Note on 't Hooft's Hamiltonian in two-dimensional quantum chromodynamics*

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The Hamiltonian of 't Hooft in two-dimensional quantum chromodynamics, defined as a natural Friedrichs extension, is shown to have a purely discrete spectrum with eigenvalues approaching infinity.

In studying two-dimensional quantum chromodynamics (QCD), 't Hooft has introduced a Hamiltonian we call H (in the limit of zero masses, H_0).[1] This Hamiltonian has been studied numerically and analytically (see, for example, Ref. 1 and Secs. 121 and 122 in Ref. 2). We do not pursue these directions, but rather use elementary Hilbert-space techniques to study the spectrum of the operators. The basic device is comparison with an exactly soluble Hamiltonian.

Our main result is that the spectrum of H^F and that of H_0^F (the Friedrichs extensions off natural initial domains) are purely discrete, with eigenvalues approaching infinity. We have not considered two obvious questions. Classify other extensions, assuming the given operators are not essentially self-adjoint. How fast do the eigenvalues approach their expected asymptotic values, $\pi^2 n$ (i.e., the spacing should approach π^2)? The simplicity of the results we have found is satisfying, and continued research is certain to provide rigorous results on further questions of interest to physicists.

We now define operators H and H_0 with domain $\mathfrak{D}$ in $L^2[0,1]$, and H_p with domain $\mathfrak{D}_p$ in $L^2[-\pi,\pi]$. Here $\mathfrak{D}$ is the set of infinitely differentiable functions with closed support in $(0,1)$, and $\mathfrak{D}_p$ is the set of infinitely differentiable functions of period 2π. That is,

$$\mathfrak{D}=\{f \mid f\in C^\infty,\ \text{closed support } f\subset(0,1)\}\,, \tag{1}$$

$$\mathfrak{D}_p=\{f \mid f\in C^\infty,\ f(x)=f(x+2\pi)\}\,. \tag{2}$$

Here the closed support of f is the closure of the set where $f\neq 0$. The operators are determined by their forms:

$$\langle\psi,H_0\phi\rangle=\tfrac{1}{2}\int_0^1 dx\int_0^1 dy\,\frac{[\bar\psi(x)-\bar\psi(y)][\phi(x)-\phi(y)]}{(x-y)^2}\,,\quad \psi,\phi\in\mathfrak{D} \tag{3}$$

$$\langle\psi,V\phi\rangle=\int_0^1 dx\,\bar\psi(x)\left(\frac{m_a{}^2}{x}+\frac{m_b{}^2}{1-x}\right)\phi(x)\,,\quad \psi,\phi\in\mathfrak{D} \tag{4}$$

$$\langle\psi,H\phi\rangle=\langle\psi,H_0\phi\rangle+\langle\psi,V\phi\rangle\,, \tag{5}$$

$$\langle\psi,H_p\phi\rangle=\tfrac{1}{2}\int_{-\pi}^{\pi} dx\int_{-\pi}^{\pi} dy\,\frac{[\bar\psi(x)-\bar\psi(y)][\phi(x)-\phi(y)]}{4\sin^2[\frac{1}{2}(x-y)]}\,,\quad \psi,\phi\in\mathfrak{D}_p\,. \tag{6}$$

It is important to us that all of these forms arise from symmetric operators, but we do not show this. We denote by H_0^F, H_p^F, and H^F the Friedrichs extensions[3] of our basic operators.

Theorem 1. H_p^F+1 has compact inverse. The functions e^{imx}, $m=0,\pm1,\pm2,\ldots$, are eigenfunctions of H_p^F with eigenvalue $\pi|m|$.

Proof of theorem 1. By the rotational symmetry of the kernel in (6), e^{imx} is necessarily an eigenfunction. It is an easy calculation (using contour integration) to show

$$\pi|m|=\frac{\langle e^{imx},H_p e^{imx}\rangle}{\langle e^{imx},e^{imx}\rangle}\,. \tag{7}$$

The compactness of $(H_p^F+1)^{-1}$ is a consequence of the spectral decomposition just exhibited.

We will characterize operators with compact resolvents, of the type interesting to us, by the following straightforward theorems.

Theorem 2. Let $A\geq 0$ be a self-adjoint operator. Then $(A+1)^{-1}$ is compact if and only if there is a basis of eigenvectors of A with eigenvalues λ_i satisfying

$$\lambda_i \xrightarrow[i\to\infty]{} \infty\,. \tag{8}$$

(We understand throughout that the λ_i are arranged in increasing order with each eigenvalue repeated a number of times equal to its degeneracy. All degeneracies are finite.)

Theorem 3. Let $A\geq 0$ be a self-adjoint operator. Let $\mathfrak{D}_{A^{1/2}}$ be the domain of $A^{1/2}$. Then

$(A+1)^{-1}$ is compact

$$\Longleftrightarrow \{\phi \in \mathfrak{D}_{A^{1/2}} \mid \langle A^{1/2}\phi, A^{1/2}\phi\rangle \le c\} \cap \{\|\phi\| \le 1\}$$

is relatively compact.

We proceed to study H (or H_0). We first embed $\mathfrak{D}$ in $\mathfrak{D}_p$ by associating to f in $\mathfrak{D}$ the function $s\circ f$ in $\mathfrak{D}_p$ defined by

$$s\circ f(x) = \begin{cases} f(x/\pi), & 0\le x\le \pi \\ f(-x/\pi), & -\pi \le x \le 0 . \end{cases} \tag{9}$$

We note that this is proportional to an isometric embedding

$$\langle s\circ f,\, s\circ f\rangle = c_1\langle f, f\rangle . \tag{10}$$

Most important to us is the following estimate.

Theorem 4. There is a constant $c_2>0$ (independent of m_a^2 and m_b^2) such that

$$\langle f, Hf\rangle \ge c_2\langle s\circ f,\, H_p s\circ f\rangle . \tag{11}$$

[Of course (11) holds with H_0 replacing H.]

We first use this theorem to prove our main result.

Theorem 5. $(H^F+1)^{-1}$ [also $(H_0^F+1)^{-1}$] is compact.

Proof of theorem 5. We use the characterization of theorem 3 and note that, from theorem 4, there follows

$$\{\phi\in\mathfrak{D}_{H^{1/2}} \mid \langle H^{1/2}\phi, H^{1/2}\phi\rangle \le c\}$$
$$\subset \{\phi \in \mathfrak{D}_{H_p^{1/2}} \mid \langle H_p^{1/2}\phi, H_p^{1/2}\phi\rangle \le c/c_2\} . \tag{12}$$

Our notation here suppresses the F, writing $H^{1/2}$ instead of $(H^F)^{1/2}$. The inclusion takes place via the identification provided by s. It is the use of the Friedrichs extension that has ensured that (12) follows from (11). (The inequality holds through the completion.)

Proof of theorem 4. We study $\langle s\circ f,\, H_p s\circ f\rangle$:

$$\langle s\circ f,\, H_p s\circ f\rangle = \tfrac{1}{2}\int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy \frac{|s\circ f(x) - s\circ f(y)|^2}{4\sin^2[\frac{1}{2}(x-y)]} \tag{13}$$

$$= \pi^2 \int_0^1 dx \int_0^1 dy \frac{|f(x)-f(y)|^2}{4\sin^2\pi[\frac{1}{2}(x-y)]} + \pi^2 \int_0^1 dx \int_0^1 dy \frac{|f(x)-f(y)|^2}{4\sin^2\pi[\frac{1}{2}(x+y)]} . \tag{14}$$

We note

$$\sin^2\pi[\tfrac{1}{2}(x+y)] \ge \sin^2\pi[\tfrac{1}{2}(x-y)] \quad \text{if} \quad 0\le x, y\le 1 \tag{15}$$

and

$$\sin^2\pi[\tfrac{1}{2}(x-y)] \ge c_3|x-y|^2 \quad \text{if} \quad 0\le x, y\le 1 , \tag{16}$$

for some $c_3>0$. The inequalities and (14) prove (11).

We finally note that we can easily show that the λ_n are asymptotically $\ge c_4|n|$, but the constant c_4 we thus find is less than π^2 (this follows from the mini-max principle and our comparison of Hamiltonians).

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