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# Phase transition in the $\sigma$ model at finite temperature*

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We study the phase transition through which the spontaneously broken symmetry of the $\sigma$ model is restored at finite temperature. The methods of nonrelativistic many-body theory, in which the equations of motion are approximated in a self-consistent manner, are applied to the $\sigma$ model in 1 time and $d$ space dimensions for $2 < d \leq 3$. We consider several different approximations of this type and discuss difficulties associated with their renormalization. The Hartree approximation predicts a second-order transition for all $d$, but breaks down at high temperatures when $d = 3$. The "modified Hartree approximation," a variant of Hartree theory which incorporates more of the effects of thermal fluctuations, predicts a first-order transition for all $d$. This result is shown to be an artifact of the approximation. The $\sigma$ model with $N$ fields [the O($N$) model] is studied in the limit of large $N$. For $2 < d < 3$ this model undergoes a second-order transition whose critical exponents are computed to $O(1/N)$. When $d = 3$, however, the large-$N$ approximation breaks down at high temperatures.

## I. INTRODUCTION

The concept of spontaneously broken symmetry has come to play an increasingly important role in relativistic field theory[1] and descriptions of high-density matter.[2] As Kirzhnits and Linde[3] first proposed, one might expect spontaneously broken symmetries in field theories to be restored at sufficiently high temperatures, analogous to the way in which nonrelativistic systems, such as superfluids, superconductors, and ferromagnets, in broken symmetry or "condensed" states, become symmetric or "normal" above a critical temperature, $T_c$. Such possible restoration of symmetry in the hot early universe has interesting cosmological consequences.[4] Finite-temperature phase transitions from broken symmetry to normal states also have importance for the behavior of hot high-density matter. For example, the pion-condensed state of neutron star matter should be destroyed at high enough temperature; an understanding of the condensation near the critical temperature is needed to assess the effects of pion condensation on the cooling of neutron stars.[5] Also of interest are possible finite-temperature phase transitions from normal to abnormal nuclear matter.[6]

Subsequent to the suggestion of Kirzhnits and Linde, several authors[7] have given descriptions of how finite-temperature field fluctuations can restore broken global as well as gauge symmetries; approximate calculations of critical temperatures have also been presented.[7] In this paper we further explore phase transitions in finite-temperature field theory, focusing on the transition expected in the $\sigma$ model. This model is particularly interesting in the study of high-density matter since it forms the basis of the description both of the pion-condensed state of neutron star matter and of abnormal nuclear matter. Limiting our considerations here only to states with no nucleons present, we shall study the extent to which various approximate calculational schemes reproduce the finite-temperature behavior physically expected in the model and the predictions they make about the order of the phase transition from broken symmetry to the normal state.

The $\sigma$ model is described by the Lagrangian density

$$L = -\tfrac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2] + \tfrac{1}{2}m_0^2(\sigma^2+\pi^2) - \tfrac{1}{4}\lambda_0(\sigma^2+\pi^2)^2 \,. \tag{1.1}$$

Here $\sigma$ and $\pi$ are real fields, $m_0^2$ is positive, and for the moment we take the $\pi$ field to have only one component; the extension to three or more $\pi$ fields is trivial. Because of the negative squared-mass term, $-m_0^2$, the complex field $\phi \equiv (\sigma + i\pi)/\sqrt{2}$ has a nonvanishing constant vacuum expectation value.

At finite temperature the behavior of the $\sigma$ model is described in terms of thermal expectation values $\langle X \rangle$, defined for any operator $X$ by

$$\langle X \rangle = \mathrm{Tr}(e^{-H/T}X)/\mathrm{Tr}(e^{-H/T}) \,, \tag{1.2}$$

where $H$ is the Hamiltonian derived from (1.1) and temperature units are chosen so as to make Boltzmann's constant unity. The trace is over all states with the internal quantum numbers of the vacuum. At sufficiently low temperatures we expect the broken-symmetry state with thermal expectation value $\langle\phi\rangle$ nonzero to persist. We choose the phases of the fields so that $\langle\phi\rangle = \langle\sigma\rangle/\sqrt{2}$ is real; $\langle\pi\rangle$ is thus zero at all temperatures. The effect of thermal fluctuations is to decrease $\langle\sigma\rangle$ monotoni-

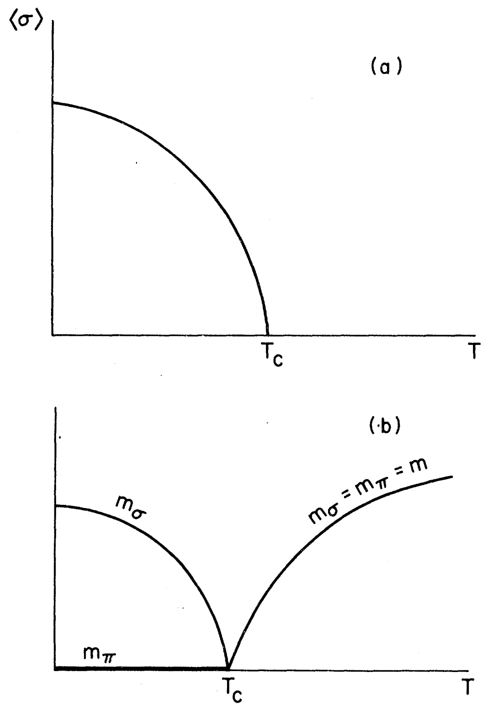


FIG. 1. (a) Qualitative behavior of $\langle\sigma\rangle$ as a function of temperature. $T_c$ is the system transition temperature. (b) Qualitative behavior of the masses $m_\sigma$ and $m_\pi$ below $T_c$, and the common mass $m_\sigma = m_\pi = m$ above $T_c$.

cally with temperature from its vacuum value. The critical temperature, $T_c$, is defined by the vanishing of $\langle\sigma\rangle$, as in Fig. 1(a); the continuous decrease of $\langle\sigma\rangle$ to zero at $T_c$ shown in that figure is characteristic of a second-order phase transition.[8] Defining the masses $m_\sigma$ and $m_\pi$ at any temperature as the poles at wave vector $\vec{k}=0$ of the thermal Green's functions for the $\sigma$ and $\pi$ excitations, respectively, we expect $m_\sigma$ and $m_\pi$ to behave as in Fig. 1(b). Below $T_c$, the $\pi$ field is the Goldstone boson[9] in the condensed state and one has $m_\pi = 0$. The mass $m_\sigma$ is nonzero below $T_c$. Since the symmetry between $\sigma$ and $\pi$ is restored at the critical temperature, $m_\sigma$ must go to zero at $T_c$, and $m_\sigma = m_\pi$ above $T_c$.

Several authors[7] have pointed out that the $\sigma$ model at finite temperature should behave qualitatively in this way. Such behavior is characteristic of a wide variety of familiar nonrelativistic systems (e.g., superfluids, superconductors, and ferromagnets) which undergo second-order phase transitions. Indeed, one can invoke the principle of "universality" (which, crudely stated, asserts that the asymptotic behavior of systems very close to a second-order phase transition is determined purely by the space dimensionality and by the symmetry of the order parameter and of the Hamiltonian and is independent of the detailed dynamics[10]) to conclude that the critical exponents characterizing the asymptotic behavior of the $\sigma$ model with $(N-1)$ $\pi$ fields are identical to those of the classical $N$-component Heisenberg ferromagnet. This correspondence is a fruitful one since a great deal is known about the critical behavior of classical magnets. In particular, Polyakov,[11] Migdal,[12] and Brézin and Zinn-Justin[13] have recently shown that for $N>2$ the nonlinear $\sigma$ model [i.e., $\lambda$, $m_0^2 \to \infty$ with $m_0^2/\lambda$ remaining finite in the notation of Eq. (1.1)] becomes asymptotically free in two dimensions. Brézin and Zinn-Justin[13] show that in $(2+\epsilon)$ dimensions with $\epsilon$ small, this model, whose critical properties are identical to those of the $N$-component Heisenberg ferromagnet, has the qualitative behavior shown in Fig. 1 with $T_c$ given (in units of $m_0^2/\lambda$) by $T_c = 2\pi\epsilon/(N-2) + O(\epsilon^2)$. They compute the critical exponents for the nonlinear $\sigma$ model to $O(\epsilon^2)$; the results should apply equally well to the linear $\sigma$ model of Eq. (1.1).

Our goal in this paper is to find simple approximate treatments of the linear $\sigma$ model that reproduce the qualitative behavior of Fig. 1 and agree as closely as possible with the exact results available in $(2+\epsilon)$ dimensions. Such approximate treatments can form a useful basis for future calculations of spontaneous symmetry breaking in high-density matter at finite temperature; the emphasis in such calculations is on having an approximately correct description over a wide range of temperatures, rather than an exact description of the critical region.

The approach we follow is to generate self-consistent approximations to the equation of motion for $\langle\phi\rangle$ and the Green's function equations derived from (1.1). Schematically written, these equations are $G_0^{-1}\langle\phi\rangle = \lambda_0\langle\phi^\dagger\phi\phi\rangle$ and $G^{-1} = G_0^{-1} - \Sigma$, where $G_0^{-1} = \Box^2 + m_0^2$. We approximate $\langle\phi^\dagger\phi\phi\rangle$ and the self-energy $\Sigma$ in terms of $\langle\phi\rangle$ and the complete Green's function, $G$, and then solve the resulting equations for $G$ and $\langle\phi\rangle$, thus obtaining expressions for $m_\sigma$, $m_\pi$, and $\langle\sigma\rangle$ as functions of temperature. One variant of this scheme was used by Kirzhnits and Linde[14] in their recent analysis of the finite-temperature phase transition in the $\sigma$ model. A similar approach was used by Chang,[15] who showed that one can induce a phase transition in the $\sigma$ model at zero temperature by varying the coupling constant. Techniques for generating self-consistent approximations directly from the equations of motion have been used extensively in nonrelativistic many-body theory,[16] notably in models of superfluid helium and superconductors.

Such self-consistent approximation methods are particularly useful in problems such as the present one, where there is no small parameter. Simple approximations to $\langle \phi^\dagger \phi\phi\rangle$ and $\Sigma$ correspond to the summation of whole classes of diagrams written in terms of the bare propagator $G_0$. Moreover, by choosing these approximations in accordance with certain prescriptions, discussed in detail in the following section, one is guaranteed that the self-consistent solutions automatically obey the Goldstone theorem or preserve the conservation laws (i.e., the Ward identities).

In this paper we examine a number of such self-consistent approximations for the $\sigma$ model. The natural starting point is the familiar semiclassical or "tree" approximation.[17] This scheme is inadequate in that thermal fluctuations are completely neglected; symmetry breaking persists to arbitrarily high temperatures. Next we examine Hartree theory,[18] an approximation which incorporates fluctuation effects in the simplest possible way. Because of its (albeit crude) treatment of fluctuations, Hartree theory does predict a finite transition temperature and a second-order phase transition. Unfortunately the structure of this theory is trivial below $T_c$; both the $\sigma$ and $\pi$ fields are free and massless.

Hartree theory provides the first illustration of a general difficulty encountered in formulating self-consistent approximations in relativistic systems: The ultraviolet properties of the theory can give rise to anomalies in the self-consistently computed Green's functions. In particular, we find that in the Hartree scheme in four space-time dimensions it is impossible to define the mass of the $\sigma$ and $\pi$ fields self-consistently at temperatures well above $T_c$. This failure is readily seen to be associated with the ultraviolet behavior of the theory; it is (both in Hartree theory and more generally) purely technical and does not, we hope, affect the critical (i.e., infrared) properties with which we are primarily concerned.

In order to circumvent this difficulty we consider the $\sigma$ model in 1 time and $d$ space dimensions, where $d$ is taken to be a continuous parameter less than or equal to 3.[19] So long as $d<3$ the theory is superrenormalizable; once mass renormalization is performed it is ultraviolet convergent, in contrast to the situation in three dimensions where infinite wave-function and coupling-constant renormalizations are required as well. One finds, correspondingly, that no high-temperature breakdown of Hartree theory occurs when $d<3$. The mass of the particles is well defined at all temperatures and the theory predicts a second-order phase transition. Except for the vanishing of $m_\sigma$ below $T_c$, the behavior of the approximation agrees with that of Fig. 1.

In an attempt to obtain a finite $m_\sigma$ below $T_c$ we next examine a slightly more complex variant of Hartree theory, incorporating more fluctuation effects. We call this the "modified Hartree approximation"; aside from trivial numerical factors, it is the approximation of Kirzhnits and Linde.[14] We find that when $d<3$ this scheme predicts a *first*-order, rather than a second-order, phase transition, that is, $\langle\sigma\rangle$ jumps discontinuously to zero at $T_c$. (Although Kirzhnits and Linde state that the transition emerging from their approximation is of second order, more detailed analysis indicates that their equations actually predict a first-order transition.)

When $d=3$ we encounter a serious difficulty in the modified Hartree theory: Removal of divergences by conventional renormalizations at zero temperature does not leave the self-consistent calculation divergence-free at finite temperature. We are forced to relax the self-consistency requirement of the scheme somewhat in order to obtain Green's functions free of divergences. The resulting approximation predicts a first-order transition, in agreement with the $d<3$ prediction.

Mean field calculations that predict first-order phase transitions are not uncommon in nonrelativistic theories. In order to gain more insight into the modified Hartree theory prediction of a first-order transition, we briefly review two such calculations: the treatment of a type-II superconductor including electromagnetic field fluctuations by Halperin, Lubensky, and Ma[20] (HLM), and the Bogoliubov approximation[21] for the interacting Bose gas. The first-order transitions predicted by these calculations arise in a manner mathematically identical to that of the modified Hartree approximation. Because the order parameter is coupled to a gauge field in the superconductor, one can argue that the prediction of a first-order transition in this system is quite plausible. However, in the Bose gas, which is very closely analogous to the $\sigma$ model, such a prediction is an artifact of the approximation used and is certainly wrong.

We next try to attain a more precise understanding of the critical behavior of the $\sigma$ model and to pinpoint the failure of the modified Hartree approximation by considering the O($N$) model, i.e., the $\sigma$ model with $(N-1)$ $\pi$ fields.[22] In the limit $N\to\infty$ the behavior of the O($N$) model for $d<3$ is exactly calculable.[23] We exhibit the large-$N$ solution; the O($N$) symmetry is spontaneously broken at low temperatures and is restored at a critical temperature via a second-order phase transition. The critical exponents are identical to those obtained for the classical O($N$) model in the large-$N$ limit[24] and are consistent with the exact results[13] available in $2+\epsilon$ dimensions.

We then apply the modified Hartree approximation to the O($N$) model for $d<3$. In the $N\to\infty$ limit this approximation agrees qualitatively with the exact large-$N$ results; in particular, the transition is predicted to be of second order. For finite $N$, however, the deficiencies of the modified Hartree scheme become clear. We first compute the $O(1/N)$ corrections to the exact $N=\infty$ results.[25] As expected, the only effect of the $1/N$ terms is to modify the critical exponents. The order of the transition remains unchanged. In contrast, the modified Hartree approximation is found to predict a first-order transition for *any* finite value of $N$. We infer that this prediction is indeed an artifact of the approximation.

Abbott, Kang, and Schnitzer[26] have argued that when $d=3$ there is no spontaneous symmetry breaking at $T=0$ in the large-$N$ limit. Turning to finite temperature we find that for sufficiently small $T$ the large-$N$ theory contains no anomalies and, as one would expect, exhibits no symmetry breaking. For $T$ sufficiently large, however, the large-$N$ limit does develop anomalies; the common mass of the $\sigma$ and $\pi$ excitations becomes complex. Thus the large-$N$ approximation, the most satisfactory of our simple mean field approximations to the critical behavior of the $\sigma$ model when $d<3$, fails at high temperatures when $d=3$.

The organization of this paper is as follows: In Sec. II we briefly review the self-consistent approximation methods of many-body theory, and examine the tree, Hartree, and modified Hartree approximations to the $\sigma$ model for $d<3$ and $d=3$. Section III is a brief discussion of the HLM treatment of a superconductor in a fluctuating electromagnetic field and the Bogoliubov approximation for the interacting Bose gas. In Sec. IV we examine the O($N$) model at finite temperature; we study the large-$N$ limit for $d<3$ and $d=3$ and include $O(1/N)$ corrections for $d<3$.

## II. MEAN FIELD THEORY

### A. Basic definitions

In terms of the complex field $\phi=(\sigma+i\pi)/\sqrt{2}$ the Lagrangian density (1.1) becomes

$$L=-\partial_\mu\phi^\dagger\partial^\mu\phi+m_0^2\phi^\dagger\phi-\lambda_0(\phi^\dagger\phi)^2\ . \tag{2.1}$$

It is convenient to introduce the shifted field $\tilde\phi(x)$, defined by

$$\phi(x)=\tilde\phi(x)+\phi^c(x)\ , \tag{2.2}$$

where $\phi^c$, the order parameter, is the expectation value of $\phi$; thus $\tilde\phi$ has vanishing expectation value at all temperatures. In the presence of a source term $(-h\phi^\dagger-h^*\phi)$ in the Lagrangian, the equation of motion for $\phi^c$ becomes

$$\Box^2\phi^c=-m_0^2\phi^c+\eta+h\ , \tag{2.3}$$

where

$$\eta=2\lambda_0\langle\phi^\dagger\phi\phi\rangle\ . \tag{2.4}$$

For nonzero $h$ the quantity $\eta(x)$ can be regarded as a functional of $\phi^c(x')$. Note that the functional $(\eta-m_0^2\phi^c)$ is simply the functional derivative with respect to $\phi^{c*}$ of the effective potential, $V(\{\phi^c\})$, frequently employed in the study of spontaneous symmetry breaking.[27] In any state with $\phi^c(x)$ uniform, the equation of motion (2.3) implies that

$$\eta/\phi^c=m_0^2 \tag{2.5}$$

when $h=0$. We shall always assume that under such circumstances the phases of the fields are chosen so that $\phi^c$ is real, i.e.,

$$\langle\sigma\rangle=\sqrt{2}\,\phi^c,\quad\langle\pi\rangle=0\ . \tag{2.6}$$

### B. Φ-derivable and gapless approximations

As Kirzhnits and Linde[14] proposed, one can study the phase transition of this model within the framework of a simple mean field theory. Starting from the equation of motion (2.3), one approximates the quantity $\eta$ in terms of lower-order correlation functions. This procedure has been used extensively in the study of superfluids, where the various approximations that have been studied are typically classified as either "Φ-derivable" or "gapless."[16]

The "Φ-derivable" approximations are formulated in terms of the quantity Φ, a functional of $\phi^c(x)$, $\phi^{c*}(x)$, and of the (time-ordered) temperature Green's function matrix $\underline{G}(xx')$, defined by

$$\underline{G}(xx')=-i\begin{pmatrix}\langle(\tilde\phi(x)\tilde\phi^\dagger(x'))_+\rangle & \langle(\tilde\phi(x)\tilde\phi(x'))_+\rangle\\ \langle(\tilde\phi^\dagger(x)\tilde\phi^\dagger(x'))_+\rangle & \langle(\tilde\phi^\dagger(x)\tilde\phi(x'))_+\rangle\end{pmatrix}$$
$$=\begin{pmatrix}\delta\phi^c(x)/\delta h(x') & \delta\phi^c(x)/\delta h^*(x')\\ \delta\phi^{c*}(x)/\delta h(x') & \delta\phi^{c*}(x)/\delta h^*(x')\end{pmatrix}. \tag{2.7}$$

Given a particular approximation for Φ, one determines $\eta$ and the self-energy matrix $\underline{\Sigma}$ through the equations

$$\eta(x)=\frac{i}{2}\left(\frac{\delta\Phi}{\delta\phi^{c*}(x)}\right)_{\phi^c,\underline{G}}\ , \tag{2.8}$$

$$\Sigma_{\alpha\beta}(xx')=\left(\frac{\delta\Phi}{\delta G_{\beta\alpha}(xx')}\right)_{\phi^c,\phi^{c*}}\ . \tag{2.9}$$

The relations

$$(G^{-1})_{\alpha\beta}=(G_0^{-1}+m_0^2)\delta_{\alpha\beta}-\Sigma_{\alpha\beta} \tag{2.10}$$

are then solved along with Eq. (2.3) to determine $\phi^c$ and $\underline{G}$ self-consistently. [Here $G_0(\vec{p}z)$ represents the free, massless propagator, $(z^2-\vec{p}^2)^{-1}$]. The $\Phi$-derivable approximations are extremely useful in many-body theory since the correlation functions derived from them preserve the conservation laws.[28] Moreover, having determined $\phi^c$ and $\underline{G}$ one can uniquely construct the effective potential, $V(\{\phi^c\})$.[28] With approximations which are not $\Phi$-derivable this is often impossible; typically one can construct two or more different effective potentials from $\phi^c$ and $\underline{G}$. The absence of such ambiguity is a great virtue of the $\Phi$-derivable approximations. Unfortunately, most of them violate the Goldstone,[9] or Hugenholtz-Pines,[29] theorem in the state of broken symmetry.

The "gapless" approximations are explicitly constructed to satisfy this theorem.[16] One starts with an approximation to $\eta$ as a functional of $\phi^c$ and $\underline{G}$. The self-energies are then computed as

$$\underline{\Sigma}(xx') = \begin{pmatrix} (\delta\eta(x)/\delta\phi^c(x'))_{\phi^{c*}} & (\delta\eta(x)/\delta\phi^{c*}(x'))_{\phi^c} \\ (\delta\eta^*(x)/\delta\phi^c(x'))_{\phi^{c*}} & (\delta\eta^*(x)/\delta\phi^{c*}(x'))_{\phi^c} \end{pmatrix}; \tag{2.11}$$

the functional dependence of $\underline{G}$ on $\phi^c$ must be taken into account in performing this functional differentiation. Equations (2.3) and (2.10) are then solved self-consistently; the resulting Green's functions satisfy the Goldstone theorem. To see this we note that under a uniform gauge transformation $\phi(x)\to e^{i\Lambda}\phi(x)$ one has $\phi^c(x)\to e^{i\Lambda}\phi^c(x)$ and $\eta(x)\to e^{i\Lambda}\eta(x)$. Thus when (2.11) is obeyed, the $k=0$ Fourier components of $\underline{\Sigma}$ satisfy

$$\Sigma_{11}(k=0)\phi^c - \Sigma_{12}(k=0)\phi^{c*} = \eta \ . \tag{2.12}$$

For $\phi^c$ uniform and real we also have

$$G_\pi = G_{11} - G_{12}, \quad G_\sigma = G_{11} + G_{12} \tag{2.13a}$$

since under these conditions $\langle\sigma\pi\rangle = 0$; also,

$$\begin{aligned} G_\sigma^{-1} &= G_0^{-1} + m_0^2 - \Sigma_{11} - \Sigma_{12} \ , \\ G_\pi^{-1} &= G_0^{-1} + m_0^2 - \Sigma_{11} + \Sigma_{12} \ . \end{aligned} \tag{2.13b}$$

Using (2.3), (2.12), and (2.13) we see that at $k=0$, $G_\pi^{-1}=0$ and thus, as expected, the $\pi$ field is the Goldstone boson of the theory. (Note though that at finite temperature, where one has a preferred frame, $G_\pi^{-1}$ does not vanish for general $k$ with $k_\mu k^\mu = 0$.) Also, $G_\sigma^{-1}(k=0) = -2\Sigma_{12}(0)$.

### C. Tree approximation

The familiar semiclassical (or tree) approximation is a "gapless" approximation which corresponds to taking

$$\eta = 2\lambda_0|\phi^c|^2\phi^c \ . \tag{2.14}$$

The self-energies are then given, according to (2.11), by

$$\underline{\Sigma}(xx') = 2\lambda_0\delta(x-x')\begin{pmatrix} 2|\phi^c|^2 & (\phi^c)^2 \\ (\phi^{c*})^2 & 2|\phi^c|^2 \end{pmatrix}. \tag{2.15}$$

From (2.5), (2.14), and (2.15) we immediately obtain

$$\langle\sigma\rangle^2 = m_0^2/\lambda_0 \ , \tag{2.16a}$$

$$G_\sigma^{-1} = G_0^{-1} - 2m_0^2 \ , \tag{2.16b}$$

$$G_\pi^{-1} = G_0^{-1} \ . \tag{2.16c}$$

We see from (2.16a) that symmetry breaking persists to arbitrarily high temperatures in the tree approximation. In order to observe the restoration of symmetry at finite temperature one must include effects of fluctuations. Any venture beyond the semiclassical treatment requires renormalizations. Since the renormalization counterterms in the Lagrangian are temperature independent, carrying out the various renormalizations to remove the infinites at any one temperature fixes the values of these counterterms uniquely. ($T=0$ is a particularly convenient temperature at which to renormalize.) Kislinger and Morley[30] have recently argued how the renormalized theory which results is free from divergences at *all* temperatures. It is not true, however, that all approximations preserve this feature. Indeed we shall exhibit a very simple self-consistent approximation which cannot be rendered free of infinities at all temperatures by the standard renormalizations. Since our prime goal is to acquire some feeling for the critical (infrared) properties of the $\sigma$ model, we shall temporarily avoid this complication by working in $d$ space dimensions, where $d$ is a continuous parameter less than 3. For any fixed $d<3$ the theory is superrenormalizable; all infinities at $T=0$ can be removed by mass renormalization alone and, at least for the elementary approximations we shall consider, no new divergences occur at finite temperature. We shall later return to the difficulties that arise when $d=3$ and coupling-constant renormalization is required.

### D. The Hartree approximation

Hartree theory, the simplest self-consistent scheme beyond semiclassical theory, amounts to approximating $\eta$ as a functional of $\phi^c$ and $\underline{G}$ by

$$\eta = 2\lambda_0[\,|\phi^c(x)|^2 + \langle\tilde{\phi}^\dagger(x)\tilde{\phi}(x)\rangle]\phi^c(x) \ . \tag{2.17}$$

This approximation can be "$\Phi$-derived" from the functional

$$\Phi = \frac{\lambda_0}{2i}\int dx[i\,\mathrm{Tr}\underline{G}(xx) + 2|\phi^c(x)|^2]^2 \tag{2.18}$$

and the general prescription (2.8) and (2.9); then

$$[-m_0^2 + 2\lambda_0|\phi^c|^2 + i\lambda_0 \,\mathrm{Tr}\underline{G}(xx)]\phi^c = 0 \qquad (2.19a)$$

and

$$\Sigma_{\alpha\beta}(xx') = \delta(x - x')\delta_{\alpha\beta} \times \lambda_0[2|\phi^c(x)|^2 + i\,\mathrm{Tr}\underline{G}(xx)] . \qquad (2.19b)$$

In the broken-symmetry state

$$\Sigma_{\alpha\beta} = \delta_{\alpha\beta}\delta(x - x')m_0^2 ;$$

thus $G_\sigma$ and $G_\pi$ are free and massless. Since $\mathrm{Tr}\underline{G} = 2G_0$ and

$$G_0(xx) = iT\sum_n \int \frac{d^dk}{(2\pi)^d} \frac{1}{(i\omega_n)^2 - \vec{k}^2} \qquad (2.20)$$

(where $\omega_n = 2\pi nT$, $n = 0, \pm 1, \pm 2, \ldots$), (2.19a) determines $\langle\sigma\rangle$ as a function of temperature:

$$\langle\sigma\rangle^2 = m_0^2/\lambda_0 - 2\int \frac{d^dk}{(2\pi)^d} \frac{1}{k}\left(\frac{1}{e^{k/T} - 1} + \frac{1}{2}\right). \qquad (2.21)$$

This equation shows how the thermal fluctuations of the field decrease $\langle\sigma\rangle^2$ and act toward restoring the symmetry. A trivial mass renormalization removes the (temperature-independent) infinity, no coupling-constant renormalization is required for $d < 3$, and (2.21) becomes

$$\langle\sigma\rangle^2 = \mu^2/\lambda_0 - 2a(d)T^{d-1} , \qquad (2.22)$$

where $\mu^2$ is the renormalized value of $m_0^2$ and $a(d)$ is the positive constant

$$a(d) = \int \frac{d^dx}{(2\pi)^d} \frac{1}{x(e^x - 1)} . \qquad (2.23)$$

The expectation value $\langle\sigma\rangle$ decreases monotonically with temperature until a critical temperature, $T_c$, defined by

$$a(d)T_c^{d-1} = \mu^2/2\lambda_0 , \qquad (2.24)$$

at which $\langle\sigma\rangle$ vanishes; $\langle\sigma\rangle^2$ decreases linearly with temperature as $T$ approaches $T_c$ from below. The phase transition is of second order and the critical exponent $\beta$ [the power of $(T_c - T)$ with which $\langle\sigma\rangle$ vanishes as $T \to T_c$] equals $\frac{1}{2}$.

Note that for $d \leq 2$ the contribution of the thermal fluctuations of the field in (2.22) is infrared divergent since the constant $a(d)$ is then infinite. Such behavior is a specific realization of the general theorems[31] that forbid the spontaneous breaking of a continuous symmetry above zero temperature when $d \leq 2$. We restrict ourselves henceforth to $d > 2$.

Above $T_c$, (2.22) has no solution and (2.3) implies that $\phi^c$ vanishes identically; still $G_\sigma = G_\pi = G$, where

$$G^{-1} = G_0^{-1} - m^2 , \qquad (2.25)$$

and from (2.19b),

$$m^2 = -m_0^2 + 2i\lambda_0 G(xx) . \qquad (2.26)$$

Evaluating $G(xx)$ as in (2.21) we have

$$m^2 = -m_0^2 + 2\lambda_0\left(\int \frac{d^dk}{(2\pi)^d} \frac{1}{2\omega_k} + I^{(d)}(m)\right), \qquad (2.27a)$$

where

$$\omega_k = (\vec{k}^2 + m^2)^{1/2} \qquad (2.27b)$$

and

$$I^{(d)}(m) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{\omega_k(e^{\omega_k/T} - 1)} . \qquad (2.27c)$$

Equation (2.27a) is the $d$-dimensional version of Dolan and Jackiw's[32] "gap equation," that describes the behavior of the O($N$) model above $T_c$ in the $N \to \infty$ limit. It is not surprising that this equation emerges from the Hartree approximation: it is well known that the large-$N$ approximation is identical to Hartree theory above the transition temperature.

After one has performed the same mass renormalization required in (2.21), Eq. (2.27a) becomes

$$m^2 + \lambda_0 b(d)|m|^{d-1} = 2\lambda_0 I^{(d)}(m) - \mu^2 , \qquad (2.28a)$$

where

$$b(d) = \frac{1}{d-1}\int \frac{d^dx}{(2\pi)^d} \frac{1}{(x^2+1)^{3/2}} . \qquad (2.28b)$$

It is trivial to verify that for all $T > T_c$, Eq. (2.28a) has a positive solution, $m = m(T)$, which increases monotonically with $T$ and vanishes at $T = T_c$. Since for small $m$ we have

$$I^{(d)}(m) = a(d)T^{d-1} - T|m|^{d-2}c(d) + \cdots , \qquad (2.29a)$$

where

$$c(d) = \frac{2}{d-2}\int \frac{d^dx}{(2\pi)^d} \frac{1}{(x^2+1)^2} , \qquad (2.29b)$$

it follows that

$$[m(T)]^2 \sim (T - T_c)^{2/(d-2)} \equiv (T - T_c)^\gamma \qquad (2.30)$$

as $T$ approaches $T_c$ from above. Hence the critical exponent $\gamma$ that describes the behavior of $m^2$ above $T_c$ is simply $2/(d-2)$ in the Hartree approximation.[33]

### E. The modified Hartree approximation: $2 < d < 3$

Hartree theory provides a simple picture of how the thermal fluctuations of the fields in the $\sigma$ model lead to a second-order phase transition. The theory is unphysical, though, in that the spectrum of $\sigma$-like excitations, described by $G_\sigma$, is that of a free massless particle for all $T < T_c$. By contrast, the tree approximation predicts a $\sigma$ mass,

$(2m_0^2)^{1/2}$, independent of $T$, and no phase transition. An interesting question is whether one can construct simple approximations that describe the phase transition while incorporating a more realistic description of the spectrum of the fields below $T_c$. Let us therefore consider a slightly more complex variant of the Hartree approximation. Aside from insignificant numerical factors, it is the mean field approximation suggested by Kirzhnits and Linde.[14] This scheme, which we shall refer to as the "modified Hartree approximation," is a hybrid between the $\Phi$-derivable and gapless methods.

We start with $\Phi$ given by (2.18); $\eta$, computed through Eq. (2.8), has the same form (2.17) as in Hartree theory. At this point, however, we deviate from the $\Phi$-derivable procedure and compute $\Sigma_{\alpha\beta}$ from (2.11), holding not only $\phi^c$ but also $\underline{G}$ constant during the functional differentiations to obtain

$$\underline{\Sigma}(xx') = 2\lambda_0\delta(x-x')\begin{pmatrix} 2|\phi^c|^2 + iG_{11}(xx) & (\phi^c)^2 \\ (\phi^{c*})^2 & 2|\phi^c|^2 + iG_{11}(xx)\end{pmatrix} . \tag{2.31}$$

This prescription differs from the gapless scheme where $\Sigma$ is computed as the *total* functional derivative of $\eta$ with respect to $\phi^c$. Fortunately the Goldstone theorem is preserved[34] despite this heresy. When $\phi^c$ vanishes (above $T_c$) this approximation is identical to self-consistent Hartree theory, as is clear from Eqs. (2.19b) and (2.31). Below $T_c$, however, the structure is that of the tree approximation (2.15) plus the fluctuation term included in Hartree theory; choosing $\phi^c$ real we find

$$G_\sigma^{-1} = G_0^{-1} - m_\sigma^2 , \tag{2.32a}$$

$$G_\pi^{-1} = G_0^{-1} - m_\pi^2 , \tag{2.32b}$$

where

$$m_\sigma^2 = 3\lambda_0\langle\sigma\rangle^2 + \lambda_0 i(G_\sigma + G_\pi) - m_0^2 , \tag{2.33a}$$

$$m_\pi^2 = \lambda_0\langle\sigma\rangle^2 + \lambda_0 i(G_\sigma + G_\pi) - m_0^2 = \eta/\phi^c - m_0^2 . \tag{2.33b}$$

Equation (2.5) implies that $m_\pi^2 = 0$ in the state of broken symmetry and

$$m_\sigma^2 = 2\lambda_0\langle\sigma\rangle^2 . \tag{2.34}$$

Thus $m_\sigma$ is determined self-consistently from

$$m_\sigma^2 + 2\lambda_0 i(G_\sigma + G_\pi) = 2m_0^2 . \tag{2.35}$$

Evaluating $G_\sigma$ and $G_\pi$ as in (2.20) and (2.27a) and carrying out a single mass renormalization we obtain

$$m_\sigma^2 + 2\lambda_0[I^{(d)}(m_\sigma) - I^{(d)}(0)] - \lambda_0 b(d)|m_\sigma|^{d-1} = 2\mu^2 - 4\lambda_0 a(d)T^{d-1} \tag{2.36}$$

for $T < T_c$, where $T_c$, defined by the vanishing of $m_\sigma$, is again given by (2.24).

Now, however, a qualitatively new feature emerges: $m_\sigma$ does *not* decrease continuously to zero as $T$ approaches $T_c$ from below. To see this we note that $|m_\sigma|^{d-2}$, which occurs in the expansion (2.29a) of $I^{(d)}(m_\sigma)$ for small $m_\sigma$, is the dominant term on the left side of Eq. (2.36) as $m_\sigma \to 0$. Thus

$$c(d)|m_\sigma|^{d-2} = -2(d-1)T_c^{d-3}a(d)(T_c - T) \tag{2.37}$$

as $T$ approaches $T_c$ from below. The left and right sides of this equation have opposite sign; hence there is no solution below $T_c$ of Eq. (2.36) for small, real $m_\sigma$.

The true behavior of $m_\sigma$ near $T_c$ in the modified Hartree approximation can be qualitatively understood by studying the effective potential, $V(\langle\sigma\rangle)$, whose derivative with respect to $\langle\sigma\rangle^2$ is simply $(\eta/\phi^c - m_0^2)/2$. In the state of equilibrium $V(\langle\sigma\rangle)$ is a minimum. Integrating (2.33b) with respect to $\langle\sigma\rangle^2$ using (2.34) we find for small $|\langle\sigma\rangle|$

$$V(\langle\sigma\rangle) = V(0) + \lambda_0 a(d)(T^{d-1} - T_c^{d-1})\langle\sigma\rangle^2 - A|\langle\sigma\rangle|^d - B|\langle\sigma\rangle|^{d+1} + \lambda_0\langle\sigma\rangle^4/4 , \tag{2.38a}$$

where

$$A = 2^{(d-2)/2}Tc(d)\lambda_0^{d/2}/d , \quad B = 2^{(d-3)/2}b(d)\lambda_0^{(d+1)/2}/(d+1) . \tag{2.38b}$$

A schematic plot of this function is shown in Fig. 2. It is clear from the figure that $T_c$ is not the true transition temperature of the system. As expected, the absolute minimum of the effective potential occurs away from the origin for all temperatures less than $T_c$; however, Fig. 2 shows that, owing to the presence of the $(-A|\langle\sigma\rangle|^d)$ term, this behavior persists even as $T$ becomes greater than $T_c$. Only at a higher temperature, $T_c^*$, does the minimum at the origin become the absolute minimum of $V$. $T_c^*$ is therefore the *true* transition temperature of the system. Furthermore, $\sigma^*$, the value of $\langle\sigma\rangle$ which minimizes $V$ just below $T = T_c^*$, is nonzero; $\langle\sigma\rangle$ jumps discontinuously to zero at the transition temperature. Thus the modified Hartree approximation predicts a *first*-order, rather than a second-order, phase transition.

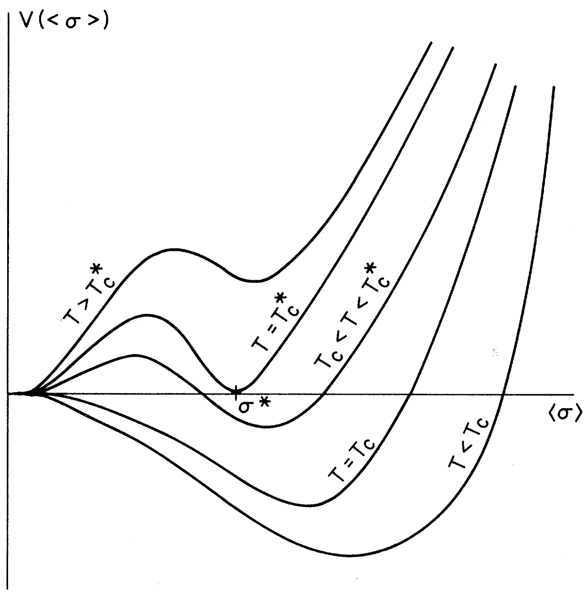


FIG. 2. Effective potential, $V(\langle\sigma\rangle)$, as a function of $\langle\sigma\rangle$ for the modified Hartree approximation with $2<d<3$ and several different values of $T$. From bottom to top these curves correspond to $T<T_c$, $T=T_c$, $T_c<T<T_c^*$, $T=T_c^*$, and $T>T_c^*$, respectively. $T_c^*$ is the true transition temperature of the system, and $\sigma^*$ is the magnitude of the discontinuity in the equilibrium value of $\langle\sigma\rangle$ at $T_c^*$.

For all $T>T_c^*$, the gap equation (2.28a) still holds and has, as we have already noted, solutions $m_\pi = m_\sigma = m$. The qualitative behavior of $m_\pi$, $m_\sigma$, $\langle\sigma\rangle$, and $m$, as predicted by this approximation, is summarized in Fig. 3.

### F. Hartree approximation in three dimensions

Before commenting further on the modified Hartree results let us consider the case $d=3$, where coupling-constant renormalization must be performed. We first examine the effects of this extra renormalization on the Hartree approximation. Above $T_c$ the unrenormalized gap equation (2.27a) takes the form

$$m^2 = -m_0{}^2 + 2\lambda_0 I^{(3)}(m) + \frac{\lambda_0 m^2}{8\pi^2}\left(1+\ln\frac{m^2}{4\Lambda^2}\right) + \frac{\lambda_0\Lambda^2}{4\pi^2}\ , \tag{2.39}$$

where $\Lambda$ is a high-momentum cutoff.

Defining the renormalized coupling constant, $\lambda$, and mass, $\mu^2$, by

$$1/\lambda_0 = 1/\lambda - [\ln(4\Lambda^2/\mu^2) - 1]/8\pi^2 \tag{2.40a}$$

and

$$m_0{}^2/\lambda_0 = \mu^2/\lambda + \Lambda^2/4\pi^2\ , \tag{2.40b}$$

we are led to a gap equation free of infinities:

$$m^2[1-\lambda\ln(m^2/\mu^2)/8\pi^2] = 2\lambda I^{(3)}(m) - \mu^2\ . \tag{2.41}$$

It is easy to verify that coupling-constant renormalization has no effect on Hartree theory below $T_c$, where the theory remains trivial (both the $\sigma$ and $\pi$ excitations are free and massless). From (2.21) and (2.40b) the condensate density varies with temperature according to

$$\langle\sigma\rangle^2 = \mu^2/\lambda - 2a(3)T^2\ , \tag{2.42a}$$

and $T_c$ is defined by

$$T_c{}^2 = \mu^2/2\lambda a(3)\ . \tag{2.42b}$$

Hartree theory in three dimensions presents the first example of an explicit breakdown of our approximation methods. To see this, note that the right side of (2.41) is positive at $m^2=0$ for $T>T_c$ and decreases monotonically with $m^2$, approaching $(-\mu^2)$ as $m^2\to\infty$; it also increases monotonically with $T$. The left side is zero at $m^2=0$, but, because of the $\ln(m^2/\mu^2)$ term, attains a maximum at $m^2=\mu^2 e^{-1+8\pi^2/\lambda}$ and goes to $-\infty$ for large $m^2$; thus for sufficiently large $T$ Eq. (2.41) has no real solution. By contrast, the left side of the gap equation in Hartree theory for $2<d<3$ does not contain the $\ln(m^2/\mu^2)$ and increases monotonically with $m$, thereby ensuring that the equation has a solution for all $T$. The dominant behavior of the $\lambda\ln(m^2/\mu^2)$ for $d=3$ signals a breakdown of the approximation and the need to include higher-order correlations. This phenomenon is an interesting illustration of the interplay in relativistic theories of the infrared behavior, which determines critical properties, and ultraviolet effects.

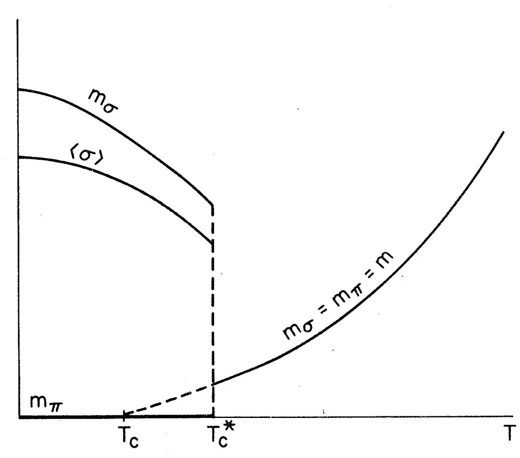


FIG. 3. Qualitative behavior of $m_\pi$, $m_\sigma$, and $\langle\sigma\rangle$ as functions of temperature. $T_c^*$ is the true transition temperature of the system. Above $T_c$, $m_\sigma$ and $m_\pi$ are identical.

### G. Modified Hartree approximation in three dimensions—renormalization difficulties

The gapless and $\Phi$-derivable approximations are not simple expansions in the coupling constant, but rather involve summing selected subsets of diagrams in a self-consistent manner. In general, however, when one must include a coupling-constant renormalization, such schemes are not consistently renormalizable. This difficulty is illustrated by the modified Hartree approximation in three dimensions, where the field equation for $T < T_c$ becomes

$$m_0^2/\lambda_0 = \langle\sigma\rangle^2 + I^{(3)}(m_\sigma) + I^{(3)}(0) + \Lambda^2/4\pi^2 + m_\sigma^2[1+\ln(m_\sigma^2/4\Lambda^2)]/16\pi^2 . \quad (2.43)$$

Since Eq. (2.34) implies that the mass $m_\sigma$ is temperature dependent, the divergent term $m_\sigma^2 \ln\Lambda^2$ in (2.43) cannot be removed by simple temperature-independent coupling-constant and mass renormalizations.

One way to circumvent these difficulties is to relax the self-consistency of the approximation so as to allow the removal of all infinities by conventional renormalization.[35] Let us write $\lambda_0 = \lambda + \delta\lambda$, where $\lambda$ is the renormalized coupling constant and $\delta\lambda$ is the corresponding counterterm, and treat the $\delta\lambda\phi^4$ term in tree approximation to obtain

$$2^{1/2}\eta = \lambda_0\langle\sigma\rangle^3 + i\lambda(G_\sigma + G_\pi)\langle\sigma\rangle . \quad (2.44)$$

At this point we abandon the self-consistency requirement used previously to determine the masses $m_\sigma$ and $m_\pi$ and simply demand that below $T_c$ these masses satisfy the tree approximation conditions

$$m_\sigma^2 = 2\lambda\langle\sigma\rangle^2, \quad m_\pi^2 = 0 . \quad (2.45)$$

The second of these equations builds in the Goldstone theorem. The first is the simplest assumption that allows the theory to be sensibly renormalized and is consistent with the tree approximation and the expectation that $m_\sigma$ and $\langle\sigma\rangle$ should vanish simultaneously as $T_c$ is approached from below. With (2.45) we have, at $T=0$,

$$2^{1/2}\eta - m_0^2\langle\sigma\rangle = \lambda_0\langle\sigma\rangle^3 + \frac{\lambda\langle\sigma\rangle}{4\pi^2}\left[\Lambda^2 + \tfrac{1}{2}\lambda\langle\sigma\rangle^2\left(1+\ln\frac{\lambda\langle\sigma\rangle^2}{2\Lambda^2}\right)\right] - m_0^2\langle\sigma\rangle . \quad (2.46)$$

Convenient definitions of the renormalized mass and coupling constant are

$$m_0^2 = \mu^2 + \lambda\Lambda^2/4\pi^2 , \quad (2.47a)$$

$$\lambda_0 = \lambda - \lambda^2[1+\ln(\mu^2/2\Lambda^2)]/8\pi^2 , \quad (2.47b)$$

whereupon the expression for $(2^{1/2}\eta/\langle\sigma\rangle - m_0^2)$ at finite temperature becomes

$$2^{1/2}\eta/\langle\sigma\rangle - m_0^2 = -\mu^2 + \lambda\langle\sigma\rangle^2 + [\lambda^2\langle\sigma\rangle^2\ln(\lambda\langle\sigma\rangle^2/\mu^2)]/8\pi^2 + \lambda[I^{(3)}((2\lambda)^{1/2}\langle\sigma\rangle) + I^{(3)}(0)] . \quad (2.48)$$

As in the case $2<d<3$, we now integrate (2.48) to arrive at the following expression for the effective potential, valid when $\langle\sigma\rangle$ is small:

$$V(\langle\sigma\rangle) = V(0) + \lambda a(3)(T^2 - T_c^2)\langle\sigma\rangle^2 - [(2\lambda^3)^{1/2}Tc(3)/3]\,|\langle\sigma\rangle^3| + \frac{\lambda}{4}\left[1+\frac{\lambda}{8\pi^2}\left(\ln\frac{\lambda\langle\sigma\rangle^2}{\mu^2} - \tfrac{1}{2}\right)\right]\langle\sigma\rangle^4 , \quad (2.49)$$

where $T_c^2 = \mu^2/2\lambda a(3)$. The qualitative behavior of this function is identical to that of the effective potential for the case $2<d<3$, plotted in Fig. 2. Again, owing to the presence of the negative $|\langle\sigma\rangle^3|$ term in $V$, $\langle\sigma\rangle$ approaches a finite value as the transition temperature is approached from below. Thus the modified Hartree approximation for $d=3$ predicts a first-order transition as well.

Above $T_c^*$, $\langle\sigma\rangle = 0$ and the common mass of the $\sigma$ and $\pi$ fields is determined by the formula

$$m^2(T) = \partial^2 V/\partial\langle\sigma\rangle^2|_{\langle\sigma\rangle=0} = 2\lambda a(3)(T^2 - T_c^2) . \quad (2.50)$$

Thus $m^2(T)$ is positive at $T_c^*$ and increases monotonically with $T$ thereafter. The behavior of $\langle\sigma\rangle$, $m_\sigma$, $m_\pi$, and $m$ as functions of $T$ is qualitatively identical to that shown in Fig. 3.

To what extent is this prediction of a first-order transition to be believed? Mean field approximations are at best marginally reliable in their predictions about critical phenomena, and the modified Hartree scheme makes approximations even beyond canonical mean field treatments. For $2<d<3$ we sacrificed some of the self-consistency of the usual gapless methods by ignoring the dependence of the Green's functions on $\phi^c$ in computing the self-energy. For $d=3$ even more self-consistency was sacrificed in order to make the theory renormalizable. Thus one should view the first-order transition that emerges from the modified Hartree approximation with a healthy dose of skepticism.

## III. ANALOGIES IN MANY-BODY THEORY

Mean field treatments that predict first-order phase transitions are familiar in nonrelativistic field theories. In trying to ascertain whether the first-order transition of the last section is qualitatively correct or whether it is simply an artifact of the approximations used, it is useful to review some of these examples briefly.

### A. Type-I superconductors

We shall first consider the type-I superconductor and then the weakly interacting Bose gas. According to BCS theory[36] the superconducting phase transition is of second order. Furthermore, owing to the very large zero-temperature coherence length in type-I superconductors, the effects of fluctuations are negligible until the temperature is so close to $T_c$ that[37] $|T-T_c|/T_c \sim 10^{-15}$.

Recently, HLM[20] showed that the presence of a fluctuating electromagnetic field in such systems can change the order of the transition from second to first. The coupling of the superconductor to a vector potential $\vec{A}(\vec{r})$ is described in their work by the classical Ginzburg-Landau "free-energy functional"

$$V(\{\psi,\vec{A}\}) = \int d^3r\left[a|\psi|^2+\tfrac{1}{2}b|\psi|^4+\gamma|(\vec{\nabla}-iq_0\vec{A})\psi|^2 + \frac{1}{8\pi\mu_0}\sum_{i>j}\left(\frac{\partial A_i}{\partial r_j}-\frac{\partial A_j}{\partial r_i}\right)^2\right], \quad (3.1)$$

where $\psi$ is the superconducting order parameter, $a=a'(T-T_c^0)/T_c^0$, $q_0=2e/\hbar c$, $\mu_0$ is the magnetic permeability of the normal metal, and $a'$, $b$, and $\gamma$ are temperature independent constants near $T_c$. Thermal expectation values, e.g., $\langle\psi(\vec{r})\psi(\vec{r}')\rangle$, are defined by the functional integral

$$\langle X\rangle = \frac{\int\delta\{\psi\}\,\delta\{\vec{A}\}Xe^{-V(\{\psi,\vec{A}\})/T}}{\int\delta\{\psi\}\,\delta\{\vec{A}\}\,e^{-V(\{\psi,\vec{A}\})/T}}, \quad (3.2)$$

where $X$ is any functional of $\psi$ and $\vec{A}$.

We may cast the HLM calculation in a form which emphasizes its similarity to our treatment of the $\sigma$ model by first writing the Ginzburg-Landau field equation which follows from (3.1):

$$-\gamma\langle(\vec{\nabla}-iq_0\vec{A})^2\psi\rangle + a\langle\psi\rangle + b\langle|\psi|^2\psi\rangle = 0. \quad (3.3)$$

For translationally invariant systems the $\vec{\nabla}$ terms in (3.3) vanish. The neglect of fluctuations in $\psi$ is equivalent to the tree approximation and reduces (3.3) to

$$(\gamma q_0^2\langle\vec{A}^2\rangle + a + b|\langle\psi\rangle|^2)\langle\psi\rangle = 0. \quad (3.4)$$

This equation is an exact analog of Eq. (2.19a).

In the superconductor, the inverse, $\kappa$, of the penetration depth is defined by

$$\kappa^2 = 8\pi\mu_0 q_0^2\gamma|\langle\psi\rangle|^2, \quad (3.5)$$

and plays the role of a photon mass; for small $|\langle\psi\rangle|$ the expectation value $\langle\vec{A}^2\rangle$ is given by [cf. (2.27c) and (2.29a)]

$$\langle\vec{A}^2\rangle = 8\pi\mu_0 I^{(3)}(\kappa) \approx 8\pi\mu_0 T^2 a(3) - (32\pi\gamma q_0^2\mu_0)^{1/2}\mu_0 T|\langle\psi\rangle|. \quad (3.6)$$

Note that $\langle\vec{A}^2\rangle$ is linear in $|\langle\psi\rangle|$, so that the field equation (3.4) has [as we found in our analogous Eq. (2.37)] no solution for arbitrarily small $|\langle\psi\rangle|$. As we have seen, such behavior indicates a first-order phase transition. Again the effective potential, $V(\langle\psi\rangle)$, obtained by integrating the left side of the field equation has the form

$$V(\langle\psi\rangle) = V(0) + a'(T-T_c)|\langle\psi\rangle|^2 - \frac{1}{6\pi}T(8\pi\mu_0 q_0^2\gamma)^{3/2}|\langle\psi\rangle|^3 + \tfrac{1}{2}b|\langle\psi\rangle|^4. \quad (3.7)$$

HLM are able to show that the transition in the superconductor occurs at a temperature, $T_c^*$, sufficiently above $T_c$ [the temperature where a second-order transition would occur in the absence of the $|\langle\psi\rangle|^3$ term in (3.7)] that the fluctuations in $\psi$ are completely negligible near $T_c^*$. Thus, for purposes of discussing the region near $T_c^*$, the initial neglect of the fluctuations in $\psi$ is justifiable. The prediction of a first-order phase transition in type-I superconductors is therefore on much firmer footing than is the analogous prediction for the $\sigma$ model. The result for the superconductor derives from the electromagnetic field fluctuations, which are significant outside the extremely narrow region about $T_c$ where the fluctuations in the order parameter are important. The $\sigma$ model, on the other hand, has only one field: $\phi$. When the low-order fluctuations in $\phi$ that we considered in the preceding section play a significant role, then fluctuation effects of arbitrarily high order become equally important. There is no justification for the omission of these higher-order fluctuations.

### B. The weakly interacting Bose gas

The weakly interacting nonrelativistic Bose gas, which can be taken as a first model for superfluid helium, is more closely analogous to the $\sigma$ model than is the superconductor. It is well known that the Bogoliubov approximation,[21] the analog in nonrelativistic Bose systems of the tree approximation in relativistic theories, provides a correct first description of the ground state and low-lying excitations of the weakly interacting Bose gas. However, when one tries to extend the approximation to describe the phase transition from the superfluid (broken symmetry) state to the normal state one finds, in a manner remarkably similar to that already observed for the $\sigma$ model and the superconductor, a first-order transition. To see this, we start with the Hamiltonian

$$\mathcal{H} = \int d^3r\left[\frac{1}{2m}(\vec{\nabla}\psi^\dagger)\cdot(\vec{\nabla}\psi) - \mu\psi^\dagger\psi + \tfrac{1}{2}\lambda\psi^\dagger\psi^\dagger\psi\psi\right], \quad (3.8)$$

where $m$ is the particle mass, $\mu$ the chemical potential, and $\psi$ the field operator, and from it derive the equation of motion for the order parameter:

$$(i\partial/\partial t+\vec{\nabla}^2/2m+\mu)\langle\psi\rangle=\lambda\langle\psi^\dagger\psi\psi\rangle\,. \quad (3.9)$$

The Bogoliubov approximation, like the tree approximation, starts with the factorization

$$\langle\psi^\dagger\psi\psi\rangle=\langle\psi^\dagger\rangle\langle\psi\rangle^2\,. \quad (3.10)$$

For a uniform, time-independent condensate, (3.9) then implies

$$\mu=\lambda|\langle\psi\rangle|^2\,. \quad (3.11)$$

The self-energy $\underline{\Sigma}$ [defined as in (2.11)] is computed by the gapless prescription (2.12), yielding

$$\underline{\Sigma}=\lambda\delta(x-x')\begin{pmatrix}2|\langle\psi\rangle|^2 & \langle\psi\rangle^2\\ \langle\psi\rangle^{*2} & 2|\langle\psi\rangle|^2\end{pmatrix}\,. \quad (3.12)$$

Then

$$G_{11}(\vec{p},z)=(z+\vec{p}^2/2m+\lambda\langle\psi\rangle^2)/(z^2-\omega_p^2)\,, \quad (3.13)$$

where $\omega_p^2=\vec{p}^2(\lambda\langle\psi\rangle^2+\vec{p}^2/4m)/m$ and $\langle\psi\rangle$ is assumed real. The conservation law for the total number of particles in the system determines $\langle\psi\rangle$ as a function of $T$. Equation (3.13) implies

$$\rho=\langle\psi\rangle^2+\int\frac{d^dp}{(2\pi)^d}\frac{1}{\omega_p}\left[\frac{\vec{p}^2/2m+\lambda\langle\psi\rangle^2}{e^{\omega_p/T}-1}+\frac{1}{2}\left(\frac{\vec{p}^2}{2m}+\lambda\langle\psi\rangle^2-\omega_p\right)\right], \quad (3.14)$$

where $\rho$ is the total particle density and $\langle\psi\rangle^2$ represents the density of particles in the condensate. For small $\langle\psi\rangle$,

$$\rho=\langle\psi\rangle^2+f(d)(2mT)^{d/2}-g(d)m^{d/2}T\lambda^{(d-2)/2}|\langle\psi\rangle|^{d-2}+\cdots\,, \quad (3.15)$$

where $f(d)$ and $g(d)$ are positive functions of the dimensionality. As we have seen before, the $|\langle\psi\rangle|^{d-2}$ term dominates for small $|\langle\psi\rangle|$, $\langle\psi\rangle$ does not vanish continuously as $T$ approaches $T_c$ [defined as the temperature where $\langle\psi\rangle=0$ solves (3.14)] from below, and the familiar first-order phase transition results. Again, the effective potential contains a negative $|\langle\psi\rangle|^d$ term, the harbinger of first-order transitions.

However, the conclusions to be drawn here are quite different from those in the superconductor since there is *no* temperature regime near $T_c$ for superfluids where the neglect of fluctuations as in (3.10) is valid. There is no reason therefore to believe this mean field result. Indeed, the vast body of well-established theoretical and experimental evidence that the $\lambda$ transition is of second order indicates that the Bogoliubov prediction is simply wrong.

We believe the situation for the $\sigma$ model to be rather similar to that for the imperfect Bose gas. The modified Hartree approximation has no apparent validity in the vicinity of the first-order transition it predicts. We have no general method for properly taking into account all of the important fluctuation effects. One can, however, acquire some feeling for the effect of higher order correlations in the $\sigma$ model by considering a related theory, the O($N$) model,[22] which for large $N$ does have a legitimate small parameter, viz., $1/N$, near $T_c$. The existence of this parameter allows us systematically to include the effects of fluctuations and thus to see how a second-order transition results.

## IV. THE O($N$) MODEL

### A. $2<d<3$

The O($N$) model, a generalization of the $\sigma$ model to $N$ fields, is described by the Lagrangian density

$$L=\tfrac{1}{2}\sum_{\alpha=1}^{N}\left[-(\partial_\mu\phi_\alpha)^2+m_0^2\phi_\alpha^2\right]-\frac{1}{2}\frac{\lambda_0}{N}\left(\sum_{\alpha=1}^{N}\phi_\alpha^2\right)^2-\sum_{\alpha=1}^{N}h_\alpha\phi_\alpha\,. \quad (4.1)$$

When $m_0^2>0$ and $h=0$ we expect the O($N$) symmetry to be spontaneously broken at low temperatures. As usual, the coupling constant is written as $\lambda_0/N$ so that the theory has a finite limit as $N\to\infty$. We consider dimension $2<d<3$, where difficulties arising from coupling-constant renormalization do not occur. It is trivial to generalize the methods of Coleman *et al.*[38] to finite temperature and so compute the effective potential in the $N\to\infty$ limit. We shall proceed slightly differently and compute the field equation and Green's functions to make contact with our earlier calculations.

We choose the "phase" of the condensate so that when $h=0$ only $\langle\phi_1\rangle$ is nonzero, and introduce the shifted fields $\tilde{\phi}_1=\phi_1-\langle\phi_1\rangle$, $\tilde{\phi}_\beta=\phi_\beta$ for $\beta\neq1$. In order to correspond with our previous notation we refer to the fields $\phi_\beta$ with $\beta\neq1$ as $\pi$ fields and $(2/N)^{1/2}\phi_1$ as the $\sigma$ field. Then $\langle\phi_1\rangle=(N/2)^{1/2}\langle\sigma\rangle$, where $\langle\sigma\rangle$ is of order unity. The equation of motion for $\langle\phi_\alpha\rangle$ is

$$(\Box^2+m_0^2)\langle\phi_\alpha\rangle=h_\alpha+2\lambda_0\sum_\beta(\langle\phi_\beta\rangle^2\langle\phi_\alpha\rangle+\langle\tilde{\phi}_\beta^2\rangle\langle\phi_\alpha\rangle+2\langle\tilde{\phi}_\beta\tilde{\phi}_\alpha\rangle\langle\phi_\beta\rangle+\langle\tilde{\phi}_\beta^2\tilde{\phi}_\alpha\rangle)/N\,. \quad (4.2)$$

Setting $h=0$ and taking $\alpha=1$ in (4.2) we find the equation for $\langle\sigma\rangle$,

$$m_0{}^2\langle\sigma\rangle = \lambda_0\left\{\langle\sigma\rangle^3 + \frac{2i}{N}[(N-1)G_\pi(xx) + 3G_\sigma(xx)]\langle\sigma\rangle + \left(\frac{2}{N}\right)^{3/2}\sum_\beta\langle\tilde{\phi}_\beta{}^2\tilde{\phi}_1\rangle\right\}, \tag{4.3}$$

since when $h=0$ the Green's functions $G_{\alpha\beta}$ assume the form

$$G_{\alpha\beta}(xx') = \delta_{\alpha\beta}[\delta_{\alpha 1}G_\sigma(xx') + (1-\delta_{\alpha 1})G_\pi(xx')]. \tag{4.4}$$

Let us now examine (4.3) in the $N\to\infty$ limit. The quantity $\langle\tilde{\phi}_\beta{}^2\tilde{\phi}_1\rangle$ in lowest order is $\sim\lambda_0 G_{\beta\beta}{}^2 G_\sigma\langle\phi_1\rangle/N$ and hence is $\sim N^{-1/2}\langle\sigma\rangle$. Equation (4.3) thus becomes

$$m_0{}^2\langle\sigma\rangle = \lambda_0[\langle\sigma\rangle^2 + 2iG_\pi(xx)]\langle\sigma\rangle \tag{4.5}$$

for $N\to\infty$. This result, exact when $N$ is strictly infinite, is identical in form to (2.19a) which was derived in the Hartree approximation for $N=2$.

The Green's functions may be evaluated by taking the first variation of (4.2) with respect to $h$:

$$(\Box^2 + m_0{}^2)G_\pi(xx') = \delta(x-x') + \lambda_0[\langle\sigma\rangle^2 + 2i(1+N^{-1})G_\pi(xx) + 2iN^{-1}G_\sigma(xx)]G_\pi(xx') + 4\lambda_0\left[\left(\frac{2}{N}\right)^{1/2}\langle\sigma\rangle\frac{\delta}{\delta h_2(x')}\langle\tilde{\phi}_1(x)\,\tilde{\phi}_2(x)\rangle + \frac{1}{2N}\sum_\beta\frac{\delta}{\delta h_2(x')}\langle\tilde{\phi}_\beta{}^2(x)\,\tilde{\phi}_2(x)\rangle\right]. \tag{4.6}$$

The last term in this equation is $\sim(\lambda_0{}^2/N)G_{\beta\beta}{}^2G_\pi{}^2$ plus higher-order terms, while the penultimate term is $\sim 1/N$; these terms can thus be neglected in the $N\to\infty$ limit, as can the terms multiplied by explicit factors of $N^{-1}$. Using the field equation (4.5) we see that in the broken symmetry state $G_\pi$ is free and massless for $N=\infty$. In this same limit $G_\sigma$ is given by

$$(\Box^2 + m_0{}^2)G_\sigma(xx') = \delta(x-x') + \lambda_0[3\langle\sigma\rangle^2 + 2iG_\pi(xx)]G_\sigma(xx') + \left(\frac{2}{N}\right)^{1/2}\lambda_0\langle\sigma\rangle\sum_\beta\frac{\delta}{\delta h_1(x')}\langle\tilde{\phi}_\beta{}^2(x)\rangle. \tag{4.7}$$

The $N-1$ terms $\delta\langle\tilde{\phi}_\beta{}^2(x)\rangle/\delta h_1(x')$ with $\beta\geq 2$ are evaluated through use of $\delta G_\pi = -G_\pi\delta G_\pi{}^{-1}G_\pi$ and the $N=\infty$ form for $G_\pi$; this produces the integral equation

$$\frac{\delta G_\pi(xx)}{\delta h_1(x')} = 2\lambda_0\int d\bar{x}\,G_\pi(x\bar{x})G_\pi(\bar{x}x)\left[\left(\frac{2}{N}\right)^{1/2}\langle\sigma\rangle G_\sigma(\bar{x}x') + i\frac{\delta G_\pi(\bar{x}\bar{x})}{\delta h_1(x')}\right]. \tag{4.8}$$

Solving (4.8) by Fourier transformation and using (4.5) we finally arrive at the exact $N\to\infty$ result for $G_\sigma$ in the broken symmetry state:

$$G_\sigma{}^{-1}(\vec{p},z) = z^2 - \vec{p}^2 - \frac{2\lambda_0\langle\sigma\rangle^2}{1+2\lambda_0 S_T(\vec{p},z)}, \tag{4.9}$$

where $S_T$ is the Fourier transform of $-iG_\pi(x\bar{x})G_\pi(\bar{x}x)$, given explicitly by

$$S_T(\vec{p},i\omega_n) = T\sum_m\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\omega_m{}^2+\vec{k}^2)[(\omega_m+\omega_n)^2+(\vec{p}+\vec{k})^2]}. \tag{4.10}$$

Since $G_\pi$ is free and massless, we see from Eq. (4.5) that $\langle\sigma\rangle$ is given as a function of $T$ by Eq. (2.22), which we write as

$$\langle\sigma\rangle^2 = 2a(d)(T_c{}^{d-1} - T^{d-1}) \quad (T<T_c), \tag{4.11}$$

with $T_c$ defined in terms of the renormalized mass $\mu$ by (2.24). The behavior of $\langle\sigma\rangle$ is thus identical to that found in Hartree theory. The transition is of second order with $\beta=\frac{1}{2}$.

In contrast to the Hartree theory result, the $\sigma$ Green's function for large $N$ is nontrivial below $T_c$. The "mass" of the $\sigma$ excitations is defined by the pole of $G_\sigma(\vec{0},z)$ that approaches the origin as $T\to T_c$. The leading behavior of $S_T(\vec{0},z)$ as $z\to 0$ is given by the $m=0$ term of (4.10) and is $\sim(z^2)^{(d-4)/2}$. (The branch cut in the $z^2$ plane is along the positive real axis.) Thus as $z\to 0$,

$$G_\sigma{}^{-1}(\vec{0},z) \sim z^2 + il(d)\langle\sigma\rangle^2(1-e^{\pi id})(z^2)^{(4-d)/2}/T + \cdots, \tag{4.12}$$

where $l(d) = 2^{d-1}\pi^{d/2-1}\Gamma(d/2)$. Owing to the presence of the massless Goldstone bosons in the theory, the $\sigma$ particle is unstable. Let us look for a zero of (4.12) of the form $z^2 = R^2e^{i\theta}$. The first sheet of the function $G_\sigma$ corresponds to $0<\theta<2\pi$. $G_\sigma{}^{-1}$ then has zeros of the form

$$R = |\,2\sin(\pi d/2)\langle\sigma\rangle^2 l(d)/T\,|^{1/(d-2)}, \tag{4.13a}$$

$$\theta = (d\pm 4\nu)\pi/(d-2), \tag{4.13b}$$

where $\nu$ can assume any integral value; $\sin(\pi d/2)$ is negative since $2<d<3$. It is clear that no value of $\nu$ corresponds to a $\theta$ on the first sheet, as is of course guaranteed by unitarity, but there are solutions on higher sheets. The "mass" of the $\sigma$ is proportional to $\langle\sigma\rangle^{2/(d-2)}$ and so vanishes like $(T_c - T)^{1/(d-2)}$ as the critical temperature is approached from below.

Above the transition, $\langle\sigma\rangle$ vanishes and the $N=\infty$ limit is identical to Hartree theory. The common mass $m$ of the $\sigma$ and $\pi$ excitations is given by the gap equation (2.28a). We recall that $m(T)\sim(T-T_c)^{1/(d-2)}$ as $T$ approaches $T_c$ from above. The critical index $(d-2)^{-1}$ is the same as that found below $T_c$.

The leading term of the $1/N$ expansion gives rise, as we have just seen, to a second-order phase transition with no anomalies when $2<d<3$. In Sec. II we observed that the modified Hartree calculation predicted a first-order transition for the $N=2$ model. It is instructive to apply the modified Hartree approximation to the O($N$) model for large $N$ in an attempt to pinpoint the source of the first-order result. For arbitrary $N$ the modified Hartree approximation corresponds to writing the self-consistency condition for $m_\sigma{}^2$ as

$$m_\sigma{}^2=2m_0{}^2+4i\lambda_0N^{-1}[G_\sigma+(N-1)G_\pi]. \quad (4.14)$$

Recall that for $N=2$, $G_\sigma$ gives rise to the term $|m_\sigma|^{d-2}$, which is ultimately responsible for the first-order transition. In the $N\to\infty$ limit, however, the offending term vanishes, leaving us with

$$\begin{aligned} m_\sigma{}^2&=2\lambda_0\langle\sigma\rangle^2\\ &=4\lambda_0a(d)(T_c{}^{d-1}-T^{d-1}), \end{aligned} \quad (4.15)$$

with $T_c$ defined by (2.24). This result is *not* the correct $N\to\infty$ answer. The order parameter $\langle\sigma\rangle$ is given correctly, but, owing to the failure of the mean field theory to include the momentum-dependent renormalization of the coupling constant arising from the $S_T$ term in (4.9), the expression for $m_\sigma{}^2$ and the analytic structure of the $\sigma$ propagator are hopelessly wrong. Nonetheless modified Hartree does predict a second-order transition for $N=\infty$, in qualitative agreement with the proper result.

As in Sec. II, (4.14) predicts a first-order transition for any finite value of $N$. The discontinuity in the condensate magnitude at $T_c$ becomes smaller with increasing $N$, disappearing when $N$ becomes strictly infinite. It is interesting to compare this prediction with that obtained by calculating the $1/N$ corrections to the true $N\to\infty$ result. The critical, i.e., infrared, properties of the quantum O($N$) theory are expected to be identical to those of the classical O($N$) model, whose critical behavior has been computed to $O(1/N)$.[25] To this order in the classical theory, Brézin and Wallace[39] have proved the exponent scaling law

$$2\beta=\gamma[d(2-\eta)^{-1}-1], \quad (4.16)$$

where $\eta$ is the critical index that describes the spatial decay of the two-point Green's function at the critical temperature. Ma's classical calculation[25] of $\gamma$ and $\eta$ to $O(1/N)$, together with (4.16), yields

$$2\beta=1-\frac{4S_d}{N}\frac{2d-5}{d-2}+O\left(\frac{1}{N^2}\right), \quad (4.17a)$$

where

$$S_d=-2[\sin(\pi d/2)/\pi(d-2)B(\tfrac{1}{2}d-1,\tfrac{1}{2}d-1)] \quad (4.17b)$$

and $B(x,y)$ is the $\beta$ function: $\int_0^1 d\alpha\,\alpha^{x-1}(1-\alpha)^{y-1}$.

It is easy to verify that to $O(\epsilon/N)$ Eq. (4.17) is in complete agreement with tbe exact calculations of Brézin and Zinn-Justin[13] on the classical nonlinear $\sigma$ model in $2+\epsilon$ dimensions. One can readily check, moreover, that the $N=\infty$ result for $T_c$ [which is identical to the Hartree theory $T_c$ and is given in (2.24)] vanishes linearly with $\epsilon$ as $d$ approaches 2 from above. Aside from a trivial normalization factor this $T_c$ is identical to that found by Brézin and Zinn-Justin.

We now briefly indicate how (4.17) follows directly from (4.3) and (4.6). The only term on the right side of Eq. (4.3) that remains to be computed to order $1/N$ is $G_\pi(xx)$. From (4.6) we obtain the $O(1/N)$ $\pi$ self-energy,

$$\begin{aligned}\Sigma_\pi(xx')=\lambda_0\{&\delta(x-x')[\langle\sigma\rangle^2+2i(1+N^{-1})G_\pi^{(1)}(xx)+2iN^{-1}G_\sigma(xx)]+4(2N^{-1})^{1/2}\langle\sigma\rangle\delta\langle\tilde\phi_1(x)\tilde\phi_2(x)\rangle/\delta\langle\phi_2(x')\rangle\\ &+2\delta\langle\tilde\phi_3{}^2(x)\tilde\phi_2(x)\rangle/\delta\langle\phi_2(x')\rangle\}\end{aligned} \quad (4.18)$$

where $G_\pi^{(1)}$ denotes the $\pi$ Green's function to $O(1/N)$. The penultimate term on the right side of (4.18) can be computed through an equation similar to (4.8); the $O(1/N)$ part of the final term is evaluated in the Appendix. The result is

$$\begin{aligned}\Sigma_\pi(\vec q,i\omega_m)=\lambda_0\Big[&\langle\sigma\rangle^2+2i(1-N^{-1})G_\pi^{(1)}(xx)+2iN^{-1}G_\sigma(xx)\\ &-4N^{-1}T\sum_n\int\frac{d^dk}{(2\pi)^d}\frac{G_\pi(\vec k+\vec q,i(\omega_n+\omega_m))G_\pi{}^{-1}(\vec k,i\omega_n)G_\sigma(\vec k,i\omega_n)}{1+2\lambda_0S_T(\vec k,i\omega_n)}\Big].\end{aligned} \quad (4.19)$$

Comparison of this result with the equation that follows from the substitution of (4.8) into (4.3),

$$m_0^2 = \lambda_0 \left[ \langle\sigma\rangle^2 + 2i(1-N^{-1})G_\pi^{(1)}(xx) + 2iN^{-1}G_\sigma(xx) - 4N^{-1}T \sum_n \int \frac{d^dk}{(2\pi)^d} \frac{G_\sigma(\vec{k}, i\omega_n)}{1+2\lambda_0 S_T(\vec{k}, i\omega_n)} \right], \tag{4.20}$$

shows that

$$\Sigma_\pi(\vec{q}, i\omega_m) = m_0^2 - 4\lambda_0 N^{-1}T \sum_n \int \frac{d^dk}{(2\pi)^d} \left\{ \frac{G_\sigma(\vec{k}, i\omega_n)}{1+2\lambda_0 S_T(\vec{k}, i\omega_n)} [G_\pi(\vec{k}+\vec{q}, i\omega_n + i\omega_m) G_\pi^{-1}(\vec{k}, i\omega_n) - 1] \right\}. \tag{4.21}$$

Since $\Sigma_\pi(\vec{0}, 0)$ is simply $m_0^2$, this equation demonstrates to $O(1/N)$ that the $\pi$ excitations are the Goldstone bosons of the O($N$) model.

Expression (4.21) enables us to compute $G_\pi^{(1)}(xx)$ in terms of the known $N=\infty$ Green's functions $G_\sigma$ and $G_\pi$. Equation (4.20) then consists entirely of known quantities; after some straightforward algebra it becomes

$$c(T_c - T) = \langle\sigma\rangle^2 \left(1 + \frac{8S_d}{N} \frac{2d-5}{d-2} \ln\langle\sigma\rangle\right) \tag{4.22}$$

for small $\langle\sigma\rangle$, where $T_c$ is defined as the temperature where $\langle\sigma\rangle$ vanishes and $c$ is a positive constant. To $O(1/N)$ this equation can be written as $(T_c - T) \sim \langle\sigma\rangle^{1/\beta}$, where $\beta$ is the exponent defined in (4.17). Thus we arrive at the expected result: The inclusion of $1/N$ terms modifies the critical exponents but does not alter the order of the transition of the O($N$) model.

Note that we were required to interpret Eq. (4.22) correctly to reach this conclusion. Had we retained in (4.22) only the $\langle\sigma\rangle^2 \ln\langle\sigma\rangle$ term, the dominant term on the right side for small $\langle\sigma\rangle$, we would have found that for $2d > 5$ the equation had no solution as $\langle\sigma\rangle$ approached zero; as in the modified Hartree approximation [cf. (4.14)] we would have concluded that the transition was of first order. In (4.14), the dominant term of $O(1/N)$ is actually a power of $m_\sigma$; try as we might we cannot interpret it as a simple correction to the leading ($N=\infty$) terms. In (4.22), on the other hand, the logarithmic $1/N$ term represents an obvious correction to the leading power. We expect inclusion of terms of higher order in $1/N$ merely to produce further corrections to the critical exponents.

The modified Hartree approximation completely neglects the $\langle\tilde{\phi}_1\tilde{\phi}_2^2\rangle$ term [the last term in (4.7)] of the $\sigma$ self-energy, $\Sigma_\sigma$. In consequence $\Sigma_\sigma$ is momentum independent and the logarithm which appears in the $1/N$ expansion is replaced by a power. A less drastic approximation to $\Sigma_\sigma$ [such as the $N=\infty$ result of Eq. (4.9)] is evidently required if the order of the transition is to be given correctly.

### B. $d=3$

In Sec. II we saw that simple Hartree theory, well-behaved when $2<d<3$, breaks down because of ultraviolet effects when $d=3$. The breakdown was manifest in the absence of a solution of the gap equation (2.41) at sufficiently high temperature. Since Hartree theory and the $N\to\infty$ limit of the O($N$) model are intimately related, we might expect to find anomalies in the large-$N$ approximation as well. Abbott, Kang, and Schnitzer[26] have argued that at zero temperature the $N\to\infty$ limit is free of anomalies and that spontaneous symmetry breaking does *not* occur. The ground state of the theory is O($N$) symmetric in the large-$N$ approximation. It is straightforward to extend the work of these authors to finite temperature. The O($N$) symmetry is of course preserved for all $T$; the $\sigma$ and $\pi$ excitations are identical and are found to have a mass given by (2.41). We conclude that the large-$N$ approximation is consistent at low temperatures but does indeed break down at temperatures sufficiently large so that (2.41) has no solution.

Thus when $d=3$ the thermodynamics of the O($N$) model cannot be reliably computed for all temperatures, even for large $N$. While we believe that spontaneous symmetry breaking does occur at low temperatures for small values of $N$ and that the restoration of symmetry occurs via a second-order phase transition whose critical exponents are identical to those of the classical O($N$) model, we know of no simple approximation whereby this prejudice can be convincingly confirmed.

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### APPENDIX: EVALUATION OF $\delta\langle\tilde{\phi}_3^2(x)\tilde{\phi}_2(x)\rangle/\delta\langle\phi_2(x')\rangle$

In this Appendix we compute to $O(1/N)$ the quantity

$$I_{32}(xyx') \equiv \delta\langle\tilde{\phi}_3^2(x)\tilde{\phi}_2(y)\rangle/\delta\langle\phi_2(x')\rangle$$

which occurs in (4.18). Recalling that

$$\langle\tilde{\phi}_3{}^2(x)\tilde{\phi}_2(y)\rangle = \frac{\delta G_3(xx)}{\delta h_2(y)}$$
$$= -\int dz\, d\bar{z} \sum_{\alpha\beta} G_{3\alpha}(xz)\frac{\delta\Sigma_{\alpha\beta}(z\bar{z})}{\delta h_2(y)} G_{\beta 3}(\bar{z}x) \qquad \text{(A1)}$$

and noting that to leading order in $1/N$,

$$\Sigma_{\alpha\beta}(z\bar{z}) = \delta_{\alpha\beta}\delta(z-\bar{z})2\lambda_0 \sum_{\delta} [\langle\phi_\delta(z)\rangle^2 + iG_{\delta\delta}(zz)]/N\,, \qquad \text{(A2)}$$

we have

$$\frac{\delta G_3(xx)}{\delta h_2(y)} = \int dz \sum_{\alpha} G_{3\alpha}(xz)G_{\alpha 3}(zx)\frac{2\lambda_0}{N}\sum_{\delta}\left[2\langle\phi_\delta(z)\rangle G_{\delta 2}(zy) + i\frac{\delta G_{\delta\delta}(zz)}{\delta h_2(y)}\right], \qquad \text{(A3)}$$

whereupon it follows that

$$I_{32}(xyx') = -\int dz\, G_{33}(xz)G_{33}(zx)\left[\frac{2\lambda_0}{N}\left(2\delta(z-x')G_{22}(zy) + (2N)^{1/2}\langle\sigma\rangle\frac{\delta G_{12}(zy)}{\delta\langle\phi_2(x')\rangle}\right) - 2i\lambda_0 I_{32}(zyx')\right]. \qquad \text{(A4)}$$

The quantity $\delta G_{12}(zy)/\delta\langle\phi_2(x')\rangle$ is readily evaluated through an equation analogous to (4.8); to leading order we find that the Fourier transform, $\Gamma_3(\vec{k}, i\omega_n; \vec{p}, i\omega_m)$, of $\delta G_{12}(zy)/\delta\langle\phi_2(x')\rangle$ is

$$\Gamma_3(\vec{k}, i\omega_n; \vec{p}, i\omega_m) = \frac{4\lambda_0}{(2N)^{1/2}}\langle\sigma\rangle\frac{G_\sigma(\vec{k}, i\omega_n)G_\pi(\vec{p}, i\omega_m)}{1 + 2\lambda_0 S_T(\vec{k}, i\omega_n)}. \qquad \text{(A5)}$$

Equation (A4) then implies that the Fourier transform, $I_{32}(\vec{k}, i\omega_n; \vec{p}, i\omega_m)$, of $I_{32}(xyx')$ is simply

$$I_{32}(\vec{k}, i\omega_n; \vec{p}, i\omega_m) = \frac{4i\lambda_0}{N}\frac{S_T(\vec{k}, i\omega_n)G_\pi(\vec{p}, i\omega_m)G_\pi{}^{-1}(\vec{k}, i\omega_n)G_\sigma(\vec{k}, i\omega_n)}{1 + 2\lambda_0 S_T(\vec{k}, i\omega_n)}. \qquad \text{(A6)}$$

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