-(x))$$
$$=-\frac{\hat{r}}{r^2}\cdot(\vec{L}_+\times\vec{L}_-)\ , \tag{A11}$$

$$(\vec{\nabla}\phi_+(x))\times\{(\vec{\nabla}\phi_-(x))\times(\vec{\nabla}\phi_+(x))\}+\{(\vec{\nabla}\phi_+(x))\times(\vec{\nabla}\phi_-(x))\}\times(\vec{\nabla}\phi_+(x))=-\frac{1}{2r^2}(\vec{\nabla}\phi_-(x))\ . \tag{A12}$$

[A similar formula is obtained on interchanging $\phi_+(x)$ and $\phi_-(x)$.]

$$\vec{\nabla}\times\{(\vec{\nabla}\phi_+(x))\times(\vec{\nabla}\phi_-(x))\} = -\frac{1}{r^2}\{(\vec{\nabla}\phi_+(x))\phi_-(x) - (\vec{\nabla}\phi_-(x))\phi_+(x)\}. \tag{A13}$$

Finally for evaluating $\{\mathrm{Tr}\vec{\mathrm{F}}^2(x)\}$ we need

$$\mathrm{Tr}\{(\vec{\nabla}\phi_+(x))\cdot(\vec{\nabla}\phi_-(x))\}^2 = \mathrm{Tr}\{(\vec{\nabla}\phi_+(x))\times(\vec{\nabla}\phi_-(x))\}^2 = \frac{1}{2r^4}. \tag{A14}$$

From (2.19) one obtains

$$\mathrm{Tr}\vec{\mathrm{F}}^2 = \sum_\epsilon \frac{1}{r^4}\left[(\eta_\epsilon{}^2+\zeta_\epsilon{}^2-1)^2 + 2r^2(\eta_\epsilon'{}^2+\zeta_\epsilon'{}^2) + (\eta_\epsilon\zeta_{-\epsilon} - \eta_{-\epsilon}\zeta_\epsilon)^2\right] \quad \left(\eta_\epsilon' \equiv \frac{d}{dr}\eta\right). \tag{A15}$$

With (2.30) one obtains

$$\mathrm{Tr}\vec{\mathrm{F}}_{(0)}{}^2 = \sum_\epsilon\left\{\left[\left(\frac{d\epsilon}{r}\right)'\right]^2 + \frac{2d^2\epsilon}{r^4}(\eta_\epsilon{}^2+\zeta_\epsilon{}^2)\right\}, \tag{A16}$$

and with (2.31),

$$\mathrm{Tr}\vec{\mathrm{F}}_{(0)}{}^2 = \left[\left(\frac{d}{r}\right)'\right]^2 + \frac{2d^2}{r^4}\left[\sum_\epsilon (\eta_\epsilon{}^2+\zeta_\epsilon{}^2)\right]. \tag{A17}$$

The corresponding expressions for $\mathrm{Tr}(\vec{\mathrm{D}}\phi)^2$ are obtained by replacing d_ϵ by c_ϵ and d by c, respectively.

APPENDIX B

In our remarks we have tried to make clear how and why we define monopoles in a certain fashion. Here we give a very simple example to illustrate some features we want to avoid.

By defining monopoles via the Higgs fields (instead of directly in terms of $F_{\mu\nu}$), it is easy to obtain various fractional charges. An example can be found in Ref. 4. Such definitions are not technically wrong. But let us try to see exactly what is involved.

Let us take the Abelian picture[11] and consider

$$W_i = (\partial_i\varphi)(1-\cos\theta)I_3,$$

$$W_3 = 0 = W_0$$

$[i = 1, 2, (\varphi, \theta)$ are the spherical angles and I_1, I_2, I_3 are the isospin matrices$]$. This is a Dirac monopole with a string.

For SU(2) the Higgs field (ϕ) had to be (const $\times I_3$). Embedding the solution in SU(3) (with $I_1 = \frac{1}{2}\lambda_1$, and so on), ϕ can be chosen, without disturbing the equations of motion, to be a suitable linear combination of λ_3 and λ_8, with constant coefficients. A pure gauge term parallel to λ_8 may also be added trivially to W_μ.

For SU(n) (with the same W_μ) ϕ can thus be taken to be a combination of I_3 and other $(n-2)$ generators of the Cartan subalgebra. In each case the role of the Higgs field (for this pointlike solution) is the same in this gauge—trivial. The magnetic field remains the same.

For each case one may apply the same gauge transformation, leading to smooth boundary conditions (without string) for all the above-mentioned cases, namely

$$U = e^{i\varphi I_3}e^{-i\theta I_2}e^{-i\varphi I_3}.$$

In ϕ only λ_3 $(=2I_3)$ is affected, while $\lambda_8, \lambda_{15}, \ldots$ remain invariant. We have now $(\vec{\mathrm{x}}\cdot\vec{\mathrm{I}})/r$ instead of I_3. When derivatives are taken this term alone survives.

But in defining the *normalized* $\hat{\phi}$, the coefficient of λ_3 takes on various fractional values according to the linear combination chosen. This leads trivially to various fractional charges for monopoles defined via $\hat{\phi}$.

It is true that the transformation used is singular and changes homotopy classes. But we do not consider this way of introducing fractional charges as really significant. The stability of a solution as a whole will of course not be affected by the definition adopted for the monopoles. In that respect the scalar field plays an essential role for finite-energy solutions.

APPENDIX C

Here we add some remarks on a work of Brihaye and Nuyts[15] (BN) from our point of view (see also Sec. IV). For relevant notations and formulas the reader should refer directly to their paper.

BN consider four ways of embedding an SU(2) representation $(E_i, i=1,2,3)$ in the SU(4) algebra.

Case A. E_i $(i=1,2,3)$ correspond to a spin-$\frac{3}{2}$ representation. The remaining generators can be grouped into a symmetric 5-plet K_{ij} and a symmetric 7-plet N_{ijk}. The three commuting operators of their little group are, in this case,

$$E = \hat{x}_iE_i, \quad K = \hat{x}_i\hat{x}_jK_{ij}, \quad N = \hat{x}_i\hat{x}_j\hat{x}_kN_{ijk}. \tag{C1}$$

The authors note, concerning pointlike solutions, that: "It is remarkable that the results are expressed most simply in the (A, B, C) basis"—where

$$A=-\tfrac{2}{5}E+\tfrac{1}{3}N, \quad B=\tfrac{3}{5}E+\tfrac{1}{3}N, \quad C=\tfrac{1}{2}K \; . \tag{C2}$$

Here we find the connection with our point of view. *It can be shown that A, $(A+B)$, and C are nilpotents* [i.e., $\alpha=0$ in (1, 3)]. This can be verified most simply as follows:

Let $V_{(x)}$ be the rotation such that

$$VEV^{-1}=E_3;$$

then (C3)

$$VKV^{-1}=K_{33}, \quad VNV^{-1}=N_{333} \; .$$

One can choose $E_3=\mathrm{diag}\tfrac{1}{2}(3,1,-1,-3)$. One then verifies easily that

$$A=\mathrm{diag}\tfrac{1}{2}(-1,-1,1,1)\;,$$
$$A+B=\mathrm{diag}\tfrac{1}{2}(1,-1,1,-1)\;, \tag{C4}$$
$$C=\mathrm{diag}\tfrac{1}{2}(1,-1,-1,1)\;.$$

In fact these are equivalent to our ϕ_+, ϕ_-, and $2\phi_+\phi_-$. *The differences arise owing to use of different $V_{(x)}$'s constructing the respective local forms.* All such cases can be unified from our point of view. (B itself has simple properties—see the comment in Refs. 5.)

Their Higgs scalar ϕ for this case is

$$\phi=\phi_E E+\phi_K K+\phi_N N$$
$$=\phi_A A+\phi_B B+\phi_C C \; . \tag{C5}$$

Regrouping we obtain in terms of the nilpotents

$$\phi=(\phi_A-\phi_B)A+\phi_B(A+B)+\phi_C C$$
$$\equiv \phi_a A+\phi_b(A+B)+\phi_c C\;, \tag{C6}$$

say. Noting now the representation (C4) the calculation of the eigenvalues become trivial. In fact, at once,

$$\phi=\mathrm{diag}\tfrac{1}{2}[(-\phi_a+\phi_b+\phi_c),\;-(\phi_a+\phi_b+\phi_c),(\phi_a+\phi_b-\phi_c),(\phi_a-\phi_b+\phi_c)]\;. \tag{C7}$$

This reproduces their result (3.A44). The equal-eigenvalue cases of BN are obtained for

(a) $\phi_b=-\phi_c\neq\phi_a$,

(b) $\phi_b=-\phi_c=-\phi_a$,

(c) $\phi_b=\phi_c=0$.

Starting with the idempotents we can construct, from our point of view, a basis for W_μ by systematically introducing their gradients and the different commutators.

Case B. This is the chain decomposition SU(4)⊃SU(3)⊃SO(3) we already noted in our first paper [Sec. 6 of Ref. 1(a)]. The extra SU(3) generators provide the quadruple 5-plet and the remaining SU(4) generators provide two triplets and one singlet.

Contracting suitably with x_i we obtain five scalars (E, F, Z, K, and N in BN notation). These can again be easily expressed as linear combinations of idempotents. The new feature will be that we will be starting with a set of idempotents which do not all commute. The non-Abelian little group of BN corresponds in our point of view to the simultaneous use of different noncommuting local transformation V_i.

This is certainly a possibility to be envisaged. But particularly interesting could be the combinations compatible with finite energy—not merely with pointlike solutions. The construction of W_μ and ϕ can be continued in our fashion starting with the idempotents.

Case C. This is the embedding we have used in Sec. III. But we have illustrated a new interesting possibility by treating *both* the SU(2)'s on an equal footing—still retaining spherical symmetry. This is suggested by the presence of two commuting SU(2)'s. This possibility has not been included by BN. Different variants corresponding to this decomposition can be easily and systematically explored from our point of view.

Case D. This corresponds to choosing the $\tfrac{1}{2}(\lambda_1,\lambda_2,\lambda_3)$ SU(2) subgroup. We already noted for the SU(3) case [1(a)] that no solution has been found which includes in the W_μ basis the generators transforming as spinor doublets (namely $\lambda_4,\lambda_5,\lambda_6,\lambda_7$). For SU(4) there are four such doublets and four singlets. These doublets have again been excluded by BN in their W_μ basis. The interesting question to be studied is whether the non-Abelian little group can contribute in a nontrivial way to finite-energy solutions.

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[3]It is interesting to point out the link with the geometrical point of view. But the reader need not be familiar with

the techniques of Refs. 2. Since we consider only the adjoint representation of SU(n), a few simple and self-contained definitions will suffice for our purpose.

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[9]The condition (2.38) has a simple interpretation. This picture is obtained by starting with two isolated SU(2) monopole-type systems and then making them undergo two *different* gauge transformations, $\exp(-i\theta_\epsilon 2\phi_+ \phi_-)$ such that $\theta_{-\epsilon}=\theta_\epsilon+(2n+1)\pi/2=$ constant. This is, of course, *not* a gauge transformation of the whole gauge field $W_\mu(x)$, and introduces nontrivial coupling. Thus, for example, $(\vec{F}_+)_{tr}$ no longer commutes with $(\vec{F}_-)_{tr}$.

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