PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Transverse-momentum distribution from the Bloch-Nordsieck method

G. Pancheri-Srivastava and Y. Srivastava*
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 15 December 1976)

The transverse-momentum distribution is studied using the Bloch-Nordsieck method. An approximate analytic form for the above distribution is found, which maintains the normalization as well as reproduces the exact result for the average (squared) transverse momentum, $\langle k_\perp{}^2\rangle$. For large $k_\perp$, our proposed approximation gives an exponential damping in $k_\perp$ which is independent of the coupling constant. The discontinuous nature of the four-momentum distribution is examined to make comparison with perturbation theory.

INTRODUCTION

In this paper we present an approximation to the transverse-momentum distribution predicted by the Bloch-Nordsieck[1] theorem for radiation due to soft-photon emission. We have looked at such a distribution for an arbitrary value of the coupling constant to see whether the sum to all orders introduces some functional dependence which can usually be neglected in QED. Our approximation to the distribution function exhibits the usual power law but also an exponential cutoff which is independent of the coupling constant and has a logarithmic dependence from the energy. This behavior shows interesting analogies with the phenomenological $p_\perp$ distribution of strong inclusive reactions at high energies,[2] a result not altogether surprising since it is already known[3,4] that the energy distribution of soft-photon emission produces a Regge-type behavior of QED cross sections. It thus seems that the Block-Nordsieck theorem in some cases can predict similar functional dependences for QED cross sections and strong inclusive distributions. The reason why strong inclusive cross sections might offer a better ground of comparison with QED than, say, elastic processes lies in the fact that a QED reaction is always an inclusive one. One is thus assured of comparing a functional behavior in the same number of invariants, that is to say to study the same type of process, i.e., scattering of on-shell particles accompanied by an energy-momentum loss. There is also another point in favor of soft-photon emission as a test behavior of some inclusive processes: Any feature in QED for which a given order calculation in α is adequate can hardly be expected to be found in strong interactions, where the coupling constant is not small and where the perturbative method breaks down. For a comparison we must then turn to those QED phenomena which require calculation to all orders in α. Soft-photon emission is one such phenomenon and the Bloch-Nordsieck method is the nonperturbative tool which deals with it.

In Sec. I the main features of the Bloch-Nordsieck method are recalled and our approximation is presented.

In Sec. II we study the behavior of our approximation for small and large values of the transverse-momentum loss and compare it with first-order QED calculations.

In Sec. III we discuss the discontinuous nature of the exact distribution.

I. THE SOFT-PHOTON MOMENTUM DISTRIBUTION

Soft-photon emission is discussed extensively in the literature,[5,6,7,8] both within the framework of perturbation theory as well as via a nonperturbative approach like the one provided by the Bloch-Nordsieck method. For the reasons stated in the Introduction, the latter approach is the one on which we shall focus our attention. We follow the notation of Ref. 9 and we start by considering the probability distribution for emission of an infinite number of real soft photons, which one assumes to have been emitted independently from each other from a classical source. The Bloch-Nordsieck theorem used in conjunction with energy-momentum conservation leads to

$$d^4P(K)=d^4K\int \frac{d^4x}{(2\pi)^4}\, e^{iK\cdot x-h(x,E)}\ , \tag{1}$$

where K is the four-momentum of the emitted radiation and

$$h(x,E)=\int_0^E d^3\bar{n}(\vec{k})(1-e^{-ik\cdot x})\ , \tag{2}$$

where $d^3\bar{n}(\vec{k})$ is the average number of real photons emitted in a momentum interval d^3k and the integration is over all directions. E is the maximum frequency allowed for single-photon emission in a given process. Following Ref. 9 we can write

$$d^3\bar{n}(\vec{k})=\beta\,\frac{dk}{k}\,f(\hat{n})d\Omega_n\ , \tag{3}$$

where β is the spectrum and $f(\hat{n})$ the angular distribution for single-photon emission. The function $f(\hat{n})$ is normalized to 1, i.e.,

$$\int f(\hat{n})d\Omega_n = 1 .$$

In the soft-photon approximation the factor β of Eq. (3) is independent of the photon's frequency k and is an invariant function of the momenta of the emitting particles. Both β as well as $f(\hat{n})$ are calculated using a classical electron current

$$j_\mu(\vec{k}) = \frac{ie}{(2\pi)^{3/2}} \sum \epsilon_i \frac{p_{i\mu}}{(p_i \cdot k)} ,$$

where e is the electron's charge, p_i's are the momenta of the emitting particles, k^μ is the single-photon momentum ($k_\mu k^\mu = 0$), and $\epsilon_i = +1$ (-1) for the destruction of a negative (positive) particle. For creation the signs are reversed. With this definition of the current, one has

$$\beta = -\int d^4n\, \theta(n_0) n_0 \delta(n_0 - 1) j_\mu(n) j^{\mu *}(n) \delta(n^2)$$

and

$$f(\hat{n}) = -\frac{1}{2\beta} k^2 j_\mu(\vec{k}) j^{\mu *}(\vec{k}) .$$

From Eq. (1) it seems hopeless to obtain an exact, close-form expression for the four-momentum distribution. However, one can integrate it in d^3K and obtain the well-known power law for the energy radiated, i.e.,

$$dP(\omega) = \beta \frac{d\omega}{\omega}\left(\frac{\omega}{E}\right)^\beta \quad \text{for } \omega < E , \tag{4}$$

where E is the cutoff energy for single photons. Note that E represents the limit beyond which the single emitted photon cannot be considered soft anymore. However, owing to the smallness of β, the distribution function is not too sensitive[7,10] to it and E may be taken as the c.m. energy (as is usual), as long as ω/E is kept small. Equation (4) is generally used to calculate the radiative correction factor to QED processes, but it also plays an interesting role when studied as the inclusive energy distribution in a reaction such as

$$A^\pm + A^- \to A^\pm + A^- + X . \tag{5}$$

It was shown[4] that the di-triple Regge limit of the cross section for process (5) suggests the relationship

$$\alpha_\gamma(t) = 1 - \beta(t)/4 , \tag{6}$$

where[11]

$$\beta(t) = \lim_{s/m^2 \to \infty,\ t \text{ fixed}} \beta(s,t,u) ,$$

and $\alpha_\gamma(t)$ is the photon's trajectory. Relation (6) implies that soft-photon emission, long thought of as a nuisance, is responsible for producing in QED a Regge behavior typical of strong interactions. The trajectory thus obtained is obviously not a linearly rising one, the nature of interaction being different, but the functional behavior is the same.

The question now arises as to whether other functional features of strong interaction distributions are also to be found from soft-photon emission. We therefore decided to study the predictions of Eq. (1) for the transverse-momentum distribution. Integrating Eq. (1) in $d\omega$ and dK_3, we have

$$d^2P(\vec{K}_\perp) = \int d\omega dK_3 \frac{d^4P(K)}{d\omega dK_3} = d^2K_\perp \int \frac{d^2x_\perp}{(2\pi)^2} e^{-i\vec{K}_\perp\cdot\vec{x}_\perp - h(x_0 = x_3 = 0, \vec{x}_\perp; E)} .$$

Using

$$h(x_0 = x_3 = 0; \vec{x}_\perp; E) \equiv h(\vec{x}_\perp) = \beta \int d\Omega_n f(\hat{n}) \int_0^E \frac{dk}{k}(1 - e^{ik(\vec{n}_\perp\cdot\vec{x}_\perp)}) ,$$

we now scale out the photon's cutoff energy E by writing $\vec{x} = \vec{x}_\perp E$ (with $\vec{x}$ a two-dimensional vector) so that

$$d^2P(\vec{K}_\perp) = \frac{d^2K_\perp}{E^2} \int \frac{d^2x}{(2\pi)^2} e^{-i(\vec{K}_\perp/E)\cdot\vec{x} - \beta g(\vec{x})} , \tag{7}$$

where

$$g(\vec{x}) = \int d\Omega_n f(\hat{n}) \int_0^{-i(\vec{n}_\perp\cdot\vec{x})} \frac{dy}{y}(1 - e^{-y}) . \tag{8}$$

Inspection of Eqs. (7) and (8) shows that, unlike the energy case, there seems to be no region in which distribution (7) can be solved exactly. Earlier attempts had to be limited to first order in the β expansion, thus simply recovering the single-photon limit. Instead, we have searched for a suitable approximation for the function $g(\vec{x})$ and then tried to integrate. By a suitable approximation we mean a function which at least bears the same large- and small-x behavior. For a process of type (5), one has

$$g(\vec{x}) \underset{|\vec{x}|\to\infty}{\sim} \tfrac{1}{2} \int d\Omega_n f(\hat{n}) \ln[(\vec{n}_\perp\cdot\vec{x})^2] \underset{|\vec{x}|\to\infty}{\sim} \tfrac{1}{2}\ln(x^2),$$

and (9)

$$g(\vec{x}) \underset{|\vec{x}|\to 0}{\sim} \tfrac{1}{4} \int d\Omega_n f(\hat{n}) (\vec{n}_\perp\cdot\vec{x})^2 ,$$

where we have made use of the normalization $\int f(\hat{n})d\Omega_n = 1$, and of the fact that $g(\vec{x}) = g(-\vec{x})$ in the c.m. frame of process (5). For such processes and in the c.m. frame, there is an equal probability that a photon be emitted forward or backward so that $f(\hat{n}) = f(-\hat{n})$. It can be shown that the asymmetric case, i.e., $f(\hat{n}) \neq f(-\hat{n})$, introduces only minor modifications in the final result. Bearing in mind the behavior given by (9), we propose the following approximate expression for $g(\vec{x})$:

$$\tilde{g}(\vec{x}) = \tfrac{1}{2}\ln\left[1 + \tfrac{1}{2}\int d\Omega_n f(\hat{n})(\vec{n}_\perp \cdot \vec{x})^2\right], \qquad (10)$$

where the function $f(\hat{n})$ has to be written in the center-of-mass frame of the incoming emitting particles and is of course a function of the momenta of all the initial and final particles. However, it should be noted that owing to the normalization, $f(\hat{n})$ does not depend any more on α, the fine-structure constant. Inserting expression (10) into Eq. (7) we obtain the approximate expression

$$d^2\tilde{P}(\vec{K}_\perp) = \frac{d^2K_\perp}{E^2}\int \frac{d^2x}{(2\pi)^2} e^{-i(\vec{K}_\perp/E)\cdot\vec{x}} \times \left[1 + \tfrac{1}{2}\int f(\hat{n})d\Omega_n(\vec{n}_\perp\cdot\vec{x})^2\right]^{-\beta/2}.$$

In order to perform the angular integration we make one more approximation. Since the integral in Eq. (10) does not show any dramatic dependence upon the direction of $\vec{x}$, we shall approximate it with its average value; i.e., we shall put

$$\int d\Omega_n f(\hat{n})(\vec{n}_\perp\cdot\vec{x})^2 \simeq \frac{1}{2\pi}\int_0^{2\pi} d\phi_x \int d\Omega_n f(\hat{n})(\vec{n}_\perp\cdot\vec{x})^2, \qquad (11)$$

which allows us to write

$$\tilde{g}(\vec{x}) \sim \tfrac{1}{2}\ln\left[1 + \tfrac{1}{4}x^2\int d\Omega_n f(\hat{n})n_\perp{}^2\right],$$

so that

$$d^2\tilde{P}(\vec{K}_\perp) = \frac{d^2K_\perp}{E^2}\int \frac{d^2x}{(2\pi)^2} e^{-i(\vec{K}_\perp/E)\cdot\vec{x}} \times (1 + Ax^2)^{-\beta/2}, \qquad (12)$$

where A is now a constant in $\vec{x}$ and is given by

$$A = \tfrac{1}{4}\int d\Omega_n f(\hat{n})n_\perp{}^2. \qquad (13)$$

We can thus proceed with the angular integration in Eq. (12) and we get

$$d^2\tilde{P}(\vec{K}_\perp) = \frac{d^2K_\perp}{(2\pi)E^2}\int_0^\infty \frac{x\,dx}{(1 + Ax^2)^{+\beta/2}} J_0\left(\frac{K_\perp x}{E}\right),$$

which allows the following expression for the approximated distribution function:

$$d^2\tilde{P}(\vec{K}_\perp) = \frac{\beta(2\pi)^{-1}}{\Gamma(1+\beta/2)}\frac{d^2K_\perp}{2E^2A}\left(\frac{K_\perp}{2E\sqrt{A}}\right)^{\beta/2-1} \times \mathcal{K}_{1-\beta/2}\left(\frac{K_\perp}{E\sqrt{A}}\right), \qquad (14)$$

where $\mathcal{K}_{1-\beta/2}$ is the modified Bessel function of the third kind which admits the integral representation

$$\mathcal{K}_\mu(z) = \frac{\sqrt{\pi}}{\Gamma(\mu+\frac{1}{2})}\left(\frac{z}{2}\right)^\mu \int_1^\infty e^{-zt}(t^2-1)^{\mu-1/2}dt, \quad \mathrm{Re}z > 0,\ \mathrm{Re}\mu > -\tfrac{1}{2}. \qquad (15)$$

In order to put forward Eq. (14) as a viable approximation to the exact $d^2P(\vec{K}_\perp)$ we must make sure that our expression satisfies two important constraints. These constraints are (i) that $d^2\tilde{P}(\vec{K}_\perp)$ be normalized to 1, (ii) that the average transverse momentum $\langle K_\perp{}^2\rangle$ be the one predicted by the exact distribution (7). We are going to show that both constraints are indeed satisfied. Notice that the satisfaction of the normalization constraint will not offer any check on the scaling variable $K_\perp/E\sqrt{A}$. In other words, the normalization cannot predict the quantity with which $K_\perp$ scales. However, it will be a ckeck on the use of the function $\mathcal{K}_{1-\beta/2}$. The second constraint, on the other hand, will provide a check on the scale, and hence a check of approximation (11) on the angular integration.

Using the integral representation for the $\mathcal{K}$ function it is actually easy to check that

$$\int d^2\tilde{P}(\vec{K}_\perp) = 1 = \int d^2P(\vec{K}_\perp), \qquad (16)$$

and we find that the normalization is fully preserved by our approximation. We can then proceed to calculate $\langle K_\perp{}^2\rangle$. Using the normalization condition on $d^2\tilde{P}(\vec{K}_\perp)$ we have

$$\langle K_\perp{}^2\rangle \int d^2\tilde{P}(\vec{K}) = \langle K_\perp{}^2\rangle = \int K_\perp{}^2 d^2\tilde{P}(\vec{K}_\perp).$$

Inserting Eq. (16) into the above leads to

$$\langle K_\perp{}^2\rangle = \frac{2^{1-\beta/2}}{\Gamma(\beta/2)}AE^2\int_0^\infty x^3dx\,x^{\beta/2-1}\mathcal{K}_{1-\beta/2}(x) = 2\beta AE^2, \qquad (17)$$

where we have made use of the integral representation for the $\mathcal{K}$ function, Eq. (15). Equation (17) gives the average $K_\perp{}^2$ as calculated through our approximated distribution function. We shall presently show that such a value is exactly the same which one obtains using the exact $d^2P(\vec{K}_\perp)$ given by Eq. (7). One has, in fact,

$$\langle K_\perp{}^2\rangle_{\rm exact} = \int K_\perp{}^2 d^2P(\vec{K}_\perp)$$

$$= \int \frac{d^2x}{(2\pi)^2} e^{-\beta g(\vec{x})} \int K_\perp{}^2 \frac{d^2K_\perp}{E^2} e^{-i(\vec{K}_\perp/E)\cdot\vec{x}}$$

$$= -E^2 \int \frac{d^2x}{(2\pi)^2} e^{-\beta g(\vec{x})} \Delta^{(2)} \int d^2K_\perp e^{-i\vec{K}_\perp\cdot\vec{x}} ,$$

where $\Delta^{(2)}$ is the Laplace operator in the (two-dimensional) $\vec{x}$ space. Since for processes of type (5) $g(\vec{x})$ is an even function of $\vec{x}$, $\operatorname{grad} g(\vec{x}=0)=0$, and one therefore gets, after an integration by parts and carrying out the integration over $d^2K_\perp$,

$$\langle K_\perp{}^2\rangle_{\rm exact} = \beta E^2 \Delta^{(2)} g(\vec{x}=0)$$

$$= \beta E^2 \tfrac{1}{2} \int d\Omega_n f(\hat{n}) n_\perp{}^2$$

or, using definition (13),

$$\langle K_\perp{}^2\rangle_{\rm exact} = 2\beta E^2 A .$$

Thus, whatever the merits of our representation, we are at least assured that it preserves the normalization and predicts the same average $\langle K_\perp{}^2\rangle$ as the exact distribution.

Before proceeding to study the small- and high-$K_\perp$ limits of $d^2\tilde{P}(\vec{K}_\perp)$, we shall calculate the quantity E^2A which appears to set the scale of such limits. We want to examine the dependence of A on the kinematic variables. The quantity A specifies also what we mean by $K_\perp$, since A, unlike β, is not a relativistic invariant. Once a certain frame has been chosen for its calculation, $K_\perp$ is defined in that frame. Let us consider the reaction

$$A^-(p_1) + A^\pm(p_2) \to A^-(p_3) + A^\pm(p_4) + X ,$$

where the p_i's are the momenta of four charged particles of equal masses. For small $M_x{}^2$ (soft-photon approximation) we are dealing with an almost elastic reaction. We work in the c.m. frame and fix the z axis along the incoming-particle direction, defining the x-z plane as the scattering plane. To calculate

$$A = \tfrac{1}{4} \int f(\hat{n}) n_\perp{}^2 d\Omega_n$$

we use the definition of $f(\hat{n})$ previously given in terms of the classical currents. Notice that there is a difference in the currents according to whether the process is

$$A^- + A^+ \to A^- + A^+ + X \qquad (18)$$

or

$$A^- + A^- \to A^- + A^- + X . \qquad (19)$$

The current will thus be written as

$$j_\mu(k) = \frac{ie^2}{(2\pi)^3}\left(\frac{p_{1\mu}}{p_1\cdot k} \mp \frac{p_{2\mu}}{p_2\cdot k} - \frac{p_{3\mu}}{p_3\cdot k} \pm \frac{p_{4\mu}}{p_4\cdot k}\right) ,$$

where the upper sign refers to process (18) and the lower sign to process (19).

The computation of A is rather long and the result is

$$A = -\frac{m^2}{s-4m^2} + \frac{1}{2}\left(-\frac{t}{s-4m^2}\right)\left(\frac{s}{s-4m^2}\left(2+\frac{3t}{s+4m^2}\right) - \left[\frac{s}{s-4m^2} + \frac{t(s+2m^2)}{(s-4m^2)^2}\right]\left(\frac{s}{s-4m^2}\right)^{1/2} \ln\frac{\sqrt{s}+(s-4m^2)^{1/2}}{\sqrt{s}-(s-4m^2)^{1/2}}\right.$$
$$\mp\left\{-\frac{t(s+2m^2)}{(s-4m^2)^2} - \frac{2m^2}{s-4m^2}\right.$$
$$\left.\left. + \frac{2m^2}{\sqrt{s}\,(s-4m^2)^{1/2}}\left[1+\frac{2t+2m^2}{s-4m^2}+\frac{6m^2t}{(s-4m^2)^2}\right]\ln\frac{\sqrt{s}+(s-4m^2)^{1/2}}{\sqrt{s}-(s-4m^2)^{1/2}}\right\}\right)$$
$$\times\left\{1 - \frac{2m^2-t}{\sqrt{-t}\,(4m^2-t)^{1/2}} \ln\frac{(4m^2-t)^{1/2}+\sqrt{-t}}{(4m^2-t)^{1/2}-\sqrt{-t}}\right.$$
$$\left.\mp\left[\frac{s-2m^2}{\sqrt{s}(s-4m^2)^{1/2}}\ln\frac{\sqrt{s}+(s-4m^2)^{1/2}}{\sqrt{s}-(s-4m^2)^{1/2}} - \frac{2m^2-u}{\sqrt{-u}\,(4m^2-u)^{1/2}}\ln\frac{(4m^2-u)^{1/2}+\sqrt{-u}}{(4m^2-u)^{1/2}-\sqrt{-u}}\right]\right\}^{-1} ,$$

where the ($\mp$) sign refers to processes (18) and (19) as specified above, and we have put

$$s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_2-p_3)^2, \quad p_i{}^2=m^2 .$$

The above expression for A simplifies considerably in the extreme relativistic limit. Neglecting terms of order m^2/s and m^2/u, one gets

$$A \underset{m^2/u,\, m^2/s\to 0}{\sim} -\frac{m^2}{s} + \tfrac{1}{2}\left(-\frac{t}{s}\right) \frac{2+\dfrac{3t}{s} - \left(1+\dfrac{t}{s}\right)\ln\dfrac{s}{m^2} \mp\left[-\dfrac{t}{s}+\dfrac{2m^2}{s}\left(1+\dfrac{2t}{s}\right)\ln\dfrac{s}{m^2}\right]}{1 - \dfrac{2m^2-t}{\sqrt{-t}(4m^2-t)^{1/2}}\ln\dfrac{(4m^2-t)^{1/2}+\sqrt{-t}}{(4m^2-t)^{1/2}-\sqrt{-t}} \mp\left(\ln\dfrac{s}{m^2}-\ln\dfrac{u}{m^2}\right)} .$$

The terms which are liable to change sign are actually negligible in the high-energy, small-scattering-angle limit and we obtain

$$E^2A \underset{s\text{ large },\ t\text{ small}}{\sim} \tfrac{1}{8}(-t)\frac{\ln(s/m^2)-2}{-1+\dfrac{2m^2-t}{\sqrt{-t}\,(4m^2-t)^{1/2}}\ln\dfrac{(4m^2-t)^{1/2}+\sqrt{-t}}{(4m^2-t)^{1/2}-\sqrt{-t}}}-\frac{m^2}{4}.$$

In the above expression we have taken E to be the c.m. energy in analogy to what had been done with the energy dependence in Eq. (4). For very small angles, the t dependence in E^2A completely disappears and one has

$$(E^2A)^{1/2} \underset{s\text{ large },\ t/m^2\text{ small}}{\sim} m\left[\tfrac{3}{8}\left(\ln\frac{s}{m^2}-\frac{8}{3}\right)\right]^{1/2},$$

which gives a $K_\perp$ scale proportional to the mass of the emitting particles and roughly of the same order of magnitude.

Even if t/m^2 is not small (but still $t/s \ll 1$), $(E^2A)^{1/2}$ does not change very much as can be seen from the above. In general we can say that both the t and s dependences are slower than logarithmic and that the scale is roughly set by m for small t values. The last occurrence is quite important as it gives a physical meaning to the limit $K_\perp/E\sqrt{A} \to \infty$. In fact, if the scale had turned out to be of the order of E [as is the case for the energy variable in Eq. (4)], the limit $K_\perp \gg E$ would not have been of interest in a soft-photon approximation where the momentum loss is assumed to be small relative to the momenta of the emitting particles.

II. LIMITING VALUES OF $d^2\tilde{P}(\vec{K}_\perp)$

We are now ready to study the small- and large-$K_\perp$ limits of the approximation, using the known behaviors of the $\mathcal{K}$ function:

$$\mathcal{K}_{1-\beta/2}(z) \underset{z\to 0}{\sim} \frac{1}{2}\left(\frac{z}{2}\right)^{\beta/2-1}\Gamma\left(1-\frac{\beta}{2}\right)$$

and

$$\mathcal{K}_{1-\beta/2}(z) \underset{z\to\infty}{\sim} \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}.$$

For small values of the arguments $K_\perp/E\sqrt{A}$, we get from Eq. (14) an isotropic distribution with the typical power-law behavior of soft-photon emission, i.e.,

$$d^2\tilde{P}(\vec{K}_\perp) \underset{K_\perp \ll E\sqrt{A}}{\sim} \frac{\beta}{2\pi}\frac{d^2K_\perp}{K_\perp^2}\left(\frac{K_\perp}{2E\sqrt{A}}\right)^{\beta}\frac{\Gamma(1-\beta/2)}{\Gamma(1+\beta/2)}.$$

The aforementioned difficulties in integrating Eq. (7) to all orders in β prevent an actual check of this limit. However, the behavior appears reasonable and it is correct to first order in β for very forward scattering, where $E\sqrt{A} \sim m$ and $K_\perp \ll m$.

The only difference of this approximation with the usually expected power law lies in the fact that $K_\perp$ scales with $2E\sqrt{A}$ rather than just the cutoff energy E, as is the case for the frequency distribution.

We now turn to the large-$K_\perp$ limit of our approximation, where we hope to encounter some interesting features. We obtain from Eq. (14)

$$d^2\tilde{P}(\vec{K}_\perp) \underset{K_\perp \gg E\sqrt{A}}{\sim} \frac{\beta}{2\sqrt{\pi}}\frac{d^2K_\perp}{\Gamma(1+\beta/2)K_\perp^2}\left(\frac{K_\perp}{2E\sqrt{A}}\right)^{(\beta+1)/2} \times e^{-K_\perp/E\sqrt{A}}. \tag{20}$$

We find that for large $K_\perp/E\sqrt{A}$ the distribution exhibits an exponential cutoff linear in $K_\perp$, in addition to a power law, which is, however, not the same as the one in the low-$K_\perp$ limit. The exponential cutoff is independent of the coupling constant, and we have already seen that in the small-t approximation it is independent of t. Equation (20) thus shows an exponential damping of the transverse-momentum distribution not unlike some strong inclusive reaction distributions.[2] The numerical value of the cutoff is of course not the one observed there, but this is to be expected since A was calculated using a (classical) electromagnetic current. However, it appears that just as in the case of Regge behavior, the collective effect of soft-photon emission contrives to produce a functional behavior in the transverse-momentum variable similar to the one observed experimentally in strong inclusive processes. The statement that the exponential cutoff is a multiphoton effect has to be understood from a nonperturbative point of view. At first glance, in fact, Eq. (20) shows the cutoff to be present at any order in β, hence also at first order. This behavior in β is hardly what one usually calls a multiphoton effect and, what seems even worse, one is led to say that our distribution does not recover the QED limit, where, at first order in β, there is obviously no exponential damping. However, one should notice that the approximation given by Eq. (14) is certainly correct on the average since $\int d^2\tilde{P}(\vec{K}_\perp) = \int d^2P(\vec{K}_\perp)$ and $\int K_\perp^2 d^2\tilde{P}(\vec{K}_\perp) = \int K_\perp^2 d^2P(\vec{K}_\perp)$. Having already seen that for very small $K_\perp$ the two distributions agree, we are led to believe that Eq. (14) cannot be too far off from the exact distribution even for $K_\perp \gtrsim E\sqrt{A}$. Now, because of the transversality of the radiation, soft photons are, on the average, confined within a cone of angle m/E around the emitting particle. This implies that in order to reach high values of $K_\perp/E\sqrt{A}$ one needs more than one photon. As $K_\perp/E\sqrt{A}$ increases, the exact distribution is no more of order β but β^2 (two-photon threshold), β^3, etc. For small β, this means that the distributions falls off extremely rapidly. What our approximation does is to describe on the average the sharp decline in the distribution as one moves from one range on $K_\perp/E\sqrt{A}$ to a higher one which requires one more photon. Our approximation is thus an analytic expression which smoothly interpolates the exact discontinuous distribution. Since the discontinuities appear as multiple-photon thresholds open up, the approximation is an average description of multiphoton effects in a truly nonperturbative way. The discontinuous behavior of the distribution function will be clarified in the next section with a specific example.

From the above discussion, we therefore claim that approximation (14) is to be a good description of the exact distribution (7) both for small as well as for large $K_\perp$ values, with the caution that for large $K_\perp$ values the approximation is fully nonperturbative and that its expansion in powers of β does not admit order by order comparison with the QED treatment.

III. DISCONTINUOUS CHARACTER OF THE PROBABILITY DISTRIBUTION

In our proposing Eq. (14) as a viable approximation, we must understand our failure in recovering the perturbative result at first order in β for large $K_\perp/E\sqrt{A}$. As we have tried to convey in the preceding section, this failure occurs because of the discontinuous character of the distribution function. To illustrate it, we have chosen to study the energy distribution, since the latter is simpler (only one positive-definite variable ω) and its exact behavior in at least one region, $\omega < E$, is known. However, it is clear that the discontinuities, if present in $dP(\omega)$, are also present in $d^4P(K)$ and hence in $d^2P(\vec{K}_\perp)$, as the latter was obtained from $d^4P(K)$ by integration.

The integral describing the energy distribution is

$$dP(\omega)=\frac{d\omega}{2\pi}\int d\tau\, e^{+i\tau\omega}e^{-h(E\tau)}\,, \tag{21}$$

where

$$h(E\tau)=\beta\int_0^{iE\tau}\frac{dy}{y}(1-e^{-y})\,.$$

E is the maximum energy that a single photon can carry away, or the limit beyond which the soft-photon treatment with a classical current is not valid anymore. For any practical calculations a solution of (21) for $\omega<E$ is more than adequate, and it is the only physically acceptable one if E has to be the c.m. energy of the reaction. However, what we are interested in is to show that $dP(\omega)$ as a function of ω/E is discontinuous. The key to the argument lies in a difference-differential equation obeyed by the function $dP(\omega)/(d\omega/E)$. If we put $x=\omega/E$, one has

$$dP(\omega)=d\,\frac{\omega}{E}\,\Pi(x)\,,$$

where

$$\Pi(x)=\int\frac{d\tau}{2\pi}\,e^{+i\tau x}e^{-h(\tau)}.$$

Taking a derivative with respect to x, we get

$$\frac{d\Pi(x)}{dx}=\frac{\beta-1}{x}\Pi(x)-\frac{\beta}{x}\Pi(x-1), \tag{22}$$

where we have made use of the fact that

$$\Pi(x)=\frac{1}{x}\int\frac{d\tau}{2\pi}e^{-i\tau}e^{-h(\tau/x)}\,.$$

Since the energy distribution must be zero for $\omega<0$, we must have $\Pi(x)=0$ for $x<0$. Then Eq. (22) for $0<x<1$ admits the solution

$$\Pi(x)\propto x^{\beta-1}\,.$$

The constant can be fixed by the normalization condition on $dP(\omega)$ and is proportional to β. Equation (22) was first written down in Ref. (9) and used to obtain in a simple way the already well-known power-law behavior of $dP(\omega)$.

However, Eq. (22) allows one to calculate $\Pi(x)$ [and hence $dP(\omega)$ eventually] also beyond the point $x=1$. Let us call $\Pi_0(x)$ the solution of Eq. (22) in the range $0\leq x\leq 1$ and $\Pi_n(x)$ the solution in the range $n\leq x\leq n+1$. Then Eq. (22) can be written as

$$\frac{d\Pi_n(x)}{dx}=\frac{\beta-1}{x}\Pi_n(x)-\frac{\beta}{x}\Pi_{n-1}(x-1)\,, \tag{23}$$

since $n-1\leq x-1\leq n$. If the solution in the preceding interval is known, Eq. (23) can be solved and one has

$$\Pi_n(x)=\Pi_{n-1}(n)\frac{\Pi_0(x)}{\Pi_0(n)}-\beta\int_n^x\frac{\Pi_0(x)}{\Pi_0(x')}\frac{\Pi_{n-1}(x'-1)}{x'}dx'\,, \tag{24}$$

where we have imposed the condition $\Pi_n(n)=\Pi_{n-1}(n)$. For the particular case of x in the interval (1, 2) one has, after some rearrangement of terms,

$$\Pi_1(x) = \Pi_0(x)\left[1 - \left(\frac{x-1}{x}\right)^{\beta} {}_2F_1\left(1, \beta; 1+\beta; \frac{x-1}{x}\right)\right].$$

An expansion in β now shows that the quantity in square brackets is of order β and hence $\Pi_1(x)$ is of order β^2. Repeated applications of Eq. (24) for higher values of n show that $\Pi_n(x)$ is of order β^{n+1}.

IV. CONCLUDING REMARKS

In this paper we have presented a nonperturbative approximation to the transverse-momentum distribution function for soft-photon emission derived from the Bloch-Nordsieck theorem. Previous calculations[3,4] have already shown the energy distribution of soft-photon emission to be responsible for a Regge-type behavior of the cross sections. We now find an exponential cutoff appearing as a collective multiphoton effect in the transverse-momentum distribution at high energies. This effect is thus one more indication that a Bloch-Nordsieck-type mechanism might be at work in producing some of the observed features of strong inclusive reaction distributions at high energies.[12]

We have also studied the discontinuities in the energy distribution within the framework of the Bloch-Nordsieck method and are presently investigating the implications of a recursion relation between the various discontinuities such as the one obtained in the last section.

ACKNOWLEDGMENTS

We are indebted to Professor L. Stodolsky for renewing our interest in this problem, and we thank Professor M. Friedman for many helpful conversations and constructive criticism.

---

*Work supported in part by the National Science Foundation.

[1] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937). For a covariant formulation see R. J. Glauber, *ibid.* 84, 395 (1951); and W. Thirring and B. Touschek, Philos. Mag. 42, 244 (1951).

[2] See the compilation of data on inclusive reactions by Particle Data Group, LBL report, 1972 (unpublished).

[3] S. C. Frautschi, in *Theoretical Physics* (IAEA, Vienna, 1963).

[4] G. Pancheri-Srivastava, Phys. Lett. 44B, 109 (1973).

[5] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta 27, 613 (1954); *Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955).

[6] D. R. Yennie and H. Suura, Phys. Rev. 105, 1378 (1957); D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[7] E. L. Lomon, Nucl. Phys. 1, 101 (1956); Phys. Rev. 113, 726 (1959).

[8] K. E. Erickson, Nuovo Cimento 19, 1010 (1961).

[9] E. Etim, G. Pancheri, and B. Touschek, Nuovo Cimento 51B, 276 (1967).

[10] E. Etim and B. Touschek, Laboratori Nazionali di Frascati Report No. LNF-66/10, 1966 (unpublished).

[11] $\beta(s,t,u)$ is the soft-photon spectrum β mentioned earlier without explicit reference to the invariants.

[12] It should be emphasized that for truly large $p_\perp$, the Bloch-Nordsieck approximation itself is no longer valid since recoil must be taken into account. For such cases strong arguments have been presented to support a power-law damping. See, for example, S. Brodsky, and J. Gunion, Phys. Rev. Lett. 37, 402 (1976), and D. Soper, *ibid.* 38, 461 (1977). We would like to thank Professor Brodsky for a correspondence on this point.