Infrared behavior of non-Abelian gauge theories. II*

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We continue and extend our earlier work on infrared singularities of non-Abelian gauge theories (exemplified by quantum chromodynamics, or QCD), emphasizing the nature of these singularities in space-time rather than in momentum space. The leading logarithms of non-Abelian gauge theories sum to an eikonal (semiclassical) form for processes involving massive particles. This form is like the corresponding Abelian (QED) form, except that the coupling constant g is replaced by the invariant coupling $\bar{g}(t)$ which depends on the momentum transfer t to the gluon. It is not known whether the nonleading logarithms exactly cancel order by order, but there are strong cancellations which are governed by Ward identities. Besides four-dimensional QCD, two analogous theories which show confinement are studied: Two-dimensional QCD is developed in the light-cone gauge without recourse to the large-N approximation of 't Hooft and is shown to have infrared singularities like those of a string theory with longitudinal modes. Three-dimensional QED is briefly discussed, and it is argued that confinement is revealed in the anomalous infrared behavior of the fermion propagator which is an entire function (of momentum) with no pole. Some speculations to this effect are made for four-dimensional QCD.

I. INTRODUCTION

In a recent series of works, some published[1,2] and some not,[3] we have examined the infrared singularities of continuum non-Abelian gauge theories (NAGT's) in perturbation theory. In much of this work, we studied on-shell processes in momentum space and regulated the singularities by giving a small mass μ to the vector gluon. The leading-logarithmic singularities appeared to sum up to a (generalized) exponential form,[4] whose structure was revealed in a differential equation, somewhat resembling a renormalization-group equation, with μ as the independent variable.

The purposes of the present work are to report in detail on the new results of Refs. 2 and 3, and to develop the whole subject in space-time, rather than in momentum space. The gluon mass is an unnecessary device in space-time, where infrared singularities are revealed in the singularities of Green's functions at very large coordinate separations. What emerges (at least for leading logarithms) is an eikonal picture, very much like the familiar eikonal results of QED,[5] with two important differences. First, the gluon propagator $D_{\alpha\beta}(q)$ is replaced by a special gauge-invariant propagator formed from the gauge- and renormalization-group-invariant charge $\bar{g}(q^2)$ in the following way[2,3,6,7]:

$$g^2 D_{\alpha\beta}(q) \to -g_{\alpha\beta}\bar{g}^2(q^2)q^{-2}, \qquad (1.1)$$

where g is the charge renormalized at an arbitrary point. Possible longitudinal ($\sim q_\alpha q_\beta$) or n-dependent terms (in the ghost-free gauges $n_\alpha A^\alpha = 0$) on the right-hand side of (1.1) may occur, but they will not yield physical effects. [An exception is two-dimensional NAGT's in the light-cone gauge in which case (1.1) must be modified; we discuss this later.] Of course, only the infrared-singular part of $\bar{g}$ need be kept in (1.1). A second important difference from QED shows up in some simple processes only at the level of nonleading logarithms, and thus might be ignored; but if $\bar{g}$ is as singular for small q as commonly supposed ($\bar{g}^2 \sim q^{-2}$), the distinction between leading and nonleading logarithms is meaningless. The difference is that the fundamental group-charge matrices for an NAGT do not commute, which necessitates a special but familiar path-ordering prescription for the eikonal exponential in order to incorporate these nonleading logarithms. (Again, two-dimensional NAGT is an exception, at least for the simplest processes.)

It is well known[8] that in QED the leading logarithms contain *all* infrared-singular effects; all nonleading logarithms cancel exactly. We do not know whether this is true for NAGT's, where some nonleading logarithms, associated with violation of naive Ward identities, may be left over (in covariant gauges) even after path-ordering. However, it is possible to imagine that the full theory (in four dimensions) is truly gauge invariant without the need for Faddeev-Popov ghosts in some mysterious way.[9] It is then further possible to imagine that the leftover nonleading logarithms are not really there, and that the modified eikonal description which we present really sums up *all* the infrared singularities. (These flights of imagination are somewhat easier in ghost-free gauges, but are more difficult to discuss from a technical point of view.)

Let us contrast the space-time eikonal picture with earlier momentum-space results,[1,2] using an infinitesimal gluon mass. In both cases, the clean separation of infrared effects from ultraviolet effects requires the hypothesis of finite mass for some of the particles [e.g., the quarks in quantum chromodynamics (QCD)]; these massive particles are supposed to propagate nearly classically. This means that $x\cdot p \gg 1$, where x is a typical coordinate and p is a typical momentum of such a particle. Such an inequality cannot necessarily be realized for a massless particle, which can travel great distances with vanishingly small momentum. A typical momentum-space approach is to calculate the S-matrix elements for the massive particles, which are strictly on-shell. It is then straightforward to show (using the usual eikonal approximations which we discuss in detail later on) that, in QED, all massive-particle propagator and vertex radiative corrections exactly cancel so that the massive particle propagates as if it were free. We show in Sec. IV that a similar cancellation holds, in leading logarithms, for an NAGT. These cancellations are a consequence of Ward identities. All that is left of radiative corrections to the massive-particle propagator is an infrared-singular wave-function renormalization constant.

The space-time eikonal picture is quite different: Vertex corrections disappear, but the massive-particle propagator appears explicitly as one of the factors of the final answer. Its momentum-space singularities (if any) are directly probed as the spatial coordinates approach infinity. It must now be recognized that the propagator for a *confined* particle will not have any singularities in momentum space[10]: Instead, it is an entire function. The concept of the mass shell, so crucial to the momentum-space calculations described in the last paragraph, becomes meaningless. The space-time eikonal picture survives this catastrophe because the existence of a mass shell in momentum space is in no way crucial to the semiclassical interpretation yielded by the eikonal techniques in three-dimensional QED to construct a propagator which is entire in momentum space. Although the concept of a mass shell is lost, the concept of a specific mass for a particle is not. (Think of a classical point particle in a harmonic-oscillator potential.) A second advantage then of a space-time picture is that it is not tied (in the case of confinement) to the meaningless idea of a mass shell.

In QED the eikonal exponential is nothing but the classical action for massive particles interacting with each other through virtual-photon exchange, evaluated along straight-line orbits. In, for example, QCD this classical picture must be slightly modified since a quark does not have definite color as it exchanges gluons with other quarks. This phenomenon of semiclassical rapid color fluctuations will be discussed briefly later; for now let us ignore it. The asymptotic properties of Schrödinger wave functions likewise depend on the exponential of classical action evaluated for large coordinate separations r. For a potential which grows at large distances only classically forbidden orbits can reach large separations, the action is pure imaginary, and all wave functions decrease exponentially. However, for short-range potentials there is (for positive energy) a term in the asymptotic wave function proportional to $r^{-1}e^{ikr}$; the coefficient is the S matrix. All these familiar ideas have their counterpart in field theory, as we discuss in Sec. II. The wave function is replaced by a Green's function, the Green's function is expressed in eikonal form, and the S matrix is again recovered as a coefficient of a term such as $r^{-1}e^{ikr}$. If there is no such term, then only bound states exist and the theory exhibits confinement. Unless the massive-particle propagators have poles, the $r^{-1}e^{ikr}$ term is missing and there is no S matrix. Even if these propagators do have poles (as they do in two-dimensional NAGT's in the light-cone gauge as we treat it), there may be no S matrix.

In this work, we study the fermionic propagators and the eikonalized fermion-antifermion Green's function for two-dimensional QCD in the light-cone gauge, for three-dimensional QED, and [with a purely phenomenological choice of the gauge-invariant gluon propagator in (1.1)[11]] four-dimensional QCD. Two-dimensional QCD in the light-cone gauge was originally studied by 't Hooft,[12] who invoked the large-N limit (N is the number of colors in QCD) to eliminate certain graphs. He adopted a particular rule for eliminating the subsidiary (and only nonvanishing) component of the gluon field, which led to infrared singularities in the quark propagator that washed out the quark pole. Callan, Coote, and Gross[13] used the same prescription in their continuation of 't Hooft's work. However, Einhorn[14] pointed out that this prescription was unnecessary, and a more natural choice left the fermion propagator with the usual free-particle pole. We use a rule very similar to Einhorn's, based on a technique for evaluating Feynman integrals in the light-cone gauge,[15] which also results in an effective free-particle quark propagator. When the rule of Ref. 15 is consistently applied, it turns out that all the graphs which were omitted by 't Hooft because they were nonleading for large N *also* carry only nonleading infrared singularities; those graphs include fermion propagator and vertex corrections, as well as crossed ladder graphs. It is reasonable to drop nonleading infrared singularities in two dimensions because they are down by powers (of a small momentum or large coordinate) rather than just down by logarithms. When these nonleading contributions are omitted and closed quark loops and quark-pair creation are disregarded, the resulting eikonalized theory is a *classical* theory of a string with quarks on the ends and capable of longitudinal oscillations.[16,17] When closed quark loops and quark-pair creation are omitted two-dimensional QCD is not a second-quantized theory but only a first-quantized one, because no vector particles can be physically realized. The transition from the classical eikonal theory to the first-quantized theory requires an improvement on the eikonalized fermion propagator; this is discussed for general dimensions in Sec. III. It amounts to the usual Feynman prescription of integrating the eikonal Green's functions over all classical paths, rather than specifying those particular classical paths which approximately minimize the classical action. Our work is complementary to that of Bars,[18] who begins with an *a priori* choice of a string theory and method of quantization and attempts to show that it is equivalent to two-dimensional QCD. We begin with QCD and show how the longitudinal string picture emerges, complete with a definite prescription for quantization. Although we do not discuss it in this paper, the quantization procedure would similarly be applied to quark-pair processes (splitting of the string), leading ultimately to the second-quantized version of a two-dimensional NAGT.

As an example of another theory which is much simpler than a four-dimensional NAGT but which apparently exhibits confinement we offer three-dimensional QED in Sec. V. The nonrelativistic one-photon potential is proportional to $\ln r$, which dominates kinetic energies at infinite distance; there is no nonrelativistic continuum. In the relativistic theory, we construct an eikonalized fermion propagator which is an entire function. There is an infrared singularity in the fermion mass which appears to be closely related to the original pseudoparticle[19] constructed by Polyakov[20] for three-dimensional QED. Because of confinement, the fermion mass itself is not observable, but it sets the mass scale for fermion-antifermion bound states.

Although we have made no progress in calculating the key quantity $\bar{g}$ in (1.1), it is folklore that it should behave like q^{-2} for small q in four dimensions. With this assumption we exhibit in Sec. V a phenomenological eikonalized fermion propagator which is an entire function in a fashion similar to three-dimensional QED. Similar work has been done by Pagels,[11] who has treated the phenomenology in some detail.

A few brief remarks on gauge invariance are in order. Off-shell propagators and Green's functions such as we study here are not gauge-invariant. The statement that there is a general eikonal form for the infrared singularities of NAGT's is gauge-invariant, however, and this is the point we want to emphasize. It is usually easy to see just what is the gauge-invariant content of any specific formula, because we consider Green's functions only at asymptotically large coordinate separations. Gauge transformations which are not themselves infrared singular will not affect this long-distance behavior. Truly gauge-invariant Green's functions are easily (in principle) constructed, but we will not take this problem up in any detail.

It is particularly interesting to use the ghost-free gauges $n_\alpha A^\alpha = 0$, because they do not suffer from ghost-line contributions to Ward identities. But except for the light-cone gauge $n^2 = 0$, it is difficult to calculate in the ghost-free gauges. It is easy to evaluate Feynman integrals in the light-cone gauge, but the results unfortunately are meaningless because of a special singularity of the type $n \cdot q = 0$, where q is a gluon momentum.[15] (These singularities do not appear in two dimensions.) However, it is not permitted to calculate directly in the light-cone gauge $n^2 = 0$; one must calculate in an axial gauge $n^2 \neq 0$ and then pass to the limit. The two procedures give quite different results for technical reasons which we will not discuss here, and only the second one is free of anomalous singularities. Granted this, one need not even calculate any graphs to know that the gluon propagator must of the the form (1.1) when all massive-particle loops are omitted.[3] The ghost-free Ward identities imply that $Z_1 = Z_2$ for all particles (including gluons) coupled to the gauge current, which leads immediately to (1.1) because $g^2 D_{\alpha\beta}(q)$ is renormalization-group-invariant. The cancellation of the massive-particle propagator and vertex corrections can also be understood as a consequence of $Z_1 = Z_2$.

It would be possible to state the new results of this paper in a somewhat more compact form than we have chosen (e.g., Ref. 3). But the motivation and structure of the theory seems to us much clearer if a certain amount of pedagogical development of well-known principles and results from Abelian gauge theories is included. Thus Sec. II is devoted to the generalization to field theory of the relation between the S matrix and the asymptotic wave function, and Sec. III develops the necessary eikonal techniques for Abelian theories.

Section IV contains our principal new result, as discussed in relation to Eq. (1.1). Although the nonleading contributions do not neatly cancel out, we discuss the uses of Ward identities in developing some of the more obvious cancellations of nonleading logarithms. Section V contains results which are new to us, at least, concerning the applications of Secs. III and IV to fermion propagators in three-dimensional QED and (at a phenomenological level[11]) in four-dimensional NAGT's. Section VI summarizes the paper, and finally there is an appendix on two-dimensional NAGT. No new results are found, but the derivation (again, new to us) avoids the large-N limit and makes clear the connection between our earlier work,[1,2] in which infrared singularities exponentiate, and the results of Refs. 12–14, which seem superficially rather remote from the ideas of Refs. 1 and 2.

II. INFRARED SINGULARITIES IN SPACE-TIME

For a short-range two-body potential the Schrödinger wave function has the asymptotic behavior ($r=|\vec{x}|$)

$$\psi(\vec{x},t)\underset{r\to\infty}{\simeq}\frac{e^{ikr-i\omega t}}{r}f(k,\theta) \tag{2.1}$$

(aside from the contribution of the unscattered wave), where $f(k,\theta)$ is an S-matrix element. Usually the form (2.1) is invoked for scattering processes, but the asymptotic behavior is the same for any spacially localized source $K(z)$ (with corresponding S-matrix element f_K). However, for a potential which is singular at large distances, such as the harmonic oscillator, there is no S matrix and the right-hand side of (2.1) decreases exponentially fast.

To transcribe these statements into field theory, consider a spatially localized source K capable of producing a massive scalar particle and its antiparticle (with field operator ϕ, mass m). Corresponding to the wave function (2.1) is the Fourier transform in z of the Green's function

$$G(x,y,z)=i\langle 0|T(K(z)\phi^\dagger(x)\phi(y))|0\rangle \tag{2.2}$$

in the limit when $x-z$, $y-z$, and $x-y$ all become large. We make this limit specific by taking $x^0=y^0$, $|\vec{x}-\vec{y}|=r$, and localizing K around the space-time origin $z=0$. The Fourier transform is written

$$G_p(x,y)=\int dz\, e^{ipz}G(x,y,z), \tag{2.3}$$

and we take p to be a forward timelike vector with $p^2>4m^2$, $\vec{p}=0$. The restriction $x^0=y^0$ is covariantly written $(x-y)\cdot p=0$.

Let $K(z)$ be a point source: $K(z)=g\phi^\dagger(z)\phi(z)$. We construct G_p as follows:

$$G_p(x,y)=\frac{e^{-ip\cdot(x+y)/2}}{(2\pi)^4}\times\int dq\,\frac{e^{-iq\cdot(x-y)}\Gamma(p,q)}{[(\frac{1}{2}p+q)^2-m^2][(\frac{1}{2}p-q)^2-m^2]}, \tag{2.4}$$

where $\Gamma(p,q)$ is the sum of all *truncated* (free propagators removed) graphs for the process $K\to\phi^\dagger\phi$. If Γ is regular on the mass shell $(\frac{1}{2}p\pm q)^2=m^2$, the integral in (2.4) is dominated by the double pinch when both propagator denominators vanish and the integral in (2.4) yields (with $\omega=p^0$, $x^0=y^0=t$)

$$G_p(x,y)\underset{r\to\infty}{\simeq}\frac{ie^{ikr-i\omega t}}{8\pi r\omega}\Gamma(p,\vec{q}), \tag{2.5}$$

where $\vec{q}$ is any mass-shell vector with $q^0=0$, $|\vec{q}|=k\equiv(\frac{1}{4}p^2-m^2)^{1/2}$. Clearly, (2.5) is analogous to (2.1) with $\Gamma(p,\vec{q})$ as the S-matrix element. The momenta $\frac{1}{2}p\pm\vec{q}$ specify the classical particle paths as they emerge from the source.

A theory with infrared singularities may not have the double pinch exhibited in (2.4) and G_p may decrease more rapidly than r^{-1}. It is tempting to say that this shows confinement, but one must be cautious if the theory is a gauge theory. The reason for caution can be expressed in two equivalent but different-sounding ways: (1) In a gauge theory $G_p(x,y)$ is not gauge-invariant; (2) infrared singularities from virtual gluons can be canceled by phase-space singularities from real-gluon emission,[21,22] order-by-order in perturbation theory. This *order-by-order* cancellation may or may not be meaningful, since gauge theories may have a perturbative and a nonperturbative phase or in some circumstances *only* the nonperturbative phase.

In QED, a gauge-invariant but path-dependent analog of G is, for a neutral current $K(z)$,

$$G_\Gamma(x,y,z)=i\left\langle 0\left|T\left(K(z)\psi(x)\bar{\psi}(y)\exp\left[-ig\int_\Gamma dz_\mu A^\mu(z)\right]\right)\right|0\right\rangle, \tag{2.6}$$

where g is the gauge charge and Γ is any path running from x to y. For an NAGT a similar expression holds except that the integral must be path-ordered (see Sec. III). A path Γ_c which runs along the *classical* electron path from x to z, thence from z to y along the classical path results in complete cancellation of all infrared singularities in QED, and this choice Γ_c is equivalent to the coherent-state picture.[21] Then G_{Γ_c} does behave like r^{-1} for large r, while for other choices of Γ $G_\Gamma \sim r^{-1-\gamma}$ with $\mathrm{Re}\gamma > 0$.

At the moment we are unable, for technical reasons, to make much use of the gauge-invariant form (2.6) for arbitrary Γ. A completely physical wave function must be path-independent as well as gauge-invariant, which might be achieved by some sort of path average over Γ; this is even further beyond our capabilities. Therefore in the following sections we discuss only the primordial, gauge-invariant form (2.2). In principle, at least this is still a useful wave function; for example, the spectrum of bound states having the quantum numbers of K will be found from the singularities in p of G_p, and these are gauge-invariant.

III. THE EIKONAL CONSTRUCTION

Here we begin by reviewing the well-known[5] eikonal development for QED, first in space-time using functional techniques and then, in momentum space, using Feynman graphs and Ward identities. The reason for using two different methods is that the summation of infrared divergences for an NAGT by functional techniques is virtually unexplored, and we are forced to use graphical methods for NAGT's (except for two-dimensional NAGT's in the light-cone gauge). But our purpose is to study the non-Abelian case in space-time, so we will need to know how to translate momentum-space results into the eikonal form in space-time.

A. The Abelian case in space-time

The Lagrangian is

$$\mathcal{L} = \bar\psi(i\gamma\cdot\partial - M + g\gamma\cdot A)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

and the generating functional Z is defined (up to an irrelevant additive constant) by

$$W(J,\eta,\bar\eta) = \exp[iZ(J,\eta,\bar\eta)] = \int (d\psi\, d\bar\psi\, dA)\exp\left[i\int dx(\mathcal{L} + J_\mu A^\mu + \bar\eta\psi + \bar\psi\eta)\right] \tag{3.2}$$

with functional integrations indicated by parentheses. A gauge-fixing term must be added to $\mathcal{L}$ but we do not write it explicitly.

Many years ago Schwinger pointed out (in somewhat different language) that the functional integral over $(d\psi d\bar\psi)$ can be evaluated. In expressing the result of this integration, we make a fundamental approximation by setting equal to one the functional determinant which results, or equivalently we ignore closed fermion loops. Such loops contribute no infrared singularities for finite fermion mass M. Then (3.2) becomes

$$W = \int (dA)\exp\left[i\int dx(\tfrac{1}{2}A^\mu D^{-1}{}_{\mu\nu}A^\nu + J_\mu A^\mu - \bar\eta S_A\eta)\right], \tag{3.3}$$

where $D^{-1}{}_{\mu\nu}$ is the inverse of the photon propagator in the chosen gauge, and S_A is the propagator for a fermion in an external field A, obeying

$$[i\gamma\cdot\partial_x - M + g\gamma\cdot A(x)]S_A(x,x') = \delta(x-x'). \tag{3.4}$$

We are interested in the Green's function

$$G(x,y,z) = \langle 0|\,T(K(z)\psi(x)\bar\psi(y))\rangle \tag{3.5}$$

for a source K of fermions and antifermions. For simplicity the following formulas will be written for a *point* source $K(z) = \bar\psi(z)\psi(z)$, but it is understood that the actual source is to be smeared out over a finite region of space-time. The formula for G is (up to irrelevant factors)

$$G(x,y,z) = \int (dA)S_A(x,z)S_A(z,y)\exp\left(\frac{i}{2}AD^{-1}A\right), \tag{3.6}$$

where we use a shorthand notation for the bilinear in A in (2.3). Omitting the factor $S_A(z,y)$ in (3.6) yields the fermion propagator $S(x,z)$. The most important step is to construct a useful approximation for S_A which allows us to evaluate the integral over A in (3.6).

B. Eikonalized fermion propagator

The fundamental ideas are found in Schwinger's paper on vacuum polarization,[23] which we adapt for our use with functional-integration techniques. The solution to (3.4) is written as a proper-time integral:

$$S_A(x,x') = -i\int_0^\infty ds\langle x|e^{+iHs}|x'\rangle, \tag{3.7}$$

where

$$H = \gamma\cdot\Pi - M + i\epsilon,\quad \Pi_\mu = i\partial_\mu + gA_\mu \tag{3.8}$$

and the states $|x\rangle$, $|x'\rangle$ are eigenstates of the coordinate operator X_μ which obeys $[\Pi_\mu, X_\nu] = ig_{\mu\nu}$. The matrix element in (3.7) can be written as a Feynman path integral over all the paths which join x and x'. However, little progress can be made until the problems of spin are disposed of. To expose these problems, we write as an alternative to (3.7)

$$S_A(x,x') = -i\,\frac{(\gamma\cdot\Pi + M)}{2M}\int_0^\infty ds\langle x|e^{i\bar H s}|x'\rangle, \tag{3.9}$$

where

$$2M\overline{H}=(\gamma\cdot\Pi-M)(\gamma\cdot\Pi+M)$$
$$=\Pi^2-M^2+\tfrac{1}{2}g\sigma_{\mu\nu}F^{\mu\nu}. \qquad (3.10)$$

We shall see that the operator ∂_μ in (3.8) is $O(M)$, so that Π^2-M^2 has a term of order gMA, where A is a characteristic field strength. We expect $A\sim r^{-1}$, where r is the (spatial) distance between x and y in (3.5), thus $F_{\mu\nu}\sim r^2$. Then the term $\frac{1}{2}g\sigma_{\mu\nu}F^{\mu\nu}$ in (3.10) is smaller by a factor $Mr\gg 1$ than the term gMA in Π^2-M^2, and we drop it. The g^2A^2 term is also small as long as $g\ll Mr$, in which case the operator $\gamma\cdot\Pi+M$ of (2.9) can be replaced by $\gamma\cdot p+M$, where p_μ is represented by $i\partial_\mu$. In short, spin does not matter for infrared phenomena: We calculate the spinless propagator with $2M\overline{H}=\Pi^2-M^2$ and then multiply by $(\gamma\cdot p+M)(2M)^{-1}$ to calculate the fermion propagator. This factor is nearly one and we will omit it for the most part.

The conversion of (3.9) to a path integral is standard.[24] Modulo a real constant, (3.9) becomes

$$S_A(x,x')\simeq -i\int_0^\infty ds\int(dp\,dz)\exp\left(-i\int[p\cdot\dot z-\overline{H}(p,z)]ds'\right)$$
$$=-i\int_0^\infty ds\int(dz)\exp\left(i\int_0^s ds'[-\tfrac{1}{2}M(\dot z^2+1)+g\dot z_\mu A^\mu]\right), \qquad (3.11)$$

where the dots are proper-time derivatives and the path integral goes over all paths beginning at x and ending at x'. We have expressed S_A as a path integral of a classical action, which differs from the usual action by the substitution of $\int\frac{1}{2}M(\dot z^2+1)$ for the reparametrization-invariant $\int M(\dot z^2)^{1/2}$.

The Gaussian path integral

$$\int(dz)\exp\left[-i\int_0^s ds'\tfrac{1}{2}M(\dot z^2+1)\right]$$
$$\sim s^{-2}\exp\left\{-\frac{i}{\hbar}\left[\frac{Ms}{2}+\frac{M(x-x')^2}{2s}\right]\right\} \qquad (3.12)$$

shows that (3.11) reduces to a well-known integral expression for a free, spinless propagator in the limit $g\to 0$. We have displayed the usually omitted $\hbar$ to facilitate discussion of the classical limit. More useful to us is the *classical eikonal* form, which is gotten by expanding the path integral in (3.11) or (3.12) around the classical path. For small g this path is just a straight line: $x_\mu-x'_\mu=v_\mu s'$, $0<s'<s$, where v_μ is a forward timelike four-velocity ($v^2=1$). In the limit $\hbar\to 0$, (3.12) becomes a δ function of the form $e^{-iMs}\delta(x-x'-vs)$, and (3.11) becomes

$$S_A^{\rm cl}(x,x')=-i\int_0^\infty ds\,e^{-iMs}\delta(x-x'-vs)$$
$$\times\exp\left[ig\int_0^s ds'v\cdot A(x-x'-vs')\right]. \qquad (3.13)$$

The Fourier transform of the free ($A=0$) propagator is

$$S_0^{\rm cl}(p)=\int dx\,e^{ipx}S_{A=0}^{\rm cl}(x)$$
$$=(v\cdot p-M)^{-1}, \qquad (3.14)$$

which differs from the free Dirac propagator by the substitution $\gamma_\mu\to v_\mu$, and is singular when $p=Mv_\mu$.

We shall refer to (3.13) as the *classical* propagator, and the path-averaged expression (3.11) as the *first-quantized* propagator, because it is constructed from a classical action by the usual methods for first quantization. Except for a brief mention in the Appendix, we use only the classical propagator in this paper. We then express infrared singularities as exponentials of classical actions; these actions may be first-quantized with the propagator (3.11).

All that remains is to put (3.13) into (3.6), and to evaluate the Gaussian functional integrals. We find the well-known result

$$G(x,y,0)=-\frac{1}{(2\pi)^8}\int_0^\infty ds\int_0^\infty ds'\int dp\,dp'\exp[-ipx+is(v\cdot p-M)-ip'y+ix'(v'\cdot p'-M)-\tfrac{1}{2}ig^2\mathcal{J}D\mathcal{J}], \qquad (3.15)$$

where

$$\mathcal{J}_\mu(z)=\int_0^s d\tau\,\delta(z-v\tau)v_\mu-\int_0^{s'}d\tau'\,\delta(z-v'\tau')v'_\mu \qquad (3.16)$$

and

$$\mathcal{J}D\mathcal{J}=\int dz\,dz'\mathcal{J}_\mu(z)D^{\mu\nu}(z-z')\mathcal{J}_\nu(z'). \qquad (3.17)$$

The sum of all infrared-singular radiative corrections to the fermion propagator has a similar form:

$$S^{\rm cl}(x) = -\frac{i}{(2\pi)^4}\int_0^\infty ds \int dp \exp[-ipx + is(v\cdot p - M) - \tfrac{1}{2}g^2\mathcal{J}(1)D\mathcal{J}(1)], \qquad (3.18)$$

where

$$\mathcal{J}_\mu(1) = \int_0^s d\tau\, \delta(z - v\tau)v_\mu \qquad (3.19)$$

is the contribution of one fermion to the total current.

The integral in (3.17) diverges for small τ, τ' because of our idealization of $K(z)$ to a point source, but this divergence has nothing to do with infrared singularities, is easily regulared by smearing $K(z)$, and we shall ignore it. An analogous noninfrared divergence is ignored in (3.18). Then (3.17) can be evaluated in the Feynman gauge:

$$G(x,y,0) = S^{\rm cl}(x)S^{\rm cl}(y)\exp\left[-\frac{g^2}{4\pi^2}\ln(x\cdot v + y\cdot v')\int_0^1 \frac{d\beta\, p'\cdot p}{(p'+p)^2\beta(1-\beta) - M^2}\right], \qquad (3.20)$$

where we have used the implicit constraints following from the p and p' integrals in (3.15) to replace the upper limits s, s' in (3.16) by $x\cdot v, y\cdot v'$ respectively, and have written $p = Mv$, $p' = Mv'$ in (3.20).

The relation with *mass-shell* calculations,[1] in which the photon is given a small mass μ and the fermions are exactly on-shell (i.e., $x, y\to\infty$) is evident: G is given, in momentum space, by the exponential factor in (3.20) with the logarithm replaced by $\ln\mu^{-1}$. This exponential factor, as it stands in (3.20), obeys a differential equation involving the operator $x\cdot\partial_x + y\cdot\partial_y$ (instead of $-\mu\partial/\partial\mu$), which is exactly of the type we have given earlier.[1]

It is clear from (3.15) and (3.16) why the gauge-invariant Green's function G_{Γ_c} [see (2.6)] has no infrared singularities. When the path integral retraces the classical path, it exactly cancels the classical current of (3.16), as one sees by evaluating the functional integrals. We have already pointed out that this may or may not be significant for confinement: In three-dimensional QED the fermions are confined even though G_{Γ_c} has no infrared singularities. We return in Sec. V to the evaluation of $S^{\rm cl}(x)$ in (3.18) for three-dimensional QED and (phenomenologically) for four-dimensional NAGT.

C. Difficulties with non-Abelian gauge theories

In the non-Abelian case it is possible to write analogs of (3.6) and (3.18), but the functional integrals over the vector-meson fields are not Gaussian. There are also some difficulties with the classical interpretation of the results, since a non-Abelian charge (e.g., color) is not a classical attribute. In this subsection we offer a classical action-at-a-distance theory of color fluctuations in quark-gluon exchange.

The Lagrangian is

$$\mathcal{L}_{\rm YM} = \bar\psi(i\gamma\cdot\partial - M + g\gamma\cdot A^a t^a)\psi - \tfrac{1}{4}G^a_{\mu\nu}G^{\mu\nu a}, \qquad (3.21)$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu. \qquad (3.22)$$

We also use the notation

$$A_\mu = \sum t^a A^a_\mu,\quad G_{\mu\nu} = \sum t^a G^a_{\mu\nu}, \qquad (3.23)$$

where the t^a are fermionic group charges.

The non-Abelian version of (3.13) for the fermion propagator in an external field is

$$S^{\rm cl}_A(x,x') = -iP\int_0^\infty ds\, e^{-iMs}\delta(x - x' - vs) \times \exp\left[ig\int_0^s ds'\, v\cdot A(x - x' - vs)\right], \qquad (3.24)$$

which differs from (3.13) only the by path-ordering symbol P. P instructs us to order the non-commuting fields A_μ from left to right in order of decreasing proper time s. Thus

$$PA_\mu(s)A_\nu(s') = A^a_\mu(s)A^b_\nu(s') \times [t^a t^b\theta(s-s') + t^b t^a\theta(s'-s)]. \qquad (3.25)$$

The eikonal Green's function is

$$G(x,y,z) = \int (dA)\exp\left(-\tfrac{1}{4}i\int dx\, G_{\mu\nu}{}^2\right)S^{\rm cl}_A(x,z)S^{\rm cl}_A(z,y), \qquad (3.26)$$

but the functional integrals cannot be evaluated because $G_{\mu\nu}{}^2$ is not quadratic in the A^a_μ. Of course, we recover the usual perturbation expansion by expanding A^a_μ around its classical value $D^{-1}{}_{\mu\nu}\mathcal{J}^{\nu a}$, but this has not yet led us to any great insights. In the next section we show, using momentum-space methods, that the sum of leading logarithms does lead to a form such as (3.26) with ${G_{\mu\nu}}^2$ replaced by a term which is quadratic in the A^a_μ, and we extract an eikonal Green's function and propagator such as (3.15)–(3.18) with a special choice for $D_{\mu\nu}$.

As we have already mentioned, these eikonal forms are expressed in terms of *classical* actions for point particles interacting through a relativistic "potential." While an Abelian charge is a classical attribute of a point particle, a non-Abelian charge is not usually considered to be one, since it will be changed by the emission or absorption of a gluon. Let us discuss briefly the minimal extension of classical QED to the non-Abelian case.

Abelian currents, such as (3.16) or (3.19), are constructed solely from mechanical quantities. But non-Abelian currents carry a group label, and it is necessary to introduce a classical group-space vector $\xi^a(\tau)$ to write the current of a given particle:

$$\mathcal{J}^a_\mu(x)=\int d\tau\,\dot{z}_\mu(\tau)\xi^a(\tau)\delta(x-z(\tau)). \tag{3.27}$$

The covariant conservation law $D^\mu\mathcal{J}^a_\mu=0$, where D_μ is the covariant derivative

$$D^{ab}_\mu=\partial_\mu\delta^{ab}+g\epsilon_{abc}A^c_\mu, \tag{3.28}$$

implies the equation of motion

$$\dot{\xi}^a+g\epsilon_{abc}\dot{z}_\mu A^{\mu b}(z(\tau))\xi^c=0 \tag{3.29}$$

for the group vector, and the solution is

$$\xi(\tau)=P\exp\left[+ig\int_0^\tau d\tau'\dot{z}_\mu(\tau')T^bA^{\mu b}(\tau')\xi(0)\right], \tag{3.30}$$

where $T^b_{cd}=-i\epsilon_{bcd}$ are the group generators of the adjoint representation. In (3.30) the field $A^{\mu b}$ is the classical field produced at the given particle by all the other particles, hence expressible in principle in terms of the mechanical motions of these particles and their group vectors ξ. Note that (3.29) implies that $\xi^a\xi^a$ is constant, and that constant is just t^at^a.

The equation of motion for the particles of mass M is

$$M\ddot{z}_\mu=g\dot{z}^\nu\xi^aG^a_{\mu\nu}, \tag{3.31}$$

and it is gauge-invariant, since both ξ^a and $G^a_{\mu\nu}$ transform homogeneously like members of the adjoint representation under gauge transformations. As in general relativity, this equation of motion is a consequence of the nonlinear field equations and is not a separate postulate.

In general it must be expected that any classical-eikonal representation of infrared singularities must involve the currents (3.27) with nontrivial $\xi^a(\tau)$, and that the corresponding classical action will be complicated with messy expressions such as (3.30). However, there are special circumstances in which these complications do not appear, and these are the only circumstances with which we deal in this paper. When exactly two particles are emitted from a group-singlet source, their initial ξ's will be parallel. Then, in (3.30), the field A^b_μ can be taken to point in the direction $\xi(0)$, the exponent vanishes by the antisymmetry of the T^b, and ξ is unchanging as τ advances. The classical-eikonal action no longer depends on $\xi^a(\tau)$, but only on the fixed group matrices t^a as we discuss in Sec. IV.

Evidently there is no fundamental difficulty in analyzing NAGT's directly in space-time, using (3.26); the difficulty is only our inexperience. In order to compare eikonal results with the analysis of Sec. IV, we have to know how these results look in momentum space, to which we now turn.

D. The Abelian eikonal in momentum space

Aside from ignoring closed fermion loops, the single approximation we need to make in the momentum-space Feynman rules is to replace γ_μ by v_μ ($v^2=1$), a four-velocity. The vertex is v_μ, and the free fermion propagator is $(v\cdot p-M)^{-1}$. Define the on-shell momentum by $\bar{p}_\mu\equiv Mv_\mu$, and define a vector q by $p=\bar{p}+q$. The effective propagator is then $M(\bar{p}\cdot q)^{-1}$, and the vertex is $\bar{p}_\mu/M$. Fermions at large distances are nearly on-shell, so $q\ll\bar{p}$ in the sense that *each* component of q is much smaller than M. For notational simplicity we drop the overbars, and reserve the symbols $p,p',\ldots$ for *on-shell* momenta; a general fermion momentum must be written $p+q, p+q'$, etc. This notation is used here and in Sec. IV *only*.

The eikonal in momentum space is often derived[5] by showing that sums of effective fermion propagators with photons attached in all possible ways add to a simply factorized form. Much more to the point is the observation that these propagators allow us to express very simply the infrared-singular parts of the Ward identities.

Consider the vertex graph in Fig. 1, where $q,q'\ll p$. A simple eikonal identity

$$\frac{1}{p\cdot(q+k)}\frac{1}{p\cdot(q'+k)}=\frac{1}{p\cdot(q-q')}\times\left[\frac{1}{p\cdot(q'+k)}-\frac{1}{p\cdot(q+k)}\right] \tag{3.32}$$

allows us to express the vertex correction in the form

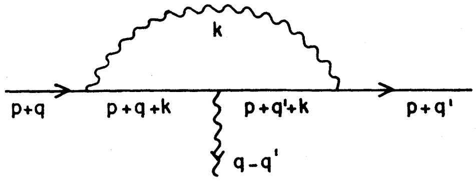

FIG. 1. Lowest-order Abelian vertex correction. Wavy lines are photons, solid lines are fermions, and p and p' are on-shell.

$$\Gamma_\mu = \frac{p_\mu}{M} + \frac{1}{p\cdot(q-q')}[\Sigma(p+q') - \Sigma(p+q)]$$
$$= \frac{p_\mu}{p\cdot(q-q')}[S^{-1}(p+q) - S^{-1}(p+q')], \quad (3.33)$$

where

$$S^{-1}(p+q) = \frac{p\cdot q}{M} - \Sigma(p+q) \quad (3.34)$$

is the inverse propagator to one-loop order, and Σ is the proper self-energy. Equation (3.33) obeys the Ward identity

$$(q-q')_\mu \Gamma^\mu = S^{-1}(p+q) - S^{-1}(p+q'), \quad (3.35)$$

which holds for the exact theory.

It is tempting to argue on general grounds that (3.33) holds, order-by-order, to all orders in perturbation theory, with an error that is at least one power of q or q' smaller than the dominant term [which is $O(g^{2N}\ln^N(q \text{ or } q'))$]. Denote the right-hand side of (3.33) as $\overline{\Gamma}_\mu$. Then the Ward identity (3.35) tells us that the difference between the *exact* vertex Γ_μ and its infrared approximation $\overline{\Gamma}_\mu$ obeys $(q-q')_\mu(\Gamma^\mu - \overline{\Gamma}^\mu) = 0$. This difference is thus expressed as

$$\Gamma_\mu - \overline{\Gamma}_\mu = i\sigma_{\mu\nu}(q-q')^\nu F_1 + [p_\mu(q-q')^2 - (q-q')_\mu p\cdot(q-q')]F_2 + \cdots, \quad (3.36)$$

where the invariant functions F_i are free of inverse power singularities in q or q', in perturbation theory. Then $\Gamma - \overline{\Gamma}$ is at least one power of $q(q')$ smaller than Γ_μ. Unfortunately this argument *fails* for NAGT's, as we show explicitly in Sec. IV.

It is straightforward to check that (3.33) does hold to all orders, using the infrared Feynman rules given earlier. Similar results hold for fermion lines with more than one infrared photon, such as the graphs of Fig. 2. Up to an overall constant, these graphs yield

$$\frac{p_\mu p_\nu}{p\cdot q_1 p\cdot q_2}[\Sigma(p+q'+q_1) - \Sigma(p+q') + \Sigma(p+q'+q_2) - \Sigma(p+q)], \quad (3.37)$$

which obeys the exact Ward identity on both photon lines, once the corresponding vertex functions are expressed as in (3.33).

In this way every one-particle irreducible graph with one fermion line is expressed in terms of self-energy parts, plus remainders which are small by powers of infrared momenta (not by powers of logarithms). When they are all added together, and graphs constructed with more than one fermion line, miraculous cancellations occur which yield eikonal formulas. Actually there is no miracle, since we already know from Secs. III A and B that the eikonal does sum up all infrared singularities.

The point of this rather lengthy section is that, to the extent that Feynman graphs of NAGT's satisfy naive Ward identities of the QED type (3.35), these graphs can be summed up into an eikonal form in space-time. Of course in the usual covariant gauges, the Ward identities of NAGT's are not the naive generalization of those in QED. Nonetheless, as we show in the next section, the leading logarithms do eikonalize but with a modified gluon propagator.

IV. NON-ABELIAN GAUGE THEORIES

Again we turn our attention to a Green's function such as (3.5), but this time in momentum space. The quark-antiquark source $K(z)$ is a group-singlet point source. To establish conventions and normalizations, we record the value of the one-loop graph of Fig. 3 in the Feynman gauge:

$$G_T(p+q, p'+q') = \frac{g^2 C_F}{4\pi^2} p'\cdot p \int_0^1 \frac{d\beta}{\beta(1-\beta)t - M^2} \times \ln[\beta p\cdot q + (1-\beta)p'\cdot q'], \quad (4.1)$$

where $t = (p+p')^2$ and C_F is the quadratic Casimir eigenvalue for the fermions, and the subscript T indicates that external propagators are truncated. Corresponding to G_T is G, the untruncated Green's

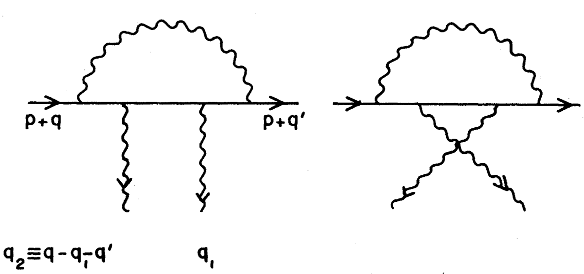

FIG. 2. Example of a four-point graph which can be expressed in terms of fermion self-energy graphs.

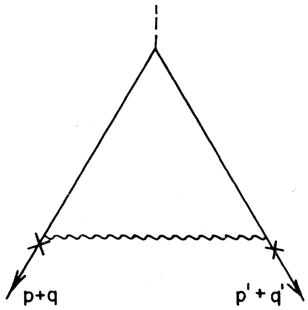

FIG. 3. Lowest-order correction to G_T, the truncated Green's function. (Truncated legs are marked with a cross.)

function. Equation (4.1) should be compared to the exponent of the space-time eikonal Green's function (3.20). In succeeding orders of g^2, graphs with leading logarithms are $O(g^{2N}L^N)$, where L is symbolic for a logarithm such as that of (4.1). We have earlier (second paper of Ref. 1) given the rule for finding leading graphs; the rule is applicable for covariant gauges which is all that we consider (for reasons mentioned later):

To find the set of leading $(N+1)$-loop skeleton graphs, add a gluon line to the *external* fermion legs of the leading N-loop skeleton graphs in all possible ways; the leading nonskeleton graphs are found by adding all vertex and propagator corrections.

It appears that this rule agrees with analyses given by other authors.[25] We note that among skeleton graphs only planar graphs (i.e., straight ladders) can be leading, and that skeleton graphs with four-gluon vertices are nonleading. However, not every planar skeleton graph is leading.

In this section we try to imitate as closely as possible the methods, using Ward identities, of Sec. III D. The new feature of NAGT is the deviation from naive Ward identities as expressed by ghosts and Λ lines.

A. Fourth order

Our objective is to illustrate, in this order, the utility of decomposing the fundamental three-gluon vertex into a part which exactly satisfies the naive Ward identity and a part which is a pure divergence and which generates the ghostlike lines called Λ lines by 't Hooft.[26] This decomposition is asymmetrical on the three lines of the vertex, and a special graphical notation will be needed for it.

We begin with the conventional three-gluon vertex as shown in Fig. 4(a), which is constructed according to the rules of Abers and Lee[24] and is denoted by $i\epsilon_{abc}V_{\alpha\beta\lambda}$. This will be split into two parts, the decomposition depending on the gauge. The gauge is specified by the free gluon propagator:

$$D_{\mu\nu}=q^{-2}[-g_{\mu\nu}+q_\mu q_\nu q^{-2}(1-\xi)], \qquad (4.2)$$

$$q^\mu D_{\mu\nu}=-\xi q_\nu q^{-2}. \qquad (4.3)$$

Then we write, assigning a special role to the line labeled k,

$$V_{\alpha\beta\lambda}=\overline{V}_{\alpha\beta\lambda}+\xi^{-1}[k'_\lambda g_{\alpha\beta}+(k'+k)_\beta g_{\alpha\lambda}], \qquad (4.4)$$

$$\overline{V}_{\alpha\beta\lambda}=-(2k'+k)_\alpha g_{\beta\lambda}+2k_\lambda g_{\alpha\beta}-2k_\beta g_{\alpha\lambda} + (1-\xi^{-1})[k'_\lambda g_{\alpha\beta}+(k'+k)_\beta g_{\alpha\lambda}]. \qquad (4.5)$$

It is easy to check that

$$k^\alpha \overline{V}_{\alpha\beta\lambda}=D^{-1}{}_{\beta\lambda}(k'+k)-D^{-1}{}_{\beta\lambda}(k'), \qquad (4.6)$$

which is the correct naive Ward identity on line k. Of course, $\overline{V}_{\alpha\beta\lambda}$ does not satisfy the naive Ward identiy on the other two lines. The difference, $V-\overline{V}$, is a pure divergence. When $\overline{V}_{\alpha\beta\lambda}$ is used in a graph, it is symbolized as in Fig. 4(b), with the special line (k, in this case) crossed with a bar.

It will simplify the discussion to use only the Feynman gauge $\xi=1$. Note that the divergence terms in (4.4), when acting on the propagators in Fig. 4, give a result independent of ξ because of (4.3).

The fourth-order vertex graphs corresponding to the process of Fig. 3 are shown in Fig. 5, except for pure self-energy graphs where all the gluons are on one or the other fermion line, and gluon propagator corrections.

Graphs (a)–(e) differ from the corresponding QED graphs by trivial group-theoretic weight factors. According to the leading-line rule all are leading [i.e., $O(g^4L^2)$] except the crossed ladder (b).

Consider now the vertex decomposition (4.4) as applied to the vertex insertion on graph 5(f) (shown in Fig. 6). Let us denote the value of the graph of Fig. 6 as $\frac{1}{2}C_A\overline{\Gamma}_\alpha$, thus explicitly displaying the group-theoretic weight factor (C_A is the Casimir eigenvalue of the adjoint representation). Accord-

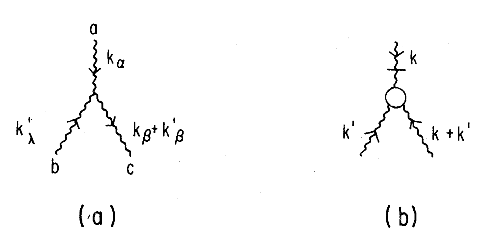

FIG. 4. (a) The fundamental three-point Yang-Mills vertex $V^{abc}_{\alpha\beta\lambda}$. (b) Graphical notation for $\overline{V}_{\alpha\beta\lambda}$ of Eq. (4.5).

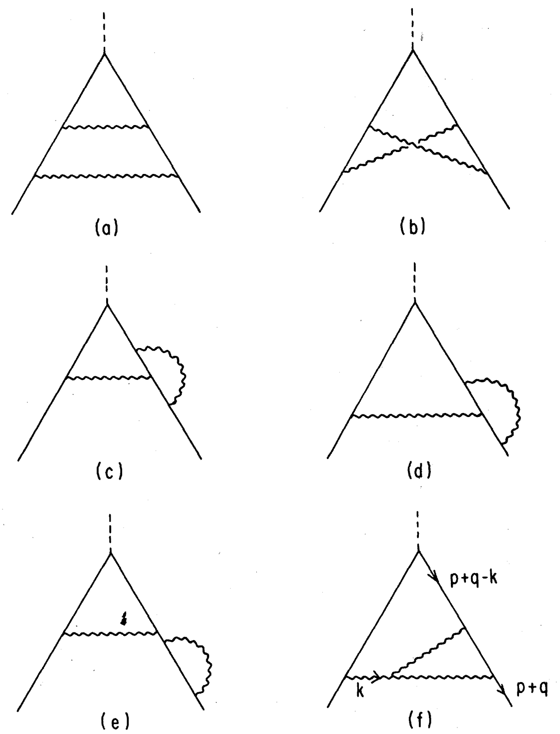

FIG. 5. Fourth-order contributions to G (excluding pure fermion self-energy graphs and gluon propagator corrections). Mirror images of nonsymmetric graphs are to be added.

ing to the general arguments of Sec. III D, $\overline{\Gamma}_\alpha$, since it obeys a naive Ward identity, should be representable as

$$\overline{\Gamma}_\alpha = \overline{\Gamma}_\alpha^W \equiv \frac{p_\alpha}{p\cdot k}[\Sigma(p+q-k) - \Sigma(p+q)], \qquad (4.7)$$

where the Σ's are one-loop fermion self-energies except that the group-theoretic weight is omitted, with an error small by at least one power of k or q. We cannot prove this as we did in QED; indeed, it is not true. We show this by calculating $\overline{\Gamma}_\alpha$ directly. To begin, note that in the Feynman gauge $\overline{V}_{\alpha\beta\lambda}$ in (4.5) has three terms. The last two terms, namely $2k_\lambda g_{\alpha\beta} - 2k_\beta g_{\alpha\lambda}$, are antisymmetric in β and λ, and when multiplied into the lowest-order fermion vertices $\sim p_\lambda p_\beta$ vanish identically. These two terms can only contribute when multiplied into a fermion vertex of $O(k$ or $q)$, and yield a contribution which actually is small by a power. We can drop these two terms as nonleading not only in the graph of Fig. 6, but in any graph. Then $\overline{\Gamma}_\alpha$ has the value, up to an overall constant,

$$\overline{\Gamma}_\alpha \sim \int \frac{dk'(2k'+k)\alpha}{k'^2(k+k')^2[(p+q+k')^2 - M^2]}. \qquad (4.8)$$

(We use the exact denominator for the fermion propagator rather than its eikonal approximation; this does not affect the infrared behavior.) On the other hand, $\overline{\Gamma}_\alpha^W$ of (4.7) can be written

$$\overline{\Gamma}_\alpha^W \sim \frac{p_\alpha}{p\cdot k}\int \frac{dk'(k^2+2k'\cdot k)}{k'^2(k+k')^2[(p+q+k')^2 - M^2]}. \qquad (4.9)$$

When the integral over k' is evaluated in (4.8) and (4.9) k' in the numerators is replaced by its shifted value. We assign Feynman parameters α_i to the denominators as shown in Fig. 6, in which case k' is replaced by $-\alpha_1(p+q) - \alpha_2 k$. Evidently q can be dropped compared to p in this shift, but k cannot necessarily be dropped since $\alpha_1 \ll \alpha_2$ in the α integration. Making the shift we find

$$\overline{\Gamma}_\alpha - \overline{\Gamma}_\alpha^W \sim \int \frac{d\alpha_1 d\alpha_2 d\alpha_3}{D}\delta(1-\textstyle\sum\alpha_i)(1-2\alpha_2) \times\left(k_\alpha - \frac{p_\alpha k^2}{p\cdot k}\right), \qquad (4.10)$$

where

$$D = \alpha_2\alpha_3 k^2 + \alpha_1\alpha_2(p+q-k)^2 + \alpha_1\alpha_3(p+q)^2 - \alpha_1 M^2. \qquad (4.11)$$

The numerator $1-2\alpha_2$ may be written $\alpha_1+\alpha_3-\alpha_2$, and the α_1 term in (4.10) is genuinely small by a power of k compared to $\overline{\Gamma}_\alpha$. The term $\alpha_3 - \alpha_2$ is *not* small by a power, but it does not have a pole at $p\cdot k = 0$ either since D is, for all practical purposes, a symmetric function under $\alpha_2 \leftrightarrow \alpha_3$ when $p\cdot k = 0$. The general argument attempted in connection with (3.36) fails, and we are unable to ignore the difference $\overline{\Gamma}_\alpha - \overline{\Gamma}_\alpha^W$ as small by a power. However, it turns out that $\overline{\Gamma}_\alpha - \overline{\Gamma}_\alpha^W$ does not contribute a leading logarithm to the Green's function G; we discuss this and other nonleading logarithms shortly.

Now consider Fig. 5(c), which is an Abelian graph except for the group-theoretic weight factor. This factor is $C_F(C_F - \frac{1}{2}C_A)$ and the second term comes from the vertex insertion. [In Fig. 5(f), the weight is $\frac{1}{2}C_F C_A$, with the $\frac{1}{2}C_A$ coming from the vertex insertion.] The vertex insertion of Fig. 5(c) is, by the analysis of Sec. III D, cor-

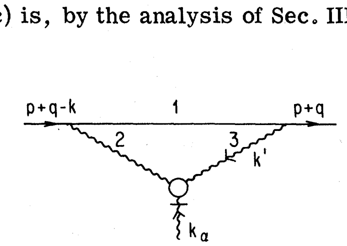

FIG. 6. The $\overline{V}$ part of the vertex insertion in Fig. 5(f). The numbers indicate assignments of Feynman parameters in Eq. (4.10).

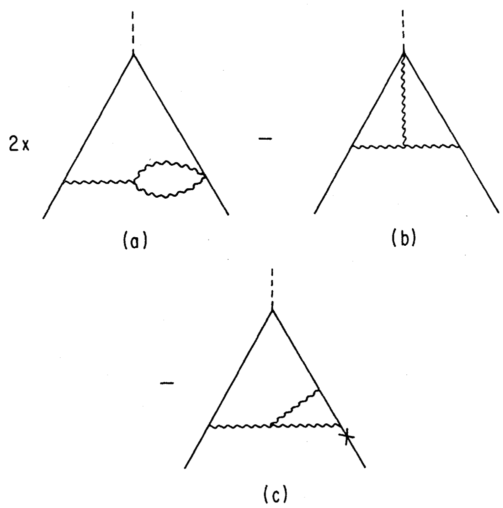

FIG. 7. Topological structure of the graphs generated by the Λ lines, or divergence parts, of the vertex [see Eq. (4.4)]. Each of these graphs appears twice in G, thus the total weight of (a) is 4. The crossed line in (c) is truncated.

rectly given by (4.7) with an error which is small by a power. It follows that, to leading order of logarithms, the sum of 5(c) and 5(f) is simply 5(c) again but with a weight C_F^2, plus whatever the term in square brackets in (4.4) yields for 5(f).

These divergence parts are easily analyzed, as far as leading logarithms are concerned. The term $k'_\lambda g_{\alpha\beta}$ in (4.4) is dotted into a vertex $M^{-1}p_\lambda$, and yields an elementary Ward identity

$$\frac{k'\circ p}{M}=S^{-1}(p+q+k')-S^{-1}(p+q), \tag{4.12}$$

where S is the eikonalized fermion propagator. Similarly, the $(k'+k)_\beta g_{\alpha\lambda}$ term is dotted into $M^{-1}p_\lambda$, and

$$\frac{(k'+k)\circ p}{M}=S^{-1}(p+q+k')-S^{-1}(p+q-k). \tag{4.13}$$

The divergence parts yield four terms, whose topological structure is shown in Fig. 7. Figure 7(a) is a propagator correction, while Figs. 7(b) and 7(c) are nonleading, as one finds by explicit computation. So the Λ lines act, in leading order, merely to correct the one-loop gluon self-energy.

Even without calculating anything, it is clear what this gluon propagator correction must be. It must change the gauge-dependent one-loop self-energy to a gauge-invariant form, and the only available object is the invariant charge $\bar{g}(k^2)$, which obeys the renormalization-group (RG) equation

$$k\cdot\frac{\partial}{\partial k}\bar{g}=-\beta(\bar{g}). \tag{4.14}$$

Explicit calculation of Fig. 7(a) (with an overall weight of 4, to account for the mirror graph) shows that the effective propagator $\Delta_{\alpha\beta}$ can be written in the form[3]

$$g^2\Delta_{\alpha\beta}(k)=-g_{\alpha\beta}\frac{\bar{g}^2(k^2)}{k^2}+\cdots, \tag{4.15}$$

$$\bar{g}^2=1+\tfrac{11}{3}C_A\frac{g^2}{16\pi^2}\Gamma(2-\tfrac{1}{2}d)(k^2)^{d/2-2}+O(g^4), \tag{4.16}$$

and $\bar{g}^2$ is indeed the RG-invariant, gauge-invariant running charge, to $O(g^2)$, obeying (4.14). The omitted terms in (4.15) are gauge-dependent and proportional to $k_\alpha k_\beta$; they come from the ordinary gluon self-energy graph. Figure 7(a) must be interpreted as contributing only to the $g_{\alpha\beta}$ part of (4.15). The gauge-dependent longitudinal terms generate nonleading contributions which are without physical significance.

It remains to discuss graphs which are purely corrections to the fermion propagator. The detailed calculations are not terribly interesting; they are summarized by saying that the propagator (to fourth order) assumes an eikonal form such as (3.18), *except that the propagator $D_{\alpha\beta}$ is replaced by $\Delta_{\alpha\beta}$ as in (4.15) and (4.16).*

So far the leading-logarithm results can be summarized as an eikonal expansion of precisely the type (3.15)–(3.18), except that the combination $g^2D_{\alpha\beta}(x-x')$ which occurs there is replaced by $g^2C_F\Delta_{\alpha\beta}(x-x')$. Later we shall see that certain nonleading logarithms lead to a simple modification, in which path-ordering is employed as in (3.24), which accounts for nonleading logarithms associated with crossed ladder graphs such as Fig. 5(b).

B. Sixth and higher order

A full discussion would be inordinately lengthy. The main point concerning leading logarithms can be appreciated from the rule given earlier: The *only* leading graphs are propagator and vertex corrections to the ladder graphs. Consider a graph such as shown in Fig. 8, where the blob is a one-gluon irreducible. We do *not* impose one-fermion irreducibility; to convert the blob into a proper vertex correction we need only truncate the external fermion legs, a procedure which we save until the end. The explicitly shown three-gluon vertex is decomposed just as before and the $\bar{V}_{\alpha\beta\lambda}$ part gives a vertex correction which obeys the naive Ward identity. (Actually, in general one must include four-gluon couplings in order that the Ward identity be strictly true; we may assume that those nonleading graphs are added to those shown in Fig. 8.) This vertex part is expressed

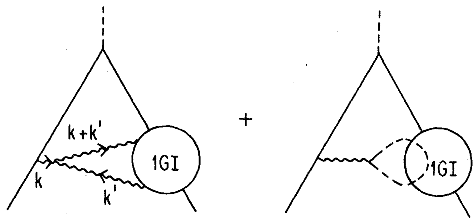

FIG. 8. General vertex insertion with a three-gluon vertex. The dashed-line loop is a ghost loop, and the blob is one-gluon irreducible but not one-fermion irreducible.

in the form $\overline{\Gamma}_\alpha^W$ of (4.7) and is used to cancel some fermion propagator corrections.

The remainder is the Λ-line parts, which act as divergences on the blob in Fig. 8. It is easy to analyze these divergences in terms of the Ward identities satisfied, not by the one-gluon-irreducible blob, but by the complete blob as shown in Fig. 9.[26,27] The Ward identity for the complete blob is shown schematically in Fig. 10; the dashed lines are ghost lines with special vertex rules for the outermost vertices which need not concern us. From this we deduce the Ward identity for the one-gluon-irreducible blob by using Fig. 9 and the Ward identity for the three-gluon blob as shown in Fig. 11. This is shown in Fig. 12.

The crux of the matter is that the only leading graph in Fig. 12 is the one-gluon-reducible graph 12(d). Figures 12(a) and 12(b) are missing a fermion denominator, as are Figs. 7(b) and 7(c); all such graphs are nonleading by the leading-graph rule. Figure 12(c) and its counterpart found by taking the divergence on the $k+k'$ line are exactly canceled by the ghost-loop graph of Fig. 8.[26] It should be evident without calculation that graphs of the type 12(d) correct the gluon propagator to the form $\Delta_{\alpha\beta}$ of (4.15), since the leading graphs are gauge-invariant.

C. Nonleading graphs

Our purpose here is to show why the nonleading graphs are nonleading, and to discuss the cancellation of certain of the nonleading graphs.

The leading-graph rule, stated at the beginning of this section, is based on the following consideration. A graph can be leading only if at least

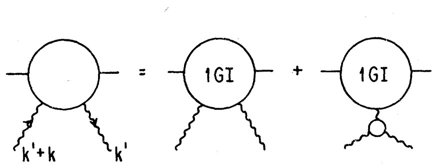

FIG. 9. Decomposition of the full blob into one-gluon irreducible and reducible pieces.

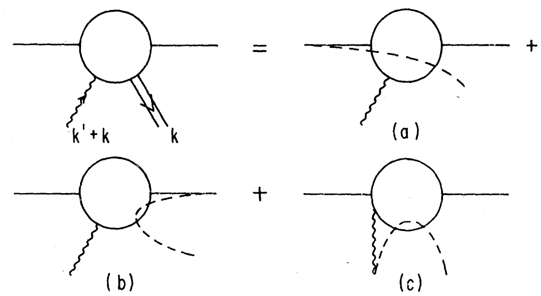

FIG. 10. Schematic depiction of the Ward identity for the full blob of Fig. 9. The gluon line whose divergence is taken is indicated by a double line.

one loop integration diverges (as the fermions go on the mass shell) independently of the other loop momenta. An example is the k loop in Fig. 5(c). It is then necessary that, when this momentum k is set equal to zero in all the other loops, at least one other loop diverges independently of the remaining loop momenta. Finally if a sequence of loop variables is found with these properties for *all* loops, the graph is leading.

This explains in a general way why graphs with a missing fermion denominator, such as Fig. 7(b), or with four-gluon vertices, are nonleading: No such sequence can be found. Figure 7(c) is special: Here an *external* denominator is truncated. Nevertheless such graphs are nonleading, as shown by an analysis (which we omit) similar to that given for the leading-graph rule in the second paper of Ref. 1. Explicit calculation shows that Fig. 7(c) has no logarithms at all.

Next we take up the difference $\overline{\Gamma}_\alpha - \overline{\Gamma}_\alpha^W$ between the vertex and its Ward identity approximation as given in (4.10). The k_α term generates graphs with missing fermion denominators, which we have already disposed of. The $p_\alpha k^2(p\cdot k)^{-1}$ term effectively substitutes a fermion line [propagator $(p\cdot k)^{-1}$] for a gluon line, and a simple analysis using the leading-graph rule as stated at the beginning of this subsection shows that this term is nonleading too.

It is still an open question whether the nonleading graphs exactly cancel, as they do in QED. It therefore seems worthwhile to point out some can-

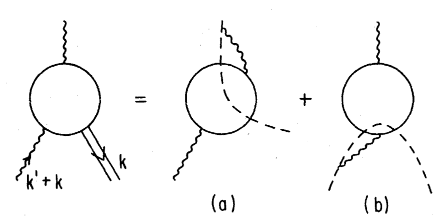

FIG. 11. Ward identity for the three-gluon blob of Fig. 9.

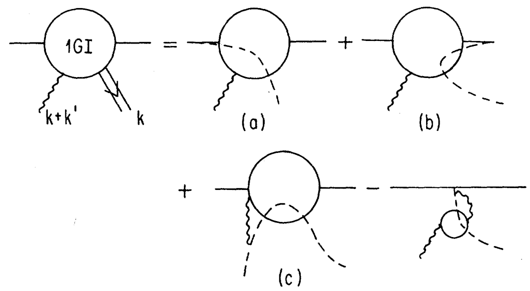

FIG. 12. Ward identity for the one-gluon irreducible blob.

cellations between nonleading graphs which could be important. Some of these graphs are shown in Fig. 13. [The crossed-H graph, such as 13(a) but with one pair of gluon legs crossed, has group-theoretic weight zero.] Graphs of this type are especially important to understand, because they are not vertex or propagator corrections and might not be expected to eikonalize. Just as we discussed for graphs 5(c) and 5(b), the leading part of 13(a) [of $O(g^6L^2)$] precisely cancels part of the weight factor for 13(b) and 13(c), leaving them with weight $\frac{1}{2}C_F{}^2C_A$. This is shown by, so to speak, solving the Ward identities for each of the graphs. Now graphs 13(b) and 13(c), with the same weight as the vertex graphs in Fig. 14, combine with them in just the right way to create an eikonal expansion, without extraneous leftovers. A similar discussion could be given for the few remaining sixth-order graphs which are not propagator or vertex corrections to ladder graphs; it is a general property that those pieces of graphs which satisfy naive Ward identities sum to an eikonal form.

We are left with, among other nonleading pieces, the Λ-line contributions. In part these lead to the replacement of gluon propagators by the gauge-invariant $\Delta_{\alpha\beta}$ in nonleading graphs, such as crossed ladders. However, there are many other pieces left over about which we know nothing. Individually, these pieces are not especially hard

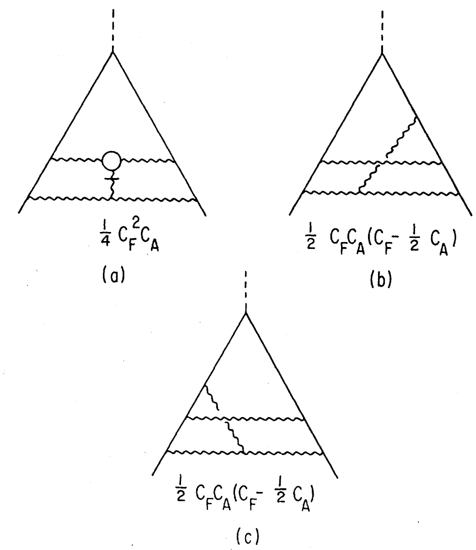

FIG. 13. Some nonleading sixth-order graphs. The group-theoretic weights ($\equiv 1$ for the Abelian theory) are shown below each graph.

to compute because it is not necessary to extract nonleading logarithms from graphs with leading logarithms; the nonleading pieces of leading graphs can be isolated as in (4.10). But there are many nonleading pieces and without a systematic understanding of why cancellations take place, there will not be much progress.

D. Summary

In summarizing in the eikonal formulas of this section, we take the liberty of incorporating some nonleading pieces into the sum of leading logarithms, for no other reason than that it is formally easy to write them down. Whether or not these nonleading terms are the only ones is not known.

The main results of this paper are the non-Abelian analogs of (3.15)–(3.18):

$$G(x,y,0)=-\frac{\mathrm{tr}PP'}{(2\pi)^8}\int_0^\infty ds\int_0^\infty ds'\int dp\,dp'\exp[-ipx+is(v\cdot p-M)-ip'y+is'(v'\cdot p'-M)-\tfrac{1}{2}g^2\mathcal{J}^a\Delta\mathcal{J}^a],\quad(4.17)$$

$$\mathcal{J}_\mu^a(z)=\int_0^s d\tau\, t^a v_\mu\delta(z-v\tau)-\int_0^{s'}d\tau'\, t^a v'_\mu\delta(z-v'\tau').\quad(4.18)$$

In (4.17), tr stands for the trace over the matrix space of the group generators t^a; of course, $\Delta_{\alpha\beta}$ is the gauge-invariant propagator (4.15). The propagator analog of (3.18) is

$$S^{\mathrm{cl}}(x,x')=\frac{-iP}{(2\pi)^4}\int_0^\infty ds\int dp\exp[-ipx+is(v\cdot p-M)-\tfrac{1}{2}ig^2\mathcal{J}^a(1)\Delta\mathcal{J}^a(1)],\quad(4.19)$$

$$\mathcal{J}_\mu^a(1)=\int_0^s d\tau\, t^a v_\mu\delta(z-v\tau).\quad(4.20)$$

To express the sum of leading logarithms only, drop the path-ordering symbols in (4.17) and (4.19), and replace t^a by $C_F^{1/2}$ in (4.18) and (4.20). Using (4.17)–(4.20) as they stand is equivalent to summing up all ladders, crossed and uncrossed, with the gluon propagator replaced by $\Delta_{\alpha\beta}$. The idea is that, in $O(g^{2N})$ in (4.17), one encounters multiple τ-ordered integrals of the type

$$\int_0^s d\tau_1 \cdots \int_0^s d\tau_N \int_0^{s'} d\tau_1' \cdots \int_0^{s'} d\tau_N' \theta(\tau_{i_1}-\tau_{i_2}) \cdots \theta(\tau'_{j_{N-1}}-\tau'_{j_N})(\cdots). \tag{4.21}$$

By working out a few examples (such as the fourth-order example in the Appendix) the reader will see that the $N!$ identical terms in (4.21) which correspond to the permutation

$$P\begin{pmatrix} i_1 i_2 \cdots i_N \\ j_1 j_2 \cdots j_N \end{pmatrix} \tag{4.22}$$

give the contribution of the crossed-ladder graphs in which the gluon vertices on one fermion line are permuted with respect to those on the other fermion line according to (4.22).

Of course, the point of worrying about nonleading logarithms is that if Δ is sufficiently singular (say, like q^{-4}), the crossed-ladder graphs will not be appreciably less singular in (4.17) than the uncrossed ones. Even so, it may be possible to ignore the effect of path-ordering. A crossed graph will differ from an uncrossed one by some integral power of $C_F^{-1}(C_F - \frac{1}{2}C_A) = -(N^2-1)^{-1}$ for the spinor representation of SU(N). So even for SU(3), as appropriate for QCD, crossed graphs are small by powers of $\frac{1}{8}$. This large-N limit has been extensively exploited in two-dimensional QCD.[12-14]

At this point there are doubtless many readers who wonder why we do not use a ghost-free gauge $n_\alpha A^\alpha = 0$, where the Ward identities are the naive ones, to obviate the mess of Λ lines that we have had to deal with. The answer is simply that it is technically difficult to demonstrate that these Ward identities can be "solved" as in (4.7), with the remainder being nonleading (one hopes by a power). The light cone $n^2=0$ suffers from the defect that although the Feynman integrals involving n are rather simple,[15] they contain certain singularities which do not appear in covariant gauges. These singularities are canceled by terms in the axial gauge ($n^2 \neq 0$) which superficially appear to vanish as $n^2 \to 0$, but actually have nonzero limits. And Feynman integrals in the axial gauges are tedious to compute. It is certainly worth considerable effort to overcome these difficulties because of the conceptual simplicity of NAGT's which obey naive Ward identities. For example, as we mentioned in the Introduction, it is a general consequence of the naive Ward identities that the gluon propagator has the form (1.1) or (4.15).

We have said little about propagators in this section; Sec. V is devoted to an exploitation of (4.19) in several circumstances where confinement is expected to take place.

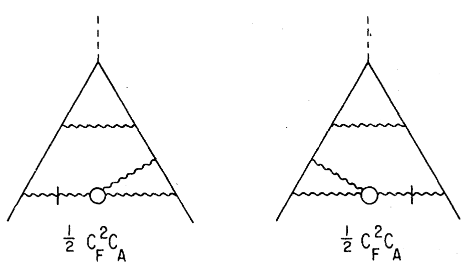

FIG. 14. Some leading sixth-order graphs whose nonleading parts eikonalize when added to the graphs of Fig. 13 and other nonleading pieces.

V. EIKONALIZED FERMION PROPAGATORS

In this section we apply the eikonal constructions (3.18) or (4.19) (in the large-N limit) to (1) three-dimensional QED; and (2) four-dimensional QCD with the assumption that $\bar{g}^2(k) \sim k^{-2}$ for small k. We indicate that the eikonal results are easily found from the Dyson equation for $S^{-2}(p)$, which becomes *linear* with the approximation (4.7) for the vertex. (We let p denote a fermion momentum near, but not necessarily on, the mass shell.)

A. Three-dimensional QED

We know of few studies of this theory, and apologize to those who may have discovered these results before us for not referring to them; there is nothing profound about them.

As we[2] and Polyakov[20] have said before, QED in three dimensions ought to be a theory with confinement of fermions, leading to a particle spectrum of massive neutral positronium states plus photons. The simplest argument is that the static potential $V(r)$ between a fermion and an antifermion grows at large distances:

$$V(r) = 2\alpha \ln \mu r, \quad \alpha = g^2/4\pi. \tag{5.1}$$

The potential is not well defined until the r in the logarithm is supplied with a scale, but changing this scale length merely shifts the zero of energy which is conventional in any case. There is, however, a physically relevant mass scale M, which measures the masses of positronium states. In order to have an intuitive grasp of M, let us suppose that $\alpha M^{-1} \ll 1$. We can think of M as the mass of a single fermion (not, of course, an observable), and the lowest-lying positronium states as having mass $\sim 2M + O(\alpha M^{-1})$.

At large distances intuition based on nonrelativistic concepts such as a potential must break down, because the potential (5.1) becomes very large, comparable to $2M$. If a fermion and an antifermion are pulled apart (by some external means), there is a distance r_S at which it becomes energetically favorable for the vacuum to create another fermion-antifermion pair. Equate the potential (5.1) to $2M$ to find

$$r_S \sim e^{M/\alpha}. \tag{5.2}$$

The exponential in α^{-1} indicates a failure of perturbation theory. A similar scale of length was found by Polyakov,[20] who studied QED without fermions on a lattice in three dimensions; in our case, M^{-1} plays the role of the lattice spacing. Polyakov's work introduced pseudoparticles into field theory for the first time, and 't Hooft[19] pointed out that for an Abelian gauge theory pseudoparticles implied a mass for fermions, even if the Lagrangian were chirally symmetric.

This connection can be seen on a more elementary level, simply by computing the one-loop contribution to the fermion self-mass. This contribution is infrared-divergent; we regulate it by introducing a small photon mass μ. Let the bare fermion mass be zero, but look for the possibility that the radiatively corrected mass M is not zero. We find, assuming that $M \gg \mu$,

$$M = \alpha \ln \frac{2M}{\mu} + \text{finite terms as } \mu \to 0. \tag{5.3}$$

$M = 0$ is not a possible solution, because the one-loop mass term is not proportional to M.

We have calculated the two-loop contributions to the mass. They are of the form

$$\sim \frac{\alpha^2}{M} \ln \frac{M}{\mu}$$

and do not have double logarithms. Thus if $\alpha M^{-1} \ll 1$ we may ignore the two-loop (and, presumably, higher-order graphs) and solve (5.3) for μ:

$$\mu = 2Me^{-M/\alpha}. \tag{5.4}$$

This shows that the effective photon mass (not necessarily the mass of the physical photon) is $O(r_S^{-1})$, that is, exponentially small in α^{-1}, for fixed M.

Let us now turn to the classical-eikonal formula for the fermion propagator, given in (3.18). The propagator $D_{\mu\nu}$ is, in the Feynman gauge,

$$D_{\mu\nu}(x) = \frac{ig_{\mu\nu}}{4\pi}(-x^2 + i\epsilon)^{-1/2}. \tag{5.5}$$

Evaluating the integrals in (3.18) yields the momentum-space propagator

$$S^{\text{cl}}(p) = -i\int_0^\infty ds \exp[is(v\cdot p - M - \alpha \ln s)], \tag{5.6}$$

where we have dropped some infinities which can be incorporated into the scale of s in $\ln s$, and have incorporated other terms into the mass M. This expression shows quite clearly the complete breakdown of perturbation theory: $S^{\text{cl}}(p)$ in (5.6) is an entire function of $v\cdot p - M$, thus has no pole at all when $v\cdot p = M$. This, of course, is the hallmark of confinement. [The expansion of (5.6) to any finite order of α does not yield an entire function.] The eikonal approximation to any Green's function factors in coordinate space, as for example (3.20) does, with one factor of $S^{\text{cl}}(x)$ for each fermion line. Since these have no mass-shell singularities, all Green's functions decrease faster than r^{-1} under the kinematical conditions discussed in Sec. II, and there are no asymptotic single-fermion states.

Note that the potential $V(r)$ appears directly in (5.6), except that s replaces r. Just as changing the scale of r in $\ln r$ shifted the zero point of energy, so changing the scale of s in (5.6) shifts the mass scale. The quantity in parentheses in (5.6) is invariant under $s \to \lambda s$, $M \to M + \alpha \ln\lambda$, or in other words, $se^{-M/\alpha}$ is invariant under this change of scale. This information can be conveyed in the form of an RG equation

$$\left(x\cdot\frac{\partial}{\partial x} + 3\right)S^{\text{cl}}(x,\alpha,M) = \left[\alpha\frac{\partial}{\partial\alpha} + (M+\alpha)\frac{\partial}{\partial M}\right]S^{\text{cl}}. \tag{5.7}$$

One of the characteristics of this equation is the invariant length $\alpha^{-1}e^{M/\alpha}$, which is identified with r_S in (5.2) and which can be used to set an invariant scale for s in (5.6). Equation (5.7) is actually a form of the infrared-singular differential equations involving the operator $\mu\partial/\partial\mu$ introduced in Ref. 1, where μ is the effective photon mass as in (5.4). There is no reason to identify μ with the actual photon mass, which as far as we can see remains at zero. The effective photon mass must be introduced only for infrared-singular objects such as S^{cl}, which are not observable; there are no infrared singularities associated with the properties of physical states, which are neutral, confined combinations of fermions and antifermions.

Three-dimensional QED is a good example of a theory where perturbation-theoretic results such as the cancellation between real- and virtual-infrared singularities are inapplicable.[21] There are no such singularities associated with real-photon emission, since the matrix element for photon emission from a neutral particle vanishes at zero photon momentum. The absence of charged particles is of course nonperturbative, as evidenced by the propagator (5.6), which is an entire function of $v\cdot p - M$.

B. Four-dimensional QCD

We use the large-N or leading-logarithm version of the non-Abelian result (4.19):

$$S^{cl}(p) = -i\int_0^\infty ds\,\exp[is(v\cdot p - M) - \tfrac{1}{2}ig^2\mathcal{J}^a(1)\Delta\mathcal{J}^a(1)], \tag{5.8}$$

$$-\frac{ig^2}{2}\mathcal{J}^a(1)\Delta\mathcal{J}^a(1) = -\frac{iC_F}{2(2\pi)^4}\int_0^s d\tau\int_0^s d\tau'\int\frac{d^4k}{k^2}\exp[-ik\cdot v(\tau-\tau')\bar{g}^2(k^2)]. \tag{5.9}$$

If we adopt the folklore that $\bar{g}^2 \sim k^{-2}$ at small k,

$$\bar{g}^2 = -m^2/k^2, \tag{5.10}$$

we find (as usual, dropping short-distance singularities)

$$S^{cl}(p) = -i\int_0^\infty ds\,\exp\left[is(v\cdot p - M) - \frac{m^2C_Fs^2}{4\pi^2}(\ln s - \tfrac{3}{4})\right]. \tag{5.11}$$

In (5.10), the purely phenomenological parameter m^2 is positive in order that $\bar{g}^2$ be positive for spacelike k^2. Just as for three-dimensional QED, $S^{cl}(p)$ is an entire function, with no mass-shell singularities.

The results (5.6) and (5.11) hold in the Feynman gauge. In a certain sense, $S^{cl}(p)$ is entire in momentum space in most covariant gauges. However, it is always possible to find a gauge (e.g., the Yennie gauge in QED) in which the infrared singularities disappear in the *propagator*, but then they will reappear somewhere else. Thus in $d=2$ QCD, 't Hooft[12] uses a special gauge in which the propagator has no singularities, but Einhorn[14] uses a slightly different gauge in which the propagator is free. Nonetheless, both authors find the same results for physical quantities. One may also raise the question whether the exact $S(p)$ in the Feynman gauge is entire, when all that is known is that $S^{cl}(p)$ is entire. We will take up this question in another publication.

The function in (5.11) is not a familiar one in the literature, but if we replace $\ln s$ in the exponent by a constant value $\ln\bar{s}$, $S^{cl}(p)$ becomes essentially the complex error function, a function widely known in plasma physics[28]:

$$S^{cl}(p) = \gamma^{-1}Z[\gamma^{-1}(M - v\cdot p)], \tag{5.12}$$

$$Z(W) = \pi^{-1/2}\int_{-\infty}^\infty dt\,\frac{e^{-t^2}}{t-W}, \tag{5.13}$$

where $\gamma = (m/\pi)C_F^{1/2}(\ln\bar{s} - \frac{3}{4})^{1/2}$. Actually, the specification of $Z(W)$ in (5.13) is not complete without a prescription for the integration contour; the usual formula of adding $i\epsilon$ to M (or W) is not consistent with the idea that a confined propagator should have no imaginary part. A better choice is to use the principal value which replaces $-i\exp[is(v\cdot p - M)]$ by $\sin s(v\cdot p - M)$ in (5.11).

In what would usually be called the mass-shell limit, $v\cdot p - M \to 0$, the principal-value propagator not only is not singular but actually vanishes:

$$S^{cl}(p) \underset{v\cdot p\to M}{\sim} -\frac{2}{\gamma}(M - v\cdot p). \tag{5.14}$$

In the opposite limit

$$|v\cdot p - M| \gg \gamma, \quad S^{cl}(p) \to (v\cdot p - M)^{-1},$$

the free propagator. Of course this limit is outside the bounds of validity of the approximation (5.10), but it is consistent with asymptotic freedom. A better scheme is to use, instead of (5.10), the correct function $\bar{g}$ for large k as well as small k. Finally, the effects of anomalous dimensions coming from ultraviolet singularities might correctly be given by multiplying (5.9) by a factor familiar from studies of the RG equation, $\exp[\int^g dg'\beta^{-1}(g')2\gamma(g')]$, where γ is the fermion anomalous dimension.

Actually the conventional RG equation is not as useful for studying infrared singularities as the sort of equation proposed in Ref. 2, which is the space-time version of the $\mu\partial/\partial\mu$ equations of Ref. 1. Using the approximation $\ln s \sim \ln\bar{s}$ again, the Fourier transform of (5.11) obeys

$$\left(x\cdot\frac{\partial}{\partial x} + 3\right)S^{cl}(x) = \frac{C_F}{2\pi^2}\bar{g}^2[(x\cdot v)^{-2}]S^{cl}(x). \tag{5.15}$$

Here $C_F\bar{g}^2/2\pi^2$ appears as an infrared-singular anomalous dimension.

We have mentioned before that Pagels[11] has studied an approximation to the fermion propagator using (5.10); his results do not quite agree with ours. Pagels's work is in momentum space, and we now take up the question of eikonalized propagators in momentum space.

C. The eikonalized Dyson equation

At first glance the eikonalized Dyson equation for the propagator appears to be highly nonlinear:

$$S^{-1}(p)=v\cdot p-M-\frac{ig^2}{(2\pi)^4}\int dk\, v_\alpha D_{\alpha\beta}(k^2)\times S(p-k)t^a\Gamma^a_\beta(p-k,p), \tag{5.16}$$

where $p_\alpha = Mv_\alpha +$ a small momentum. But the Ward identity "solution"

$$\Gamma^a_\beta(p-k,p)\simeq t^a\frac{p_\beta}{p\cdot k}[S^{-1}(p)-S^{-1}(p-k)], \tag{5.17}$$

coupled with the replacement of $D_{\alpha\beta}$ by $\Delta_{\alpha\beta}$ [see (4.15)], which accounts for the leading logarithms, yields a linear integral equation

$$S(p)=\frac{1}{v\cdot p-M}\times\left\{1-\frac{iMC_F}{(2\pi)^4}\int\frac{dk}{k^2}\bar{g}^2(k^2)\frac{[S(p-k)-S(p)]}{p\cdot k}\right\}. \tag{5.18}$$

Pagels,[11] on the other hand, has used a more drastic approximation to the Ward identity by expanding (5.17) as

$$\Gamma^a_\beta(p-k,p)\simeq\Gamma^a_\beta(p,p)=t^a\frac{\partial S^{-1}(p)}{\partial p_\beta}, \tag{5.19}$$

which leads to a nonlinear Dyson equation.

The eikonal propagator (5.8) is a solution of (5.18). To show this, integrate (5.8) by parts, taking $e^{is(v\cdot p-M)}$ as the factor to be integrated, and using (5.9) to find the derivative of the remaining factor. The result will be of the form (5.18). This is the simplest demonstration of the equivalence between the eikonal construction in space-time and the momentum-space eikonal, which depends heavily on the Ward identity "solutions" such as (5.17). Equations similar to (5.18) hold for three- and four-dimensional QED, giving the space-time eikonal results. Linear Dyson equations can also be derived for the vertices in (3.15) or (4.17) by integrating by parts. We will not consider applications of the propagator (5.11) to hadronic physics in this paper. For some first steps in this direction, see Ref. 11.

VI. SUMMARY AND DISCUSSION

The main point of this paper is displayed in Eqs. (4.15)–(4.20): For the leading logarithms of an NAGT, there is an eikonal structure like that of the Abelian case except that in place of the free gluon propagator $g^2D_{\alpha\beta}$ the combination $g^2C_F\Delta_{\alpha\beta}$ appears, where $g^2\Delta_{\alpha\beta}$ is constructed from the invariant charge as in (4.15).

Unlike QED, where it is rather straightforward to show that all the nonleading logarithms exactly cancel, it is still an open question for NAGT's whether the nonleading logarithms all cancel. Some of them—equivalent to crossed-ladder graphs with modified gluon propagators—can be summed up into a modified eikonal form as in (4.17) and (4.19). Others can be subsummed into the nonleading logarithms necessary to build-up the eikonal structure. The fate of the rest is unknown, and this question is of much more than academic interest, if $\bar{g}^2(q^2)$ is as singular for small q as people commonly believe. Mere logarithms are insignificant compared to the power singularities which develop.

The reason that there are no nonleading logarithms in QED is that the Ward identities can be solved as in (3.33) or (3.37), with errors genuinely small by a power of an infrared momentum q. This is simple because of the eikonal identity in (3.32) which works for fermion propagators. Unfortunately neither this identity nor the naive Ward identity holds for the gluon vertices in an NAGT. It is possible to solve the Ward identities (when one includes the Λ-line parts and ghost loops, as in Sec. IV), but the error made is (barring cancellations) only small by a power of $\ln q$.

It is clear that naive Ward identities play an essential role in building up an eikonal structure. This points unmistakably to the conceptual advantages of the ghost-free gauges, where the Λ lines are missing and it is automatic that the gluon propagator has the form (4.15). But for purely technical reasons it is not clear exactly how the cancellation of noncovariant pieces goes, nor whether the infrared approximation to the Ward identities [e.g., (4.7)] is valid up to nonleading powers of q, as in QED. We are currently investigating these problems of the ghost-free gauges. Another possibly fruitful line of investigation, which we have avoided in this paper, is the direct investigation of the functional integral in (3.26). In this way one sidesteps our detour into momentum space and works directly in space-time.

We expect that any eikonal approximation to a field theory will be expressed in terms of a classical action function for the massive particles involved, and this is indeed so for NAGT's. In

QED, the massive particles interact through a kind of (relativistic) potential, and are fully described with only the usual classical attributes of momentum, mass, and charge. In the non-Abelian case the potential (i.e., gluon exchange) can transfer group charge, and it is necessary to introduce a new classical degree of freedom $\xi^a(s)$ with its own equation of motion (3.29). It has not been necessary to consider this phenomenon in detail for the two-particle processes discussed here. It should be relevant for three-quark states in QCD.

The classical action for an NAGT is much more singular than that for QED (assuming that $\bar{g}^2 \sim q^{-2}$). It may be that this action has something to do with a string with fermions at the ends, and carrying both transverse and longitudinal modes.[16-18] This classical action comes with its own prescription for first quantization, based on the path integral (3.11) (and its non-Abelian counterpart) for the fermion propagator. We have shown in the Appendix that for two dimensions the classical action is precisely that of a longitudinal string, in the light-cone gauge.

Of course, in two dimensions confinement is virtually automatic, given the singular nature of the Coulomb potential. A somewhat less obvious situation is three-dimensional QED, where we have constructed the fermion propagator and found that it has no singularities in momentum space. This is a clear-cut signal for confinement of the fermions. In four dimensions, although it is possible to construct a phenomenological confined fermion propagator, we can say nothing further until more is known about $\bar{g}^2(q^2)$.

In earlier work[1,2] we found an apparently logically independent signal for confinement. This was the exponential decrease of vertices describing emission of soft gluons, *even without* appealing to the singularities of $\bar{g}$. Since the resulting failure of real-gluon singularities to cancel virtual-gluon singularities was nonperturbative, there was no conflict with the perturbative calculations of Ref. 22. We have not studied such vertices in this paper, nor processes which do not involve heavy particles. It is reasonable to guess that one should replace (in effect) g by $\bar{g}$ in the earlier works[1,2] dealing with these processes, but how this really works for pure gluon graphs is not known. If it were, we would probably know how to calculate $\bar{g}$.

APPENDIX: TWO-DIMENSIONAL QCD

This model was originally studied by 't Hooft[12] in the large-N limit (N is the number of colors in QCD), then by Callan *et al.*[13] and Einhorn[14] in the same limit, always in the light-cone gauge. The ghost-free gauges are distinguished for two-dimensional QCD because the field strength tensor

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu \tag{A1}$$

is linear in the potentials A^a_μ, just as in QED. One can solve the field equations for the single nonvanishing component of A^a_μ and so express the theory as one of potential interactions between fermions.

In this appendix, we study a limited section of the full theory, in which no pair-creation processes are allowed (e.g., no closed fermion loops). This sector, which fully illustrates the leading infrared singularities, is purely a first-quantized theory since no particle production of any kind takes place. By using classical (eikonal) fermion propagators we show that the leading infrared singularities yield a classical theory of strings with longitudinal modes and quarks on the ends (of course, expressed in the light-cone gauge for the string action). No recourse to the large-N limit is used. This theory is promoted to a first-quantized theory by the path-integral prescriptions of Sec. III for the fermion propagators; no independent method of quantization should be introduced. The limited sector which we study corresponds to strings which do not break, and it will be an interesting task for the future to study the correspondence between independently conceived string theories[16-18] and QCD in the pair-production sectors.

A two-dimensional NAGT is an exception to nearly all the generalities of confinement which we raise for other dimensions. The fermion propagator is free, the gluon "propagator" is free, the light-cone gauge works, etc. Therefore it may be illusory to make any generalizations (e.g., four-dimensional QCD is a string theory with longitudinal modes) to other situations. The light-cone gauge works because there are no three- or four-gluon couplings, and our experience is that only Feynman graphs with such couplings are treacherous.

The light-cone gauge is partly specified by the condition

$$\eta_\mu A^\mu = 0 \tag{A2}$$

with $n^2 = 0$. Choose $\eta_\mu = 2^{-1/2}(1, 1)$; with the notation

$$v_\pm = 2^{-1/2}(v_0 \pm v_1) \tag{A3}$$

for any four-vector v_μ the gauge condition (A2) is $A_- = 0$. The orthogonal component A_+ is determined by

$$\partial_-{}^2 A_+ = -g\mathcal{J}_-, \tag{A4}$$

where $\mathcal{J}_\mu$ is the quark color current. The gauge is completely specified by choosing the parameters a, b in the solution to (A4):

$$A_+(x) = -g \int d^2y\, \delta(x_- - y_-)\{\tfrac{1}{2}|x_+ - y_+| + a + b(x_+ - y_+)\}\mathcal{J}_-(y). \qquad \text{(A5)}$$

The standard choice is $a = b = 0$, which we (and Einhorn[14] and others) adopt. It is not the choice of 't Hooft[12] and of Callan *et al.*,[13] who considered a limit where b became arbitrarily large. In momentum space, our choice corresponds to the principal-part prescription

$$\frac{1}{n\cdot k} = \tfrac{1}{2} i \int_{-\infty}^{\infty} d\lambda\, \epsilon(\lambda) e^{-i\lambda n\cdot k}, \qquad \text{(A6)}$$

$$\frac{1}{(n\cdot k)^2} = -\tfrac{1}{2} \int_{-\infty}^{\infty} d\lambda\, |\lambda|\, e^{-i\lambda n\cdot k}. \qquad \text{(A7)}$$

With $a = b = 0$, the effective propagator (really an instantaneous current-current interaction) is

$$D_{\mu\nu}(k) = -\frac{n_\mu n_\nu}{(n\cdot k)^2}. \qquad \text{(A8)}$$

The Feynman rules are those of QED, except that the quark-gluon vertex has a factor t^a which is the color matrix of the quarks.

Consider the one-loop correction to the fermion propagator. It has the value (no eikonal approximation is made)

$$\Sigma(p) = -\frac{2ig^2 C_F}{(2\pi)^2} \not{n} \int \frac{d^2k}{(n\cdot k)^2} \frac{n\cdot(p-k)}{[(p-k)^2 - M^2]}. \qquad \text{(A9)}$$

According to the general rules[15] for integrating (A9), k is replaced by p in each of the n-dependent factors, resulting in $\Sigma(p) \equiv 0$. These rules are ambiguous in two dimensions, because we encounter expressions of the type

$$0 \overset{?}{=} 0 \times \int \frac{d^2K}{(p-k)^2 - M^2}, \qquad \text{(A10)}$$

that is, $0\times\infty$. Einhorn[14] has argued that $\Sigma(p)$ simply becomes a mass shift (independent of p) so that the propagator still remains free. Clearly $\Sigma(p)$ is a constant (possibly zero) for all rainbow graphs.

Next consider the one-loop vertex correction (Fig. 1). It has the value (dropping q, q' compared to p, p' when this is permitted)

$$\Lambda_\mu = \frac{g^2}{4\pi}(C_F - \tfrac{1}{2}C_A)\gamma_\mu (n\cdot Q)^2 \int_0^1 \frac{d\alpha\, \alpha(1-\alpha)}{n\cdot[\alpha p + (1-\alpha)p']^2[\alpha(1-\alpha)Q^2 - M^2]} + (\text{terms} \sim n_\mu), \qquad \text{(A11)}$$

where $Q = q - q'$. We need not record the terms $\sim n_\mu$; they are annihilated by the propagator (A8). Equation (A11) is found[15] by replacing the integration variable k by its shifted value, and this time there is no ambiguity about applying the rules because there are two fermion denominators. Such vertex corrections need not be saved in the large-N limit $[C_F - \tfrac{1}{2}C_A = -(2N)^{-1}]$, but they are nonleading even without appealing to this limit. The reason is that the $(n\cdot Q)^2$ factor in (A11) cancels out the infrared-singular propagator which will be joined to Fig. 1 in completing a graph for a physical process. As long as p, p' are near the mass shell the integral (A11) is $O(Q^2)$ and nonleading by two powers of x in coordinate space.

It is now easy to see that all the propagator and vertex corrections which were previously rejected by virtue of the large-N limit have nonleading infrared singularities, down by powers of x (not of $\ln x$ as in four dimensions). These corrections are, so to speak, doubly small.

Now we can calculate the standard Green's function

$$G(x,y,z) = \langle 0|T(K(z)\psi(x)\bar\psi(y))|0\rangle \qquad \text{(A12)}$$

for a colorless point source $K(z)$. The results is in the usual path-ordered eikonal form, with no propagator or vertex corrections. First we give the results using *classical* eikonal propagators, assigning an orbit $x_\mu(s)$ to the quark and an orbit $y_\mu(s')$ to the antiquark. The point source $K(z)$ is at the origin $z = 0$. G turns out to be

$$G(x,y,0) = -pp' \int_0^\infty ds \int_0^\infty ds'\, \delta(x - x(s)) \times \delta(y - y(s'))e^{iS}, \qquad \text{(A13)}$$

$$S = -M\int_0^s d\tau(\mathring{x}^2)^{1/2} - M\int_0^{s'} d\tau'(\mathring{y}^2)^{1/2} - \tfrac{1}{2}g^2 \int_0^s d\tau \int_0^{s'} d\tau'\, t^a t'^a \mathring{x}_- \mathring{y}_- |x_+ - y_+|\, \delta(x_- - y_-). \qquad \text{(A14)}$$

In (A14) the argument of x is τ, that of y is τ', and we have written the free action terms $-Ms - Ms'$ in a reparametrization-invariant form.

We show that the path-ordering PP' in (A13) can be dropped, and $t^a t'^a$ replaced by C_F in (A14). The proof will be given only for straight-line orbits $x = v\tau$, $y = v'\tau'$ but the result is more general; it only depends on the velocities $\mathring{z}, \mathring{z}'$ being forward timelike, and on the single-valuedness of the classical paths. There are, in $O(g^{2N})$, $(N!)^2$ different terms corresponding to the $N!P$ orderings and $N!P'$ orderings; only $N!$ of these terms are distinct, and they correspond to the distinct ways of drawing a generalized ladder graph with N rungs. Just as in four dimensions, all the crossed ladder graphs are nonleading, but they are nonleading by powers. This may be verified directly by evaluating the Feynman-graph integrals, or more instructively by evaluating the integrals in (A14). In fourth-order, a typical "crossed" ordering leads to the integral

$$\int_0^s d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^s d\tau_2' \int_0^{\tau_2'} d\tau_1' \delta(v_-\tau_1 - v_-'\tau_1') \times \delta(v_-\tau_2 - v_-'\tau_2')(\cdots), \quad (A15)$$

where the omitted factors are irrelevant. The δ functions tell us that

$$\tau_1' = \frac{v_-}{v_-'}\tau_1, \quad \tau_2' = \frac{v_-}{v_-'}\tau_2. \quad (A16)$$

For forward timelike vectors v_μ, v_μ' it is always the case that $v_\pm, v_\pm' > 0$ and thus $\tau_2' > \tau_1'$ implies $\tau_2 > \tau_1$. But in (A15) only $\tau_1 > \tau_2$ contributes, and (A15) is zero.

In $O(g^{2N})$ there are $N!$ "uncrossed" orderings which do not vanish; each one has a value $(N!)^{-1}$ times the Nth power of the g^2 term. Therefore $t^a t'^a$ is replaced by C_F in (A14) and the path-ordering dropped.

It only remains to show that the integral in (14) is the action for a string in the light-cone gauge.[16-18] In its general form the string action is

$$A = \int_0^\pi d\sigma\, d\tau [(z_\tau \cdot z_\sigma)^2 - z_\tau{}^2 z_\sigma{}^2]^{1/2}, \quad (A17)$$

with $z(\sigma=0,\tau)=x(\tau)$, $z(\sigma=\pi,\tau)=y(\tau)$, $z_\tau = dz/d\tau$, etc. The chosen gauge is $z_- = f(\tau)$, for which

$$A = \int d\tau f_\tau \int_0^\pi d\sigma |z_{+\sigma}| = \int d\tau\, \mathring{z}_- |x_+(\tau) - y_+(\tau)| \quad (A18)$$

(we assume that the string has no folds). An integration over the δ function in (A14) yields the same result. It is tempting to suppose that the gauge-invariant eikonalized Green's function would be directly expressed in terms of the action in (A17), but we have not shown this.

The prescription for first-quantizing the theory follows directly from the prescription (3.11) for first-quantizing the eikonal propagator. One thus derives Bethe-Salpeter equations for wave functions such as those already described.[12-14] The classical form (A14) yields, of course, the WKB approximation to the wave functions and energy levels, giving a linear asymptotic (mass)^2 spectrum[12-14,18]

$$M_n{}^2 \simeq \pi g^2 C_F n \quad (n \text{ a large integer}) \quad (A19)$$

for the mesons.

We have not completed a study of baryon wave function or of pair-production processes, which will show how strings join and split. It is most natural for the baryons to have a Δ-shaped configuration rather than a Y-shaped configuration, and Bars[18] has already discussed the Δ-shaped baryon in the context of a specific string theory. We will return to these questions at a later date.

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[1]J. M. Cornwall and G. Tiktopoulos, Phys. Rev. Lett. 35, 338 (1975); Phys. Rev. D 13, 3370 (1976).

[2]J. M. Cornwall and G. Tiktopoulos, in *New Pathways in High-Energy Physics II: New Particles—Theories and Experiments,* proceedings of the third Orbis Scientiae, Univ. of Miami, 1976, edited by A. Perlmutter (Plenum, New York, 1976), p. 213.

[3]J. M. Cornwall, work presented at the U.S.–Japan Seminar on Geometric Models of the Elementary Particles, Osaka, 1976, Report No. UCLA/76/TEP/10 (unpublished).

[4]See also J. J. Carazzone, E. R. Poggio, and H. R. Quinn, Phys. Rev. D 11, 2286 (1975); 12, 3386(E) (1975); E. R. Poggio and H. R. Quinn, *ibid.* 12, 3279 (1975), who find exponentiation for some simple processes. J. Frenkel and J. C. Taylor [Nucl. Phys. B116, 185 (1976)] also find exponentiation for on-shell processes in the Coulomb gauge.

[5]The literature on eikonalized QED is vast. In addition to standard works on QED such as Ref. 8, H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1972), has a full set of references, especially to the functional methods of Schwinger. A sample of older works follows: S. J. Chang and S. Ma, Phys. Rev. Lett. 22, 1334 (1969); H. D. I. Abarbanel and C. Itzykson, *ibid.* 23, 53 (1969); F. Englert *et al.*, Nuovo Cimento 64A, 561 (1969); M. Lévy and J. Sucher, Phys. Rev. 186, 1656 (1969); H. Cheng and T. T. Wu, *ibid.* 186, 1611 (1969); Y.-P. Yao, Phys. Rev. D 1, 2971 (1970); S. J. Chang, *ibid.* 1, 2977 (1970).

[6]J. Frenkel, R. Meuldermans, I. Mohammad, and J. C. Taylor, Phys. Lett. 64B, 211 (1976); Report No. 67/76 (unpublished).

[7]A replacement of the type (1.1) can be inferred from the calculation by C. P. Korthals-Altes and E. de Rafael of the fourth-order anomalous magnetic moment of the quark in QCD [Marseille Report No. 76/P 820, 1976

(unpublished)]. The same calculation has been done by P. Cvitanović [Phys. Lett. 65B, 272 (1976)], who discusses the general properties of the infrared singularities of the magnetic moment in QCD.

[8]See, e.g., D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[9]The idea here is that the only known truly gauge-invariant formulation of NAGT's, i.e., on the lattice [see K. G. Wilson, Phys. Rev. D 10, 2445 (1974)], does not need ghosts or any sort of gauge-fixing term.

[10]For example, G. Preparata, Phys. Rev. D 7, 2973 (1973); in *Lepton and Hadron Structure,* Proceedings of the 1974 Erice Summer School, edited by A. Zichichi (Academic, New York, 1975), Vol. 12. See also Refs. 12 and 13.

[11]H. Pagels has also studied hadronic structure in a similar vein [private communication and Phys. Rev. D 14, 2747 (1976)].

[12]G. 't Hooft, Nucl. Phys. B75, 461 (1974).

[13]C. G. Callan, Jr., N. Coote, and D. J. Gross, Phys. Rev. D 13, 1649 (1976).

[14]M. B. Einhorn, Phys. Rev. D 14, 3451 (1976).

[15]J. M. Cornwall, Phys. Rev. D 10, 500 (1974); A. Chakrabarti and D. Darzens, *ibid.* 9, 2484 (1974).

[16]A. Patrascioiu, Nucl. Phys. B81, 525 (1974).

[17]W. A. Bardeen, I. Bars, A. J. Hanson, and R. D. Peccei, Phys. Rev. D 13, 2364 (1976).

[18]I. Bars, Phys. Rev. Lett. 36, 1521 (1976).

[19]The concept of the pseudoparticle was discussed by G. 't Hooft [Phys. Rev. Lett. 37, 8 (1976)] in his discussion of the classical solutions to an NAGT discovered by A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu S. Tupkin, Phys. Lett. 59B, 85 (1975). Note that the effective Lagrangian introduced by 't Hooft reduces to a fermion mass term for QED.

[20]A. M. Polyakov, Phys. Lett. 59B, 82 (1975).

[21]For the Abelian case see D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961), who formulate the problem in conventional graphical terms. More recently it has been proposed to use a coherent-state framework: See V. Chung, Phys. Rev. 140, B1110 (1965); T. Kibble, J. Math. Phys. 9, 315 (1968); Phys. Rev. 173, 1527 (1968); 174, 1882 (1968); 175, 1624 (1968). Further references are found in Ref. 1.

[22]For the non-Abelian case see T. Appelquist, J. Carazzone, H. Kluberg-Stern, and M. Roth, Phys. Rev. Lett. 36, 768 (1976); 36, 1161(E) (1976); Y.-P. Yao, *ibid.* 36, 653 (1976); L. Tyburski, *ibid.* 37, 319 (1976); D. R. Crewther, Report No. CALT-68-556, 1976 (unpublished).

[23]J. Schwinger, Phys. Rev. 82, 664 (1951).

[24]See, e.g., E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

[25]P. Cvitanović and T. Kinoshita, Phys. Rev. D 10, 3991 (1974); T. Kinoshita and A. Ukawa, paper submitted to the International Conference on High Energy Physics, Tbilisi, 1976 (unpublished).

[26]G. 't Hooft, Nucl. Phys. B33, 173 (1971).

[27]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 7, 1049 (1973).

[28]B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic, New York, 1961).