Consequences of vacuum instability in quantum field theory*

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When the effective potential of a quantum field theory has nonglobal minima, the question arises as to whether they may correspond to the vacuum state. The conventional response is negative because of vacuum instability and also on the basis of an analogy with many-body theory where the effective potential is compared to the Helmholtz free energy for which certain convexity properties may be proved using Van Hove's theorem. In pursuing an alternative possibility, it is pointed out here that when a nonglobal minium of the potential occurs, the field equations can sustain a pseudoparticlelike classical solution in four-dimensional Euclidean space-time and a corresponding soliton solution in three-dimensional space. The existence of these solutions is not inconsistent with Derrick's theorem which assumes that the vacuum is an absolute minimum. The solutions are interpreted physically in Minkowski space as vacuum bubbles created by quantum tunneling from the metastable vacuum to one of lower energy. Thus explicit calculations of the decay probability are rendered feasible. A simple theory where the phenomenon occurs is the skewed Goldstone model, which is studied in detail. Extension to Higgs-Kibble gauge theories is straightforward. As an example, in the Weinberg-Salam theory the lower limit on the Higgs mass derived by Weinberg and Linde, and the limit derived by Gildener and Weinberg for the modified version incorporating a dimensional transmutation mechanism, can be significantly modified. For the original version of the theory, the lower limit is changed from 4.91 GeV to about 3.5 GeV, for mixing angle $\theta_W = 35°$. For this new range of masses the vacuum is totally secure, for practical purposes, against spontaneous bubble formation by vacuum fluctuations.

I. INTRODUCTION: THE PROBLEM

In a quantum field theory, the vacuum state is usually defined as that state of lowest energy or that state which is annihilated by all particle destruction operators.[1] The identification of the vacuum is an important step in finding the physical predictions of the theory; for example, in a theory with spontaneous breakdown of symmetry the symmetry properties of the vacuum are as significant as those of the defining Lagrangian. In the present paper, we study a situation where the normal procedure to locate the vacuum leads to ambiguities.

A convenient device for finding the vacuum state is to calculate the effective potential $V(\phi)$ which is the generating functional for one-particle irreducible Green's functions with vanishing external momenta. The ground state of the theory corresponds to a minimum of $V(\phi)$. In the tree approximation to a theory with a real scalar field ϕ the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi - V(\phi) \ . \tag{1}$$

The quantum corrections from loop diagrams may be calculated from

$$V = iS_{1\,\mathrm{PI}}(p_\mu = 0) \ , \tag{2}$$

where the subscript 1PI denotes that only the contributions of one-particle-irreducible diagrams to the S matrix are included.

It can be proved (see Symanzik, Ref. 2) that $V(\phi)$ is the expectation value of the energy density in a state for which the expectation value of the field is ϕ.

If there is only one nondegenerate minimum, no ambiguity exists. Also, if there is no minimum, the energy is unbounded below and no theory exists.

There may be degenerate vacuums, leading to spontaneous symmetry breaking. In this case, if the degeneracy is not accidental but the result of a symmetry, the choice of vacuum is arbitrary; physical predictions do not depend on this choice. If the broken symmetry is continuous (and in the absence of local gauge invariance, discussed later) there will be Goldstone bosons. In the U(1) Goldstone model[3] the Lagrangian density ($m^2 < 0$)

$$\mathcal{L} = \partial_\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi - \frac{\lambda}{6}(\phi^*\phi)^2 \ , \tag{3}$$

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{4}$$

is invariant under

$$\phi \to e^{i\theta}\phi \ . \tag{5}$$

In the tree approximation the effective potential is

$$V(\phi, \phi^*) = m^2\phi^*\phi + \frac{\lambda}{6}(\phi^*\phi)^2 \ , \tag{6}$$

with minima when

$$|\phi|^2 = -3m^2/\lambda = |\chi|^2 \ . \tag{7}$$

The choice of phase of χ is arbitrary. Taking χ to be real and shifting the fields to

$$\phi_1 = \phi_1' + \sqrt{2}\,\chi \,, \tag{8}$$

$$\phi_2 = \phi_2' \,, \tag{9}$$

then

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi_1' \partial_\mu \phi_1' + 2m^2 \phi_1'^2) + \tfrac{1}{2}\partial_\mu \phi_2' \partial_\mu \phi_2' - \sqrt{2}\,\frac{\lambda\chi}{6}\,\phi_1'(\phi_1'^2 + \phi_2'^2) - \frac{\lambda}{24}(\phi_1'^2 + \phi_2'^2)^2 + \frac{3m^4}{2\lambda}\,; \tag{10}$$

so ϕ_1' has real mass $(-2m^2)^{1/2}$ and ϕ_2' is the massless Goldstone boson. Goldstone[3] also considered the case of a real scalar ($m^2 < 0$)

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi \partial_\mu \phi - m^2\phi^2) - \frac{\lambda}{24}\,\phi^4 \,. \tag{11}$$

Shifting the field to

$$\phi = \phi' + \chi, \quad \chi = (-6m^2/\lambda)^{1/2} \,, \tag{12}$$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi' \partial_\mu \phi' + 2m^2\phi^2) - \tfrac{1}{6}\lambda\chi\phi'^3 - \frac{\lambda}{24}\,\phi'^4 + \frac{3m^4}{\lambda}\,. \tag{13}$$

The reflection symmetry $\phi \to -\phi$ is lost. The potential has minima at $\phi = \pm\chi$, and we may arbitrarily choose $\pm\chi$ as the vacuum state.

Suppose now that we destroy all symmetry by skewing the potential to

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \partial_\mu \phi - V(\phi) \,, \tag{14}$$

$$V(\phi) = \tfrac{1}{2}m^2\phi^2 - \frac{\phi}{3}\,\phi^3 + \frac{\lambda}{4}\,\phi^4 \,, \tag{15}$$

with $m^2 < 0$, $0 < \delta/|m| \ll \lambda \ll 1$. The minima are now at $[V = (-m^2/\lambda)^{1/2}]$

$$\phi_0^{\pm} = \pm v + \delta/2\lambda \,, \tag{16}$$

$$V(\phi_0^{\pm}) = -\frac{\lambda}{4}\,v^4 \mp \frac{\delta}{3}\,v^3 \,, \tag{17}$$

as indicated in Fig. 1. We shall refer to this theory as the skewed Goldstone model.

The essential physics of the problem addressed in this paper is all contained in the skewed Goldstone model. The question is: As soon as $\delta \neq 0$, does the metastable minimum cease to be an acceptable choice for the vacuum? In other words, must the physical vacuum be an *absolute* minimum of $V(\phi)$?

Once the skewed Goldstone model can be understood, more complicated and interesting models such as the Higgs model,[4] the σ model,[5] the O(N) ϕ^4 model[6] for $N \to \infty$, the Weinberg-Salam model,[7] quantum chromodynamics(?),[8] etc. may be considered. In the present paper we shall study in detail only the skewed Goldstone model and the Weinberg-Salam model.

In Sec. II we discuss the conventional dogma concerning the problem. In Sec. III we introduce an alternative possibility. The skewed Goldstone model and the Weinberg-Salam theory are treated as examples in, respectively, Secs. IV and V. Section VI briefly considers nonspontaneous vacuum bubbles and finally Sec. VII gives some discussion.

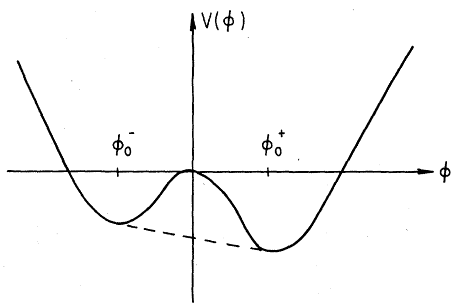

FIG. 1. Effective potential for skewed Goldstone model.

II. CONVENTIONAL WISDOM

It is common to assume that the vacuum must correspond to an absolute minimum of the energy density. The reason for this assumption is presumably that the vacuum is taken to be absolutely stable. But if the decay lifetime of the unstable vacuum is sufficiently long, then a nonglobal minimum is an equally good candidate for the vacuum.

The effective potential is often compared to the Helmholtz free energy (A) of statistical mechanics. Consider, for example, the Van der Waals equation of state for a liquid-gas system. The isotherm for this equation does not satisfy the convexity property $dP/dV \le 0$ that follows from Van Hove's theorem.[9] By requiring that the free energy be minimized as we vary the pressure P *and* that the Van Hove inequality be respected, one arrives at the Maxwell construction of a straight-line (constant P) segment with equal areas above and below the straight line. For the straight section, the equilibrium situation is a mixture of the liquid and gas phases.

The analogy requires that we add to $V(\phi)$ a term with a Lagrange multiplier (or external source)

$$\Gamma = V - J\phi \,, \tag{18}$$

whereupon

$$J = -\frac{\delta\Gamma}{\delta\phi} \,. \tag{19}$$

Now the 1PI Green's function is

$$\frac{\delta^2\Gamma}{\delta\phi^2} = -\frac{\delta J}{\delta\phi} = M^2 \ge 0 \tag{20}$$

for no tachyons; thus Γ should be convex downwards. In comparison to the liquid-gas system Γ, J, ϕ correspond to A, P, V, respectively.

If we apply this analogy to the skewed Goldstone model, then the region between the two minima may be a result of the bad approximation and we should draw a straight-line segment (constant J, equal areas in J-ϕ) tangential to the calculated V-ϕ curve. With this construction, the nonglobal minimum is excluded from the potential curve since it is argued that a lower-energy state is available as a mixture of the two possible phases of the system.

The trouble with such arguments is that they are circular to the extent that they presuppose that the vacuum state is in precise equilibrium. In the statistical-mechanics analogy, the nonglobal minimum corresponds to a superheated liquid which is metastable but which may have a long lifetime. Thus for the field-theory case it is essential to evaluate a lifetime before accepting arguments based on equilibrium situations.

Before proceeding to this, in Sec. III below, it is appropriate to mention certain interesting papers on vacuum instability.

The first is by Lee and Wick,[10] who consider a local finite-volume excitation of an abnormal degenerate or nondegenerate (higher energy density) vacuum. For example, if there is a scalar field strongly coupled to nucleons, then an abnormal nuclear state may occur for high nuclear density in, say, a high-energy collision between two heavy nuclei. This work assumes that the physical vacuum is an absolute minimum.

There are two papers[11] by Kobsarev, Okun, and Zeldovich which point out that in a theory with spontaneous breakdown the choice of vacuum state (mentioned in Sec. I above) may be made randomly at spacelike-separated points in the early universe. This implies a domain structure, and by making a model for the motion of the domain walls these authors suggest that this would lead to such inhomogeneity of the universe that it would be inconsistent with the observed isotropy (to within 0.1%) of the 2.7°K background radiation. We mention the work of Kobsarev *et al.* here not because it is relevant to Secs. III through VI below but because we shall return briefly to a related question at the end of this paper.

Finally, we must mention the important work on vacuum instability by Voleshin, Kobsarev, and Okun,[12] which was complete except for the correct handling of Lorentz invariance.

III. ANOTHER POSSIBILITY: VACUUM BUBBLES

Let us suppose hypothetically that the skewed Goldstone Lagrangian describes the world and that the universe, or at least a large domain containing the Earth, is built upon the nonglobal minimum as ground state. According to the statistical-mechanics analogy discussed above in Sec. II, this domain corresponds to a large finite volume of superheated liquid whose boiling may be nucleated somehow by, e.g., fluctuations, impurity, or a cosmic ray, forming a bubble of gas. This bubble has negative volume energy and positive surface-tension energy; hence a bubble is spherical with a critical radius above which it will survive and expand.

In the present case, we picture a vacuum bubble with surface energy arising from the difficulty of quantum tunneling through the potential barrier. If this picture is correct then there exist corresponding solutions to the classical field equations. It is technically convenient to treat the quantum tunneling as a classical motion in imaginary time.[13,14] Therefore, given a Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi - V(\phi) , \tag{21}$$

we seek solutions of the Euclidean equation of motion

$$\left(\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\phi = \frac{dV(\phi)}{d\phi} . \tag{22}$$

This equation is O(4) symmetric[15] so that, introducing $\rho^2 = x^2 + y^2 + z^2 + t^2$ and specializing to the skewed Goldstone model, it becomes

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho}\right)\phi(\rho) = \frac{d}{d\phi} V(\phi) \tag{23}$$

$$= \phi(m^2 - \delta\phi + \lambda\phi^2) \tag{24}$$

for $0 \le \rho \le \infty$ subject to the boundary conditions that

$$\phi(0) \simeq \phi_0^+ , \tag{25}$$

$$\phi(\infty) = \phi_0^- . \tag{26}$$

Mathematically, this resembles a pseudoparticle[16] but the physics is quite different from in Ref. 16. This solution corresponds at times $t = \pm\infty$ to no bubble, while at $t = 0$ there is a static bubble in unstable equilibrium. This static soliton solution will solve the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\phi(r) = \frac{d}{d\phi} V(\phi) , \tag{27}$$

where $r^2 = x^2 + y^2 + z^2$.

The existence of such a solution is of interest in connection with an impossibility theorem of Derrick.[17] This theorem states that the time-independent solutions of scalar field theory cannot exist for space-time dimensionality $d = (D+1) \ge 3$. The proof of the theorem is very short and proceeds by defining, in D space dimensions,

$$I_1 = \tfrac{1}{2}\int d^D x (\vec{\nabla}\phi)^2 \ , \qquad (28)$$

$$I_2 = \int d^D x\, V(\phi) \ . \qquad (29)$$

The energy E is the sum of these two integrals. Suppose that $\phi(x)$ is a static solution, then consider the functions $\phi(\lambda x)$. The corresponding energy $E(\lambda)$ and its derivative at $\lambda = 1$ are given by

$$E(\lambda) = \lambda^{2-D} I_1 + \lambda^{-D} I_2 \ , \qquad (30)$$

$$E'(1) = (2-D)I_1 - D I_2 \ . \qquad (31)$$

If $\phi(x)$ is a solution, its energy must be stationary and hence $E'(1) = 0$. But $I_1 \geq 0$ and, if we *assume* $I_2 \geq 0$ ($V = 0$ being the ground state), then the only solution is $I_1 = I_2 = 0$ corresponding to a constant ϕ.

In the present case, it is no longer true that $I_2 \geq 0$ since the vacuum state (taken to be time-independent) must be that for which $V(\phi) = 0$. Thus, for the purposes of this theorem, we are *not* allowed to add an arbitrary constant to $V(\phi)$. Both the soliton solution of Eq. (27) *and* the pseudoparticlelike solution of Eqs. (23) and (24) (the latter may, of course, be regarded as a static solution in $d = D + 1 = 5$ dimensions) evade Derrick's theorem because $I_2 < 0$. In fact, we may turn the theorem around by stating that nontrivial static solutions of scalar field theories are possible *only* if the vacuum is a nonglobal minimum of $V(\phi)$.

Returning to the vacuum bubble in Euclidean space, Eqs. (23)–(26), its classical action gives the logarithm of the probability for tunneling through the potential barrier to form a bubble with the equilibrium radius. This is the square of the amplitude, since the action for $-\infty < t < +\infty$ is double the action for $-\infty < t < 0$ because of the symmetry $t \to -t$. The pseudoparticle corresponds at $t = -\infty$ to the unstable vacuum in which a bubble grows to its equilibrium radius at $t = 0$, then shrinks and disappears for $t \to -\infty$. Once the equilibrium bubble is formed, it is classically unstable in Minkowski space; we do not pursue the ensuing vacuum disaster here; the dynamics of the vacuum bubbles requires explicit evaluation of $\phi(\rho)$ and will be discussed elsewhere.

In what follows, we shall make the approximation (to be justified *a posteriori*) that the bubble radius is large compared to its surface thickness. In this approximation we may write the bubble action A as a sum of a (negative) volume and a (positive) surface contribution for Euclidean radius $\rho = R$,

$$A(R) = \tfrac{1}{2}\pi^2 R^4 \epsilon + 2\pi^2 R^3 S \ , \qquad (32)$$

which is stationary for the value $R_m = 3S/\epsilon$ with value

$$A(R_m) = -\frac{27\pi^2}{2}\left(\frac{S^4}{\epsilon^3}\right). \qquad (33)$$

For this radius the energy of the three-dimensional bubble vanishes:

$$E = -\tfrac{4}{3}\pi R_m{}^3 \epsilon + 4\pi R_m{}^2 S \qquad (34)$$

$$= 0 \ . \qquad (35)$$

In Eq. (32) ϵ and S are the volume and surface energy densities, respectively.

The probable number N of vacuum bubbles formed in space-time volume V can now be estimated in the leading exponential approximation as

$$N = \left(\frac{V}{L^4}\right)\exp(-A_m) \ . \qquad (36)$$

Here L is a characteristic length of the theory; its value is computable in next order in h of the exponent. For the present, we shall use the estimate $L = R_m$.

To obtain an upper limit for N, we use the largest value one can reasonably contemplate, $V = 10^{164}$ F^4, which is that volume corresponding to the age of the universe, 10^{10} yr. The relevant domain volume may be much smaller, but our general conclusions will not be at all sensitive to the precise value of V, fortunately.

Finally, the criterion that the nonglobal minimum be a viable candidate for the physical vacuum will be that N be exceedingly small: $N \ll 1$.

IV. SKEWED GOLDSTONE MODEL

As a simple theory where the phenomenon occurs, we return to the model already mentioned in the Introduction. The classical potential is $(0 < \delta/|m| \ll \lambda \ll 1)$

$$V(\phi) = \tfrac{1}{2}m^2\phi^2 - \frac{\delta}{3}\phi^3 + \frac{\lambda}{4}\phi^4 \ . \qquad (37)$$

Let us suppose hypothetically (we consider a more realistic example in the next section) that this is the theory of the world and that the universe, or at least the domain of which we are a part, is built upon the nonglobal minimum at $\phi = \phi_0^-$. We wish to calculate the probability that there is a spontaneous vacuum-bubble formation leading to collapse into the lower-energy state available when $\phi = \phi_0^+$.

The volume energy density ϵ of a bubble is given by

$$\epsilon = V(\phi_0^-) - V(\phi_0^+) \qquad (38)$$

$$= \tfrac{2}{3}\delta v^3 \ . \qquad (39)$$

The surface energy density S can be looked at in two quite different ways. This quantity is the energy per unit area of the surface of the 3-sphere;

for an area element $d^2\sigma$ the action associated with expansion for time dt is $(S\,d^2\sigma)dt$. The same quantity also represents the action per unit hyperarea of the surface of the 4-sphere in Euclidean space; thus $\exp(-S)$ gives the tunneling probability through the potential barrier. Since δ is very small we may thus evaulate S as

$$S=\int_{-v}^{+v} d\phi\{2[V(\phi)-V(v)]\}^{1/2} \quad (40)$$

$$=2\sqrt{2}\int_0^v d\phi[\tfrac{1}{2}m^2\phi^2+\tfrac{1}{4}\lambda(\phi^4+v^4)]^{1/2} \quad (41)$$

$$=2\sqrt{2\lambda}\;v^3/3\;. \quad (42)$$

The equilibrum bubble radius is

$$R_m=3S/\epsilon=3\sqrt{2\lambda}\;\delta\;. \quad (43)$$

The probable number of vacuum bubbles forming in space-time volume V is therefore

$$N=\frac{V\delta^4}{324\lambda}\exp\left(-36\pi^2\frac{\lambda^2v^3}{\delta^3}\right). \quad (44)$$

First we can check that this expression has sensible limits: as $\delta\to 0$, $N\to 0$ as we expect for degenerate vacuums (Derrick's theorem[17] now applies and no vacuum-bubble pseudoparticles occur).

The point is that for any V we may make N arbitrarily small by making δ small. Thus $N\ll 1$ may be satisfied and the nonglobal minimum is equally as good a candidate for the vacuum state as the global minimum.

Our interpretation thus differs from the conventional one discussed in Sec. II where there would be an abrupt difference between $\delta=0$ and $\delta\neq 0$. In the present picture there is a smooth change since, even for $\delta\neq 0$ but sufficiently small, one can choose either minimum as vacuum, and the resulting physical predictions will depend on this choice, at order δ.

We now turn to a more physically interesting case, the Weinberg-Salam model of weak interactions.

V. WEINBERG-SALAM MODEL[18]

The Weinberg-Salam (WS) model[7] is the simplest unification of weak and electromagnetic interactions which is renormalizable. Since there has recently been some work[19] on the question of under what conditions the asymmetric vacuum is stable, it is amusing to apply the present considerations to this case.

With a single complex doublet of Higgs fields, it is customary to induce the spontaneous breaking of $SU(2)\otimes U(1)$ by putting quadratic and quartic terms into the Lagrangian so that the classical potential reads

$$V(\phi)=\tfrac{1}{2}\mu^2\phi^2-\lambda\phi^4\;. \quad (45)$$

The vacuum value of ϕ is related to Fermi's constant G by

$$\langle\phi\rangle=v=(\sqrt{2}\,G)^{-1/2}=248\text{ GeV}\;. \quad (46)$$

In the tree approximation $v=(\mu^2/2\lambda)^{1/2}$, and it appears that one can make the Higgs mass arbitrarily small by reducing λ. However, if $\lambda\sim e^4$ (e = gauge coupling constant) then the gauge vector loop competes with the ϕ^4 term and significantly alters the form of $V(\phi)$. Scalar and fermion loops are negligible by comparison. Including the vector loop gives[19]

$$V(\phi)=\tfrac{1}{2}\mu^2\phi^2-\lambda\phi^4+B\phi^4\ln\frac{\phi^2}{v^2}\;, \quad (47)$$

where

$$B=\frac{3}{64}\alpha^2\left(\frac{2+\sec^4\theta_W}{\sin^4\theta_W}\right) \quad (48)$$

$$=9.7\times 10^{-5} \quad (49)$$

for the empirical mixing angle $\theta_W=35°$.

The Higgs mass is given by

$$m_H{}^2=V''(v)=4Bv^2+8v^2(B-\lambda)\;. \quad (50)$$

The difference in energy density between the symmetric ($\phi=0$) and asymmetric ($\phi=v$) minima of $V(\phi)$ is

$$\epsilon=V(v)-v(0)=-(B-\lambda)v^4\;. \quad (51)$$

If $B>\lambda$, therefore, the asymmetric vacuum is absolutely stable; this implies, by Eq. (50), a lower limit on the Higgs mass

$$m_H\geq m_{cR}=(4Bv^2)^{1/2}=4.91\text{ GeV}\;. \quad (52)$$

This bound has been advocated recently by Weinberg.[19] In a modified version of the model, where the ϕ^2 term is disallowed because all masses are generated by the Coleman–E. Weinberg dimensional transmutation mechanism,[20] Gildener and S. Weinberg[21] find that the lower bound is increased by a factor $\sqrt{2}$ to give $m_H\gtrsim 7$ GeV.

If vacuum instability is allowed, then these bounds are significantly altered.

We shall treat the potential, Eq. (47), as though it were the classical potential and again keep only the leading exponential of the tunneling probability; this is presumably justified because for small couplings the corrections are also small. We should clearly warn the reader that this reasoning, although plausible, has not been proved and must be verified or refuted by a subsequent, more complete treatment of quantum corrections.

The surface energy density S of the vacuum bubble is given by

$$S=\int_{\phi_1}^{v} d\phi\{2[V(\phi)-\epsilon]\}^{1/2} . \quad (53)$$

Putting $\phi^2=xv^2$ and $r=\delta M^2/m_{cR}^2$, where $\delta M^2 = m_{cR}^2 - m_H^2$, this may be rewritten

$$S=v^3\sqrt{2B}\, I(r) , \quad (54)$$

$$I(r)=\tfrac{1}{2}\int_{x_1(r)}^{1} dx[(1+r) - (1+\tfrac{1}{2}r)x + x\ln x - r/2x]^{1/2} , \quad (55)$$

with the lower limit $x_1(r)$ satisfying the transcendental equation

$$(1+r)-(1+\tfrac{1}{2}r)x_1 + x_1\ln x_1 - r/2x_1 = 0 . \quad (56)$$

In terms of $I(r)$ the equilibrium bubble radius is given by

$$R_m = 24\sqrt{2B}\, vI(r)/(\delta m^2) . \quad (57)$$

If this is evaluated in fermis (F) and S is in F^{-3}, the probable number of bubbles in the backward light cone is

$$N=\left(\frac{10^{164}}{R_m^{\,4}}\right)\exp\left(-\frac{\pi^2}{2}R_m^{\,3}S\right) . \quad (58)$$

The results of evaluating $I(r)$ numerically, and the corresponding values of R_m and N, are given for several values of the Higgs mass m_H in Table I.

With these results at hand, we conclude that $N \ll 1$ for values of the Higgs mass down to about 3.5 GeV. For the mass range $4.91 > m_H > 3.5$ GeV, the vacuum is unstable but is, for practical purposes, totally secure against spontaneous bubble formation.

Concerning the approximation that the bubble radius is large compared to its surface thickness, this corresponds roughly to the requirement that $m_H R_m \gg 1$. This is reasonable down to $m_H = 3.6$ GeV, but below this mass the estimates given in Table I would become more accurate if the equation

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{3}{\rho}\frac{\partial}{\partial\rho}\right)\phi(\rho)=V'(\phi) \quad (59)$$

$$= \phi(m^2 - 4\lambda\phi^2 + 2B\phi^2 + 4B\phi^2\ln\phi^2/v^2) \quad (60)$$

is solved exactly, but the new lower limit on m_H will not be significantly altered thereby.

TABLE I. The results of evaluating $I(r)$ numerically, and the corresponding values of R_m and N for several values of the Higgs mass m_H.

m_H	$I(r)$	R_m	N
4.91 GeV	0.21	∞	0
4.0 GeV	0.064	0.13 F	$\sim 10^{-80,000}$
3.6 GeV	0.041	0.060 F	$\sim 10^{-365}$
3.5 GeV	0.034	0.047 F	$\sim 10^{-39}$
3.45 GeV	0.032	0.043 F	$\sim 10^{+20}$
3.4 GeV	0.030	0.039 F	$\sim 10^{+63}$
3.0 GeV	0.017	0.018 F	$\sim 10^{+164}$

VI. VACUUM CATASTROPHE: HIGGS MASS

So far, we have considered only spontaneous bubble formation through vacuum fluctuations. As an alternative, we may consider triggering the process in, say, an ultrahigh-energy collision of two elementary particles; but a reliable calculation of the relevant probability is exceedingly difficult. Let us take the Weinberg-Salam model as a concrete example. For $m_H \sim 4.9$ GeV the local maximum of $V(\phi)$ is

$$V_{\max} = \tfrac{1}{2}Bv^4 f^2(1-f^2) \quad (61)$$

where

$$1-f^2+2f^2\ln f^2 = 0 . \quad (62)$$

Solving these equations gives $f=0.534$ and $V_{\max} = 4\times 10^4$ GeV^4. Thus if this energy density is exceeded in some space-time volume, the potential barrier disappears and there is a possible classical path to the lower-energy minimum. The difficulty is what volume to use. If we take 1F^4 then, although $V_{\max}$ is not attained yet in artificially induced collisions (e.g., CERN ISR), it does occur in cosmic-ray collisions since cosmic rays up to 10^{11} GeV have been observed. Thus vacuum instability might be ruled out this way.

Once a vacuum bubble is formed, with zero energy at the equilibrium radius relative to the unstable vacuum, it may now gain further energy by expansion which will rapidly approach the speed of light. The dynamics of the vacuum bubbles may be studied by solving, for example, Eqs. (24) and (60) for the skew Goldstone model and the Weinberg-Salam model, respectively.

Returning to the Higgs mass of the WS model, the present situation may be summarized as follows.

$m_H > 7$ GeV supports gauge field theory ideas

$7 > m_H > 4.91$ GeV bare mass $\mu \neq 0$ *or* vacuum instability

$4.91 > m_H > 3.5$ GeV vacuum instability.

3.5 GeV $> m_H$?

Of course, we should bear in mind that Higgs bosons may not exist and that the symmetry-breaking mechanism may be dynamical in origin.

VII. SUMMARY

We have queried the usual dogma that the vacuum state in quantum field theory must be an absolute minimum of the effective potential.

When the vacuum corresponds to a nonglobal minimum, the classical field equations in Euclidean space-time $\Box\phi = -V'(\phi)$ possess vacuum-bubble pseudoparticle solutions. The solutions seem mathematically interesting both because they evade Derrick's theorem and because they are simpler examples of pseudoparticles than the non-Abelian-gauge pseudoparticles[14,16] (characterized by winding number).

The physical interpretation of these solutions in Minkowski space is that they represent tunneling from the metastable vacuum, and quantitative estimates of the vacuum-vacuum transition are made possible. There are consequences for the Higgs mass in gauge theories. A similar analysis can be made for other field theories; to cite just one example, in the O(N) model of Refs. 6, 22, the analysis is changed in an important way if we allow that the vacuum may be a nonglobal minimum.

It is not clear, at least to the author, whether the vacuum-bubble analysis given here or the discussion given above in Sec. II, based on the many-body-theory analogy, is the more compelling. But it does seem that the statistical-mechanics analogy is to some extent circular in that it *assumes* equilibrium situations and thus precludes the consideration of instability. As a further argument in our favor, we should say that for small coupling constants it seems improbable that the classical approximation to $V(\phi)$ should be so much in error as would be implied by the straight-line construction indicated in Sec. II.

Once one accepts vacuum instability and the related phase transitions, one is soon led to consider the cosmological ramifications. One would, of course, like to know by what mechanism our particular vacuum was selected during the early universe. There is also difficulty raised in Ref. 11 about the likely formation of phase domains in space-time, although there exists no observational evidence that the universe is anything but a single domain. It would be interesting if the vacuum-bubble methods can shed any light on these profound questions which are beyond the scope of the present article.

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