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Composite operators in non-Abelian gauge theories

Richard A. Brandt*
Department of Physics, New York University, New York, New York 10003

Ng Wing-chiu†
Department of Physics, University of Maryland, College Park, Maryland 20742

Kenneth Young
Department of Physics, Chinese University of Hong Kong, Hong Kong
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The renormalization of the composite gauge field product operators $A_\mu^a(x)A_\nu^b(x)$ is carried out in detail in asymptotically free non-Abelian SU(n) gauge theories. Upon renormalization, these operators mix with similar operators obtained by Lorentz and SU(n) group rotations and with other composite operators formed from ghost fields or derivatives of A. It is shown, using renormalization-group and SU(n)-projection techniques, that this renormalization problem is completely soluble. The renormalization-group equations satisfied by the composite renormalization-constant matrix Z are deduced and solved using the computed second-order expression for Z. For SU(2), Z is put in triangular form so that the effective anomalous dimension eigenvalues can be read off. For the general SU(n) group, it is more convenient to use group projection operators and crossing matrices to explicitly diagonalize the renormalization-group equations. The main results can be most simply stated as an explicit short-distance operator expansion which expresses the product $A_\mu^a(x)A_\nu^b(0)$ for $x\to 0$ in terms of the finite composite operators $:A_\alpha^c(0)A_\beta^d(0):$. The leading singularity is seen to be associated with the singlet operator $\delta^{ab}g_{\mu\nu}:A\cdot A:$. The results are used to study the invariance of the models under the Abelian gauge transformations $A_\mu^a\to A_\mu^a+\partial_\mu\Lambda^a$.

I. INTRODUCTION

Consider a non-Abelian gauge theory (NAGT) involving (unrenormalized) Yang-Mills (YM)[1] vector-meson fields a_μ^a, ghost fields c_i^a, and fermion fields ψ, where $a=1,\ldots,N$ for a gauge group of dimension N and $i=1,2$. The corresponding renormalized fields A_μ^a, C_i^a, Ψ are formally obtained from the unrenormalized fields by simply multiplying by the wave-function renormalization constants $Z_3^{-1/2}$, $\tilde{Z}_3^{-1/2}, Z_2^{-1/2}$, respectively.[2] The renormalization of unrenormalized composite operators such as $a_\mu^a a_\nu^b$, $c_1^a c_2^b$, $\bar\psi\Gamma\psi$, etc., is more complicated because of the need to incorporate additive renormalizations involving lower-dimensional operators and because of the fact that operators of the same dimension in general mix under renormalization. In this paper we will be concerned with this latter mixing problem. This is an important problem which occurs in connection with all of the observable gauge-invariant operators such as $\bar\psi\gamma_\mu\psi$, $\bar\psi\gamma_\mu\mathfrak{D}_\nu\psi$, $f_{\mu\alpha}f^{\alpha}{}_{\nu}$, etc.

In this paper we will study explicitly the renormalization of the lowest-dimensional composite operator $a_\mu^a a_\nu^b$ and the operators (including ghosts) of the same dimension (2) with which it can mix. This study should be worthwhile, in spite of the fact that these operators are not observable, for a number of reasons. The low dimension of these operators makes the mixing problem relatively simple and transparent. Furthermore, there is no ghost-mixing to contend with. Also, the field product of interest occurs in the fundamental covariant field operator

$$f_{\mu\nu}^a=\partial_\mu a_\nu^a-\partial_\nu a_\mu^a+g_0 f^{abc}a_\mu^b a_\nu^c, \tag{1.1}$$

where g_0 is the unrenormalized charge and f^{abc} are the antisymmetric structure constants of the gauge Lie algebra, and in the gauge-invariant products such as $f_{\mu\nu}^a f_{\kappa\lambda}^a$. Finally, as we will return to later, our field product occurs when the field product $a_\mu a^\mu a_\nu$, which occurs in the field equations, is subjected to an "R transformation" $a_\mu\to a_\mu+r_\mu$, $r_\mu=$ constant, or to an Abelian gauge transformation $a_\mu\to a_\mu+\partial_\mu\Lambda$.[3,4]

If the non-Abelian gauge theory is asymptotically free (AF),[5] the only case we will consider in this paper, the renormalization problem is completely and exactly soluble. Because of the asymptotic freedom, the renormalization-group (RG) equations for the renormalization-constant matrix can be exactly solved to give the exact[6] expression for the matrix in terms of its behavior in the lowest nontrivial order of perturbation theory. These lowest-order calculations are straightforward but often tedious.

In principle, there are three distinct sets of operators that can mix with $a_\mu^a a_\nu^b$: (i) the bilinear operators

$$M^{ab}_{cd} a^c_\mu a^d_\nu, \quad g_{\mu\nu} M^{ab}_{cd} a^c \cdot a^d, \tag{1.2}$$

where M^{ab}_{cd} is a gauge-group-invariant matrix (constructed out of δ^{ab}, structure constants f^{abc}, etc.); (ii) the linear operators

$$N^{abc} \partial_\mu a^c_\nu, \tag{1.3}$$

where N^{abc} is a gauge-group-invariant matrix, and (iii) the ghost bilinear operators

$$g_{\mu\nu} M^{ab}_{cd} c^c_1 c^d_2. \tag{1.4}$$

Our analysis is somewhat simplified because the operators (1.4) cannot, in fact, occur, as follows from the structure of the YM Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} f^a_{\mu\nu} f^{a\mu\nu} - \frac{1}{2\alpha_0}(\partial \cdot a)^2 + \partial^\mu c^a_1 \mathfrak{D}^{ab}_\mu c^b_2 + i\bar{\psi}\gamma_\mu D^\mu \psi,$$
$$\mathfrak{D}^{ab}_\mu \equiv \delta^{ab}\partial_\mu + g_0 f^{abc} A^c_\mu, \tag{1.5}$$
$$D_\mu \equiv \partial_\mu - i g_0 A^a_\mu T^a,$$

where T^a are the fermion representation matrices. The presence of the operators (1.3) is easily accounted for since they are multiplicatively renormalized (by Z_3). Thus the heart of our analysis will be the solution of the mixing problem associated with the operators (1.2). The tools will be elementary SU(n) group theory and the RG with the dynamical input supplied by single-loop Feynman diagrams. In principle, directionally dependent singularities (e.g., $A_\mu A_\nu x^\mu x^\nu / x^2$) can also occur, but these will be seen to be less singular than (1.2).

Composite operators were first precisely defined in perturbation theory by point-separation[7,8] and normal-product[8,9] techniques. The original applications of the renormalization group to study the renormalization of composite operators was made by Symanzik in a series of beautiful and important papers.[10] This work was extended to other operators and models and to light-cone operator-product expansions by Christ, Hasslacher, and Mueller[11] and by Mitter.[12] For NAGT's, renormalization and mixing treatments of composite operators in leading light-cone expansions were given in the original studies of AF.[5] In this work, it was assumed that the anomalous dimensions of gauge-invariant operators could be correctly computed by ignoring the mixing with operators involving ghost fields, but the validity of this assumption has not been rigorously established. The severity of the ghost-mixing problem was subsequently illustrated by Kainz, Kummer, and Schweda.[13] These authors studied the renormalization and mixing of $f_{\mu\alpha} f^\alpha{}_\nu$ and $\bar{\psi}\gamma_\mu \mathfrak{D}_\nu \psi$ in AF theories and found a crucial ghost dependence. Their conclusions were further supported by Kluberg-Stern and Zuber,[14] who studied the renormalization of $f_{\mu\nu} f^{\mu\nu}$ and the operators (including those involving ghost fields) with which it mixes, using renormalization-group methods. It was again found that ghost mixing could not be ignored.

For us, as we have stated, the relevant ghost operators (1.4) are rigorously ignorable. However, they can be included for illustrative purposes since the operators (1.2) *do* occur in the renormalization of (1.4). In either case, the same result is obtained for the renormalization of (1.2).

Our main result is most simply expressed as the explicit short-distance operator-product expansion (OPE)

$$A^a_\mu(x) A^b_\nu(0) \underset{x\to 0}{\sim} \sum_l [\ln(x^2)^{-1/2}]^{\delta_l} \times M^{abcd,\alpha\beta}_{l\mu\nu} : A^c_\alpha(0) A^d_\beta(0):, \tag{1.6}$$

which expresses the singularities of the product of renormalized fields A^a_μ in terms of c-number functions of the space-time separation and finite local composite operators $:A^c_\alpha A^d_\beta:$ symmetric under the simultaneous interchanges $c \leftrightarrow d$, $\alpha \leftrightarrow \beta$. The numbers δ_l and matrices M_l will be given explicitly in Sec. V for the SU(n) gauge group. The leading singularity will be seen to be carried by the identity projection

$$M^{abcd,\alpha\beta}_{I\mu\nu} = \delta^{ab}\delta^{cd} g_{\mu\nu} g^{\alpha\beta}(n^2-1)^{-1}. \tag{1.7}$$

We begin our analysis in Sec. II with a review of the general operator-mixing formalism. The connection between the cutoff dependence of the renormalization mixing matrix Z_{ij} and the singularity structures of associated OPE's is stressed. In Sec. III, after a review of the RG equations for Z_2 and Z_3, RG equations are deduced for Z_{ij}. For AF theories, the forms of the solutions to these equations are given. The renormalization of the product $A^a_\mu A^b_\nu$ for the SU(2) gauge group is taken up in Sec. IV. The simplicity of this group enables the renormalization problem to be discussed in the absence of group-theoretic complications. The mixing operators (1.2)–(1.4) are explicitly enumerated and the renormalization constant matrix is given in triangular form. The simplicity of the operators (1.3) and the irrelevance of the operators (1.4) is discussed. The general SU(n) gauge group is considered in Sec. V. Group-theoretic projection operator and crossing matrix techniques are used to diagonalize the RG equations. The resulting uncoupled equations are then solved and the general singularity structure is exhibited in essentially the form (1.6). The final Sec. VI summarizes our results and applies them to the study of the R invariance of NAGT's.

II. OPERATOR MIXING

In this section we will renew in a general context the way in which renormalization mixes together operators with the same quantum numbers and naive dimensions. Let $U_i \equiv U_i(x)$, $i=1,\ldots N$ be a set of such unrenormalized local field operators which mix upon renormalization, and denote by $R_i \equiv R_i(x)$ the corresponding renormalized operators. We suppress the implicit dependence of the U_i on a cutoff parameter K^2. [Matrix elements of U_i are cutoff dependent and generally possess no finite limit when the cutoff is removed ($K^2 \to \infty$).] The multiplicative renormalization matrix $Z_{ij} \equiv Z_{ij}(K^2)$ is defined by[15]

$$\sum_{i=1}^{N} R_i Z_{ij} = U_j , \tag{2.1}$$

or

$$R_i = \sum_j U_j (\underline{Z}^{-1})_{ji} , \tag{2.2}$$

where $\underline{Z}$ denotes the matrix with matrix elements Z_{ij}. More precisely, (2.2) should be written

$$R_i(x) = \lim_{K^2 \to \infty} \sum_j U_j(x\,;K^2)[\underline{Z}^{-1}(K^2)]_{ji} . \tag{2.3}$$

We will use the simpler symbolic notation of (2.2) throughout this paper.

Because of the mixing (2.2), the U_j are not multiplicatively renormalizable in the usual sense. The linear combinations

$$U_i' \equiv \sum_j V_{ji} U_j , \tag{2.4}$$

which are multiplicatively renormalizable, are determined by the matrix $\underline{V}$ which diagonalizes $\underline{Z}^{-1}$:

$$\underline{V}^{-1}\underline{Z}^{-1}\underline{V} = \underline{z} , \tag{2.5}$$

where $\underline{z}$ is diagonal, with entries z_k, $k=1,\ldots,N$, the eigenvalues of $\underline{Z}^{-1}$. Thus

$$R_j' \equiv \sum_j V_{ji} R_j = z_i U_i' . \tag{2.6}$$

The renormalization matrix will have the general form

$$Z_{ij}(K^2) = \sum_k (e^{(\ln t)\underline{d}})_{ik} C_{kj} , \tag{2.7}$$

where, for scale-invariant theories, $t = K^2/\mu^2$ (μ^2 is the normalization mass) and $\underline{d}$ is the "anomalous dimension" matrix, and for AF theories or in finite orders of perturbation theory,

$$t = \ln K^2/\mu^2 , \tag{2.8}$$

and $\underline{d}$ is the "effective anomalous dimension matrix." In matrix notation,

$$\underline{Z} = e^{(\ln t)\underline{d}} \underline{C} . \tag{2.9}$$

In terms of projection matrices $\underline{P}_i$, $i=1,\ldots,N$ which satisfy

$$\underline{P}_i \underline{P}_j = \delta_{ij} \underline{I} , \quad \sum_{i=1}^{N} \underline{P}_i = \underline{I} , \tag{2.10}$$

where $\underline{I}$ is the $N \times N$ unit matrix, we can write

$$\underline{d} = \sum_l d_l \underline{P}_l , \tag{2.11}$$

where $\{d_l ; l=1,\ldots,N\}$ are the eigenvalues of $\underline{d}$. Thus

$$e^{(\ln t)\underline{d}} = \sum_l e^{(\ln t) d_l} \underline{P}_l = \sum_l t^{d_l} \underline{P}_l , \tag{2.12}$$

and (2.9) becomes

$$\underline{Z} = \sum_l t^{d_l} \underline{P}_l \underline{C} , \tag{2.13}$$

so that (2.1) becomes

$$U_i = \sum_l t^{d_l} \sum_j R_j (\underline{P}_l \underline{C})_{ji} . \tag{2.14}$$

The leading degree of divergence of U_i for $K^2 \to \infty$ is then given by the largest eigenvalue d_L among the $\{d_l\}$:

$$U_i \underset{K^2 \to \infty}{\longrightarrow} t^{d_L} \sum_j R_j (\underline{P}_L \underline{C})_{ji} . \tag{2.15}$$

If $U_i(x)$ is expressible as the product

$$U_i(x) = a_i(x) b_i(x) \tag{2.16}$$

of unrenormalized fields a_i, b_i, and if the corresponding renormalized fields are

$$A_i = Z_{A_i}^{-1} a_i , \quad B_i = Z_{B_i}^{-1} b_i , \tag{2.17}$$

then (2.1) becomes

$$A_j(x) B_j(x) = \sum_i R_i(x) \tilde{Z}_{ij}(K^2) , \tag{2.18}$$

where

$$\tilde{Z}_{ij} = Z_{ij} Z_{A_j}^{-1} Z_{B_j}^{-1} . \tag{2.19}$$

Note that although (2.17) are cutoff independent, the local product $A_i(x)B_i(x)$ is not in general cutoff independent but rather has singularities for $K^2 \to \infty$ given by $\tilde{\underline{Z}}(K^2)$. The nonlocal product $A_i(x)B_i(0)$, on the other hand, *is* cutoff independent for $K^2 \to \infty$ but has singularities for $x \to 0$. If the $K^2 \to \infty$ and $x \to 0$ limits commute, then (2.18) can be recast as the OPE[16]

$$A_j(x)B_j(0) \underset{x\to 0}{\sim} \sum_i R_i(x)\tilde{Z}_{ij}(1/x^2)\,. \qquad (2.20)$$

We will see in the following section that (2.20) is in fact the correct OPE. If

$$Z_{A_i}{}^{-1}Z_{B_i}{}^{-1} \to t^{\delta_i}\,, \qquad (2.21)$$

(2.19), (2.20), and (2.13) give the explicit OPE

$$A_i(x)B_i(0) \underset{x\to 0}{\sim} \sum_l u^{d_l+\delta_i} \sum_j R_j(x)(\underline{P}_l\underline{C})_{ji}\,, \qquad (2.22)$$

where

$$u = \ln(-x^2\mu^2)\,. \qquad (2.23)$$

III. RENORMALIZATION-GROUP EQUATIONS

In this section we shall deduce the renormalization-group equations satisfied by the renormalization matrix $\underline{Z}$. We begin by recalling the simpler equations satisfied by the elementary renormalization constants Z_2 and Z_3 in NAGT's.[17] The renormalized YM vertex functions satisfy[18,5,19]

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n\gamma(g)\right]\Gamma^{(n)}(p_1,\ldots,p_n;g,\mu) = 0\,, \qquad (3.1)$$

where

$$\beta(g) = \mu\frac{\partial}{\partial\mu}g\bigg|_{g_0\ \text{fixed}}\,, \qquad (3.2)$$

$$\gamma(g) = \mu\frac{\partial}{\partial\mu}\ln Z_3\bigg|_{g_0\ \text{fixed}}\,, \qquad (3.3)$$

with g_0 the unrenormalized charge, g the renormalized charge, and μ the subtraction mass. The finiteness of $\Gamma^{(n)}$ implies the cutoff independence of β and γ. Expressing the derivative in (3.3) in terms of derivatives at fixed g gives

$$\left[-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}\right]\ln Z_3(t,g) \underset{t\to\infty}{\longrightarrow} \gamma(g)\,, \qquad (3.4)$$

which implies the RG equation[20]

$$\left[-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} - \gamma(g)\right]Z_3(t,g) = 0\,. \qquad (3.5)$$

Similarly, using[2]

$$g = g_0 Z_3^{3/2}/Z_1\,, \qquad (3.6)$$

(3.2) gives

$$\tfrac{3}{2}g\gamma - g\left(-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial g}\right)\ln Z_1 \underset{t\to\infty}{\longrightarrow} \beta\,, \qquad (3.7)$$

which implies the RG equation

$$\left[-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial g} + \left(\tfrac{3}{2}\gamma - \frac{\beta}{g}\right)\right]Z_1 = 0\,. \qquad (3.8)$$

Equations (3.5) and (3.8) can be summarized as

$$\left[-\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + \gamma_i(g)\right]Z_i(t,g) = 0\,, \qquad (3.9)$$

$$\gamma_i(g) = \lim_{t\to\infty}\frac{\partial}{\partial t}\ln Z_i(t,g)\,, \quad i = 1,3 \qquad (3.10)$$

so that

$$\gamma_1 = -\gamma\,, \quad \gamma_3 = \tfrac{3}{2}\gamma - \beta/g\,. \qquad (3.11)$$

It should be emphasized that the existence of the limits in (3.10) is the crucial ingredient in the derivation of (3.9).

The solutions to (3.9) have been discussed elsewhere.[17,21] For our purposes, we will only need the result

$$(Z_1/Z_3)^2 \underset{t\to\infty}{\sim} t^{-n(\alpha_c+3)/2\bar{b}} \qquad (3.12)$$

for the SU(n) gauge group. Here $\bar{b}$ is related to the (positive) lowest-order coefficient b in the expansion of $\beta(g)$:

$$\beta(g) = -bg^3 + O(g^5)\,, \quad \bar{b} = 16\pi^2 b \qquad (3.13)$$

and α_c is the limiting value of the effective gauge parameter $\bar{\alpha}(t,g)$,[17]

$$\bar{\alpha}(t,g) \underset{t\to\infty}{\longrightarrow} \alpha_c\,. \qquad (3.14)$$

This function arises in connection with the term $\delta(g)\partial/\partial\alpha$ in the RG equations[5] which we have suppressed. In terms of the constants C_1 and C_2, defined by

$$f^{acd}f^{bcd} = 2C_1\delta^{ab}\,, \qquad (3.15)$$

$$\mathrm{tr}(T^aT^b) = 2C_2\delta^{ab}\,, \qquad (3.16)$$

where T^a are the fermion representation matrices, one has

$$\alpha_c = 0 \text{ for } C_2/C_1 > \tfrac{13}{8} \qquad (3.17)$$

and

$$\alpha_c = \tfrac{13}{3} - \tfrac{8}{3}C_2/C_1 \text{ for } C_2/C_1 < \tfrac{13}{8}\,. \qquad (3.18)$$

We recall that[5]

$$b = \frac{1}{8\pi^2}\left(\tfrac{11}{3}C_1 - \tfrac{4}{3}C_2\right)\,, \qquad (3.19)$$

so that for AF ($b > 0$) one needs $C_2/C_1 < \frac{11}{4}$.

We consider next the renormalization of composite operators. For generality we treat a renormalized operator product $A(x)B(0)$, where inessential indices have been suppressed, which can be expanded as

$$A(x)B(0) \underset{x\to 0}{\sim} \sum_{n,i} E_i^{(n)}(x)R_i^{(n)}(0)\,. \qquad (3.20)$$

Here n denotes the Lorentz and internal attributes carried by $R^{(n)}$, and i enumerates the operators with identical quantum numbers.

We define the renormalization constants as usual by

$$a = Z_A A\,, \tag{3.21a}$$

$$b = Z_B B\,, \tag{3.21b}$$

where a, b are the unrenormalized field operators. Under renormalization the set $\{R_i^{(n)} | i = 1, \ldots\}$ can mix among themselves, as described by a renormalization matrix $Z^{(n)}(t)$, where

$$R_i^{(n)} = \sum_j U_j^{(n)} (\underline{Z}^{(n)-1})_{ji}\,, \tag{3.22}$$

in terms of the unrenormalized operators $U_j^{(n)}$. We can thus write the unrenormalized OPE

$$a(x)b(0) \underset{x\to 0}{\sim} \sum_{n,i,j} E_i^{(n)}(x) U_j^{(n)}(0) (Z^{(n)-1})_{ji} Z_A Z_B\,. \tag{3.23}$$

By varying the renormalization point μ in Eq. (3.23), we obtain[18,22]

$$\sum_j \left[\delta_{ij}\left(\mu \frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) + \gamma_{ij}(g)\right] E_j^{(n)}(x;\mu) = 0\,, \tag{3.24}$$

where

$$-\gamma_{ij} = \sum_k \mu \frac{\partial}{\partial\mu} (Z_A^{-1} Z_B^{-1} Z^{(n)})_{ik}\bigg|_{g_0\ \text{fixed}} \times Z_A Z_B (Z^{(n)-1})_{kj}\,. \tag{3.25}$$

The fact that (3.21) and (3.22) renders all operators cutoff independent implies that $\gamma_{ij}(g)$ is also cutoff independent. It therefore follows from (3.25) that the matrix

$$\underline{\tilde{Z}}^{(n)} = \underline{Z}^{(n)} Z_A^{-1} Z_B^{-1} \tag{3.26}$$

satisfies the RG equation

$$\sum_j \left[\delta_{ij}\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) + \gamma_{ij}(g)\right] \tilde{Z}_{jk}^{(n)}\left(\frac{K}{\mu}\right) = 0\,, \tag{3.27}$$

an equation identical in form to the RG equation (3.24) satisfied by the singular functions $E_j^{(n)}(x;\mu)$. Thus these equations imply that the behavior of Z as $K^2 \to \infty$ is the same as the behavior of E as $x^2 \to 0$.

We shall now specialize to the case in which $U_i^{(n)}$ is bilinear in some fields, say

$$U_i^{(n)}(x) = c_i(x) d_i(x)\,. \tag{3.28}$$

We write the contribution of $R_i^{(n)}$ to the renormalization of $U_j^{(n)}$ as

$$U_j^{(n)}(x) = Z_{C_i} Z_{D_i} R_i^{(n)}(x) \zeta_{ij} + \cdots\,, \tag{3.29}$$

so that

$$Z_{ij}^{(n)} = \zeta_{ij}^{(n)} Z_{C_i} Z_{D_i} \tag{3.30}$$

and

$$\tilde{Z}_{ij}^{(n)} = \zeta_{ij}^{(n)} Z_{C_i} Z_{D_i} Z_A^{-1} Z_B^{-1}\,. \tag{3.31}$$

In the case where $C_i = A$, $D_i = B$ we then have

$$\tilde{Z}_{ij}^{(n)} = \zeta_{ij}^{(n)}\,. \tag{3.32}$$

The input for the RG calculation is the second-order evaluation of the matrix $\underline{\tilde{Z}}$. Suppose we would like to determine the element of $\underline{\tilde{Z}}$ for the contribution

$$ab = {:}CD{:}\ \ Z_C Z_D \zeta + \cdots\,, \tag{3.33}$$

where $:CD:$ stands for the renormalized operator R. Taking matrix elements of (3.33) between $\langle C|$ and $|D\rangle$ one-particle states to lowest order isolates the contribution of $:CD:$, giving

$$\langle C|AB|D\rangle^{(2)} = \langle C|{:}CD{:}|D\rangle^{(0)} (Z_A^{-1} Z_B^{-1} Z_C Z_D \zeta)^{(2)} + \langle C|{:}CD{:}|D\rangle^{(2)} (Z_A^{-1} Z_B^{-1} Z_C Z_D \zeta)^{(0)}\,. \tag{3.34}$$

The superscripts on the matrix elements indicate the order in g to which they are to be evaluated. Since $:CD:$ is a finite operator, the singular contributions to (3.34) originate only from the first term on the right side, and so the calculation determines the contribution $Z_{ijAB} = \zeta_{ij} Z_{C_i} Z_{D_i} Z_A^{-1} Z_B^{-1}$ [see (3.31)] directly to second order.

For $C = A$, $D = B$, (3.34) simplifies to

$$\langle A|AB|B\rangle^{(2)} = \langle A|{:}AB{:}|B\rangle^{(0)} \zeta^{(2)} + \langle A|{:}AB{:}|B\rangle^{(2)}\,, \tag{3.35}$$

and Z_{AB} is now simply ζ in (3.35).

Returning to Eq. (3.27), we find its solution to be ($t = \ln K/\mu$)

$$\tilde{Z}_{ik}^{(n)}(t,g) = \sum_j \left\{ T \exp\left[-\int_0^t dt' \underline{\gamma}^{(n)}(\bar{g}(t',g))\right]\right\}_{ij} \times \tilde{Z}_{jk}^{(n)}(0, \bar{g}(t,g))\,, \tag{3.36}$$

where T is the time-ordering operator and $\bar{g}(t,g)$ is the effective charge. We define

$$\underline{\gamma}^{(n)}(g) = \underline{\gamma}_0^{(n)} g^2 + \underline{R}^{(n)}(g)\,. \tag{3.37}$$

Then

$$\tilde{Z}_{ik}^{(n)}\left(\frac{\Lambda}{\mu}, g\right) \underset{t\to\infty}{\sim} \sum_j \left[\exp\left(-\frac{\ln t}{b}\underline{\gamma}_0^{(n)}\right)\underline{M}^{(n)}\right]_{ij} \times \tilde{Z}_{jk}^{(n)}(1, \bar{g}(t,g))\,, \tag{3.38}$$

where we need not exhibit the expression for the "mixing" matrix $\underline{M}$.

IV. SU(2) GAUGE GROUP

In this section we shall illustrate the RG techniques for determining operator-product singularities by calculating explicitly the renormalization required for the composite operator $A_\mu^m A_\nu^n$ for the gauge group SU(2). Thus we must first determine the set of operators with identical quantum numbers that mix with $A^m_\mu A^n_\nu$ under renormalization. The first kind consists of the bilinears in A, of which there can be five independent combinations: (1) $:A^m_\mu A^n_\nu:$, (2) $:A^m_\nu A^n_\mu:$, (3) $g_{\mu\nu}:A^m_\lambda A^{n\lambda}:$, (4) $\delta^{mn}:A^l_\mu A^l_\nu:$, (5) $g_{\mu\nu}\delta^{mn}:A^l_\lambda A^{l\lambda}:$. (We will see that directional-dependent singularities are nonleading.) The second kind is made from the ghost operators: (1) $g_{\mu\nu}(\overline{C}^m C^n + \overline{C}^n C^m)$, (2) $g_{\mu\nu}\delta^{mn}\overline{C}C$. We have here written $\overline{C}=C_2$, $C=C_1$. The third kind is linear in the derivative of A: $\epsilon^{lmn}(\partial_\mu A^l_\nu - \partial_\nu A^l_\mu)$.

We shall then be required to calculate the eigenvalues of the renormalization matrix of this set of operators to lowest nontrivial order. The task is greatly simplified by observing that the renormalization matrix can be chosen to be triangular in this case. For the composite operators bilinear in A, we set

$$A^m_\mu A^n_\nu = K_1 :A^m_\mu A^n_\nu: + K_2 :A^m_\nu A^n_\mu: + K_3 g_{\mu\nu}:A^m_\lambda A^{n\lambda}: + K_4\delta^{mn}:A^l_\mu A^l_\nu: + K_5 g_{\mu\nu}\delta^{mn}:A^l_\lambda A^{l\lambda}: + (\text{other operators}), \tag{4.1}$$

and we learn from (4.1) that, apart from ghost and ∂A contributions,

$$[A^m_\mu A^n_\nu]^+ = (K_1+K_2):[A^m_\mu A^n_\nu]^+: + K_3 g_{\mu\nu}:A^m_\lambda A^{n\lambda}: + K_4\delta^{mn}:A^l_\mu A^l_\nu: + K_5 g_{\mu\nu}\delta^{mn}_\lambda:A^l_\lambda A^{l\lambda}:, \tag{4.2a}$$

$$[A^m_\mu A^n_\nu]^- = (K_1-K_2):[A^m_\mu A^n_\nu]^-:, \tag{4.2b}$$

$$g_{\mu\nu}A^m_\lambda A^{n\lambda} = (K_1+K_2+4K_3)g_{\mu\nu}:A^m_\lambda A^{n\lambda}: + (K_4+4K_5)g_{\mu\nu}\delta^{mn}:A^l_\lambda A^{l\lambda}:, \tag{4.2c}$$

$$\delta^{mn}A^l_\mu A^l_\nu = (K_1+K_2+3K_4)\delta^{mn}:A^l_\mu A^l_\nu: + (K_3+3K_5)g_{\mu\nu}\delta^{mn}:A^l_\lambda A^{l\lambda}:, \tag{4.2d}$$

$$g_{\mu\nu}\delta^{mn}A^l_\lambda A^{l\lambda} = (K_1+K_2+4K_3+3K_4+12K_5):A^l_\lambda A^{l\lambda}:, \tag{4.2e}$$

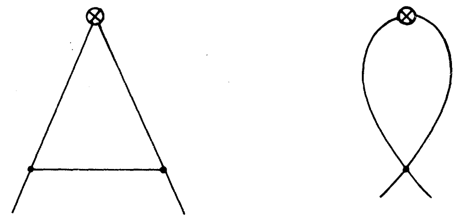

FIG. 1. Second-order Feynman diagrams giving the $:AA:$ terms in the expansion of the product AA of gauge fields. The solid lines between vertices are the gauge field propagators, the dots are the cubic ($\sim g$) or quartic ($\sim g^2$) gauge field couplings, and the cross is the AA vertex. The loop integrals are logarithmically divergent and so give rise to the factor $\ln K^2$ when a cutoff $\boldsymbol{K}$ is introduced. The Feynman rules are given, for example, in Ref. 5.

where

$$[A^m_\mu A^n_\nu]^\pm = \tfrac{1}{2}(A^m_\mu A^n_\nu \pm A^m_\nu A^n_\mu). \tag{4.3}$$

This sector of the renormalization matrix is thus clearly triangular. As for the sector involving ghost operators, we shall define the renormalization constants by

$$\overline{C}^m C^n + \overline{C}^n C^m = N_0:(\overline{C}^m C^n + \overline{C}^n C^m): + N_0'\delta^{mn}:\overline{C}\cdot C: + N_1:A^m_\lambda A^{n\lambda}: + N_2\delta^{mn}:A^l_\lambda A^{l\lambda}: \tag{4.4}$$

and

$$A^m_\mu A^n_\nu = [\text{right-hand side of (4.1)}] + M_1 g_{\mu\nu}:\overline{C}^m C^n + \overline{C}^n C^m: + M_2 g_{\mu\nu}\delta^{mn}:\overline{C}\cdot C:. \tag{4.5}$$

We can now calculate these constants to second order in g. The $K_1, \ldots, K_5$ are determined by the diagrams in Fig. 1. Explicit evaluation gives

$$(\text{Fig. 1}) = L\{(-2\delta^{mn}\delta^{ab} + \delta^{ma}\delta^{nb} + \delta^{na}\delta^{mb})[-\tfrac{1}{4}(1+\alpha)g_{\mu\nu}g_{\alpha\beta} - \tfrac{1}{4}(1+\alpha)(g_{\mu\alpha}g_{\nu\beta} + g_{\nu\alpha}g_{\mu\beta})] + \tfrac{3}{2}(-\delta^{ma}\delta^{nb} + \delta^{mb}\delta^{na})(g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta})\}, \tag{4.6}$$

from which we obtain

$$K_1 = -\tfrac{1}{4}(7+\alpha)L, \tag{4.7a}$$

$$K_2 = \tfrac{1}{4}(+5-\alpha)L, \tag{4.7b}$$

$$K_3 = -\tfrac{1}{4}(1+\alpha)L, \tag{4.7c}$$

$$K_4 = +\tfrac{1}{2}(1+\alpha)L, \tag{4.7d}$$

$$K_5 = +\tfrac{1}{4}(1+\alpha)L. \tag{4.7e}$$

We have used the abbreviation

$$L = \frac{g^2}{16\pi^2}\ln K\,. \tag{4.8}$$

(Directional-dependent singularities are seen to be absent.)

This sector of the $\underline{\gamma}$ matrix is thus

$$\underline{\gamma} = \begin{array}{c} \\ + \\ - \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} + & - & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} K_1+K_2 & 0 & 0 & 0 & 0 \\ 0 & K_1-K_2 & 0 & 0 & 0 \\ K_3 & 0 & K_1+K_2+4K_3 & 0 & 0 \\ K_4 & 0 & 0 & K_1+K_2+3K_4 & 0 \\ K_5 & 0 & K_4+4K_5 & K_3+3K_5 & K_1+K_2+4K_3+3K_4+12K_5 \end{bmatrix} \end{array} \times \frac{1}{\ln K}\,, \tag{4.9}$$

and the diagonal entries are

$$\begin{aligned} \gamma_{++} &= -\tfrac{1}{2}(\alpha+1)\frac{g^2}{16\pi^2}\,, \\ \gamma_{--} &= -3\frac{g^2}{16\pi^2}\,, \\ \gamma_{33} &= -\tfrac{3}{2}(\alpha+1)\frac{g^2}{16\pi^2}\,, \\ \gamma_{44} &= (\alpha+1)\frac{g^2}{16\pi^2}\,, \\ \gamma_{55} &= 3(\alpha+1)\frac{g^2}{16\pi^2}\,. \end{aligned} \tag{4.10}$$

By a well-known theorem, the eigenvalues of such a matrix are just the diagonal elements, and so for our purposes it is never necessary to know the off-diagonal elements.

Composite ghost operators such as $:\bar{C}^m C^n + \bar{C}^n C^m:$, which can by quantum-number considerations occur in the expansion of AA, do not actually contribute. The structure of the ghost Lagrangian, $\partial_\mu \bar{C}\mathfrak{D}^\mu C$, means that only $:(\partial_\mu \bar{C})C:$ can occur in the expansion, and that is of dimension 3. This observation can be explicitly verified from the Feynman diagrams to all orders in g. (See, for example, Fig. 2.) We can thus exclude the ghost operators from consideration in the renormalization matrix. On the other hand, the $\bar{C}C$ expansion does involve contributions from $:AA:$ (see Fig. 3), in addition to $\bar{C}C$ (see Fig. 4). We can of course consider this larger renormalization matrix for the set of operators $\{AA\}\cup\{\bar{C}C\}$. The result of the computation, using this Z, for the AA singularity is of course the same as if the ghost operators are ignored in the first place. We have explicitly verified this.

The contribution from terms of the type ∂A merits a separate discussion. They do occur in the expansion of AA, but they are trivial to renormalize. In particular, no $:AA:$ terms occur on the right-hand side of ∂A. Thus these contributions occur in the same triangular configuration. The diagonal element in the $\underline{Z}$ matrix is of course

$$Z_{\partial A \partial A} = Z_3^{\,1/2}\,. \tag{4.11}$$

We write schematically

$$AA \to E_1 :AA: + E_2 \partial A\,, \tag{4.12}$$

and then the functions E_1 and E_2 satisfy

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)E_1 = Ag^2 E_1\,, \tag{4.13}$$

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right)E_2 = BgE_1 + Cg^2 E_2\,. \tag{4.14}$$

Some care is needed in solving this pair of RG equations because here the coefficient Bg of E_1 is only of first order in g. We report here the solution[23]:

$$E_1 \sim (\ln x^2)^{-A/2\bar{b}}\,, \tag{4.15}$$

$$E_2 \sim \max\{E_1, (\ln x^2)^{-C/2\bar{b}+1/2}\}\,. \tag{4.16}$$

In reality E_1 includes only those components of $:AA:$ that are antisymmetric under the internal

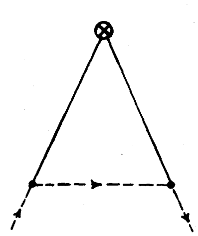

FIG. 2. Second-order Feynman diagram giving the ghost field product $:\bar{C}C:$ terms in the expansion of the product AA of gauge fields. The dashed line between vertices is the ghost field propagator and the dots are the ghost-gauge field couplings. The loop integral is convergent, corresponding to the fact that it is actually the operator $:\partial_\mu \bar{C}C:$ of dimension 3 which occurs.

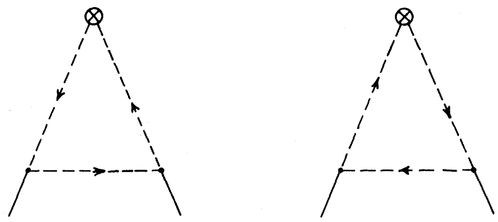

FIG. 3. Second-order Feynman diagrams giving the $:AA:$ contributions to the product $\bar{C}C$. The cross now represents the $\bar{C}C$ vertex. The loop integral is logarithmically divergent.

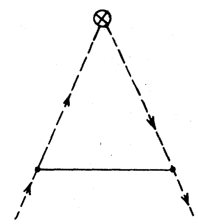

FIG. 4. Second-order Feynman diagram giving the $:\bar{C}C:$ contributions to the product $\bar{C}C$. The loop integral is logarithmically divergent.

symmetry, since ∂A contributes only to that sector. The second quantity in the max $\{\}$ can be recognized as just Z_3/Z_1, using $Z_1/Z_3{}^{3/2} \sim (\ln x^2)^{-1/2}$. For SU(2), the antisymmetric part γ_{--} in (4.10) is negative, and so E_1 actually vanishes. E_2 is then simply $= Z_3/Z_1$.

This analysis works for any gauge group. The ∂A contribution always provides a minimum singularity of Z_3/Z_1 to the complete AA expansion. Thus our subsequent statements about the singularity of the AA operator product should be understood in that sense.

From the foregoing analysis, the maximum singularity of the operator product is determined to be $[b = (1/16\pi^2)\bar{b},\ \beta(g) = -bg^3 + \cdots]$

$$A_\mu^m(x) A_\nu^n(0) \underset{x\to 0}{\sim} [\ln(x^2)^{-1/2}]^{3(\alpha_c+1)/2\bar{b}} O_{\mu\nu}^{mn}(0)\,, \quad (4.17)$$

with $O_{\mu\nu}^{mn}$ finite. The limiting value α_c for a theory depends in the usual way (3.17) and (3.18) on the representation content of other fields.

V. SU(n) GAUGE GROUP

We have chosen the group SU(2) for illustrative purposes in Sec. IV. The simple relation

$$\epsilon^{abl}\epsilon^{lcd} = \delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \quad (5.1)$$

for SU(2) enables the enumeration of all the operators that can mix, and the renormalization matrix can be put in triangular form. What we need for SU(n) is a group-theoretic analysis analogous to (5.1) that enumerates the representation content of the various operators that can mix.

The calculation of the Feynman diagrams (Fig. 1) is no more complicated, and we get similar results for a general SU(n) group. Following the discussion in Sec. IV we need only concern ourselves with the $:AA:$ contribution to the AA expansion. The lowest-order result is

$$A_\mu^a A_\nu^b \to (E_1^{abcd}:A_\mu^c A_\nu^d: + E_2^{abcd} g_{\mu\nu}:A^c\cdot A^d:)L\,, \quad (5.2)$$

where

$$E_1 = -f^{abl}f^{cdl} + f^{adl}f^{bcl} + \tfrac{1}{4}(\alpha - 1)(f^{acl}f^{bdl} + f^{adl}f^{bcl})\,, \quad (5.3)$$

$$E_2 = \tfrac{1}{8}(\alpha + 1)(f^{acl}f^{bdl} + f^{adl}f^{bcl})\,, \quad (5.4)$$

and we have used the abbreviation (4.8).

Equation (5.2) can be easily diagonalized in its space-time indices to give two independent equations:

$$\delta^{ac}\delta^{bd}A^c\cdot A^d \to (E_1 + 4E_2)^{abcd}:A^c\cdot A^d:L\,, \quad (5.5)$$

$$\delta^{ac}\delta^{bd}(A_\mu^c A_\nu^d - \tfrac{1}{4}g_{\mu\nu}A^c\cdot A^d) \to E_1^{abcd}:(A_\mu^c A_\nu^d - \tfrac{1}{4}g_{\mu\nu}A^c\cdot A^d):L\,. \quad (5.6)$$

The coefficient in (5.5) is

$$E_1 + 4E_2 = -f^{abl}f^{cdl} + \tfrac{1}{4}(3\alpha + 1)f^{acl}f^{bdl} + \tfrac{1}{4}(3\alpha + 5)f^{adl}f^{bcl}\,. \quad (5.7)$$

The aim now is to decompose (5.5) and (5.6) according to the internal-group symmetry into a number of diagonal, independent equations, each of which can be the input to an RG equation. The key to the solution of the problem is to realize that the tensorial forms occurring in these equations are just proportional to projection operators for the identity and the regular representations in either the s, t, or u channels, where s corresponds to $ab \to cd$, t to $ac \to bd$, and u to $ad \to bc$. Thus if we were to express all the projection operators in the t and u channels in terms of those in the s channel, we would arrive at a number of independent s-channel projections of the original equations. Each of these projected equations can then be the lowest-order inputs to the RG equations. The singularity of the original expansion then coincides with the highest singularity found among these projections.

The projection operators P_β^t in the t and P_β^u in the u channels onto the representation β are in general expressed in terms of those in the s channel ($P_\alpha^s \equiv P_\alpha$) by means of crossing matrices. Specifically,

$$P_\beta^t(ab\,|\,cd) = \sum_\alpha C_{\beta\alpha}^{ts} P_\alpha^s(ab\,|\,cd)\,, \quad (5.8)$$

$$P_\beta^u(ab\,|\,cd) = \sum_\alpha C_{\beta\alpha}^{us} P_\alpha^s(ab\,|\,cd)\,, \quad (5.9)$$

where C is the crossing matrices. They exist in principle for any group, and have been computed in the literature for the groups SU(n).[24]

In Eqs. (5.5) and (5.6) we are dealing with (regular representation)-(regular representation) "scattering" in SU(n). The s-channel $(ab \to cd)$ projection operators are[24] $P_I, P_R, P_{R'}, P_{AS}, P_{SA}, P_{AA}, P_{SS}$, where, for instance,

$$P_{SS}(abcd) = \sum_{ijkl} T_{SS}(ab)^{ik}_{jl}[T_{SS}(cd)^{ik}_{jl}]^\dagger \,. \tag{5.10}$$

The tensors T_{SS}, T_{SA}, T_{AS}, T_R, $T_{R'}$, and T_I correspond to the $\underline{27}$, $\underline{10}$, $\underline{10}$, $\underline{8}$, $\underline{8}'$, and $\underline{1}$, respectively in the case of $\mathrm{SU}(3)$. T_{AA} first makes its appearance in SU(4). The exact forms of the T's need not concern us, except for the I and R' representations:

$$T_I(ab) = \frac{1}{(n^2-1)^{1/2}} + \mathrm{tr}(\lambda^a\lambda^b) = \frac{1}{(n^2-1)^{1/2}}\, 2\delta^{ab} \,, \tag{5.11}$$

$$T_{R'}(ab)^i_j = \frac{1}{\sqrt{2n}}\,[(\lambda^a\lambda^b)^i_j - (\lambda^b\lambda^a)^i_j] = \frac{1}{\sqrt{2n}}\, 2if^{abc}(\lambda^c)^i_j \,. \tag{5.12}$$

Thus

$$P_I(ab\,|\,cd) = \frac{4}{n^2-1}\,\delta^{ab}\delta^{cd} \tag{5.13}$$

and

$$P_{R'}(ab\,|\,cd) = \frac{4}{n} f^{abl} f^{cdl} \,. \tag{5.14}$$

Now Eqs. (5.5) and (5.6) can be expressed via Eqs. (5.13) and (5.14) entirely in terms of P_I and $P_{R'}$ in the s, t, and u channels:

$$(n^2-1)P_I(ac\,|\,bd)A^c\cdot A^d \to n\left[-P_{R'}(ab\,|\,cd) + \tfrac{1}{4}(3\alpha+1)P_{R'}(ac\,|\,bd) + \tfrac{1}{4}(3\alpha+5)P_{R'}(ad\,|\,bc)\right]:A^c\cdot A^d:L \,, \tag{5.15}$$

$$(n^2-1)P_I(ac\,|\,bd)(A^c_\mu A^d_\nu - \tfrac{1}{4}g_{\mu\nu}A^c\cdot A^d) \to n\{-P_{R'}(ab\,|\,cd) + P_{R'}(ad\,|\,bc) + \tfrac{1}{4}(\alpha-1)[P_{R'}(ac\,|\,bd) + P_{R'}(ad\,|\,bc)]\}:(A^c_\mu A^d_\nu - \tfrac{1}{4}g_{\mu\nu}A^c\cdot A^d):L \,. \tag{5.16}$$

The decomposition of (5.15) and (5.16) into s-channel projection operators is easily made with the help of the tables of crossing matrices.[24]

t channel	P^t_I	P^t_R	$P^t_{R'}$	P^t_{SA}	P^t_{AS}	P^t_{AA}	P^t_{SS}
P_I	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$
$P_{R'}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{n}$	$-\frac{1}{n}$

u channel	P^u_I	P^u_R	$P^u_{R'}$	P^u_{SA}	P^u_{AS}	P^u_{AA}	P^u_{As}
P_I	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{-1}{n^2-1}$	$\frac{-1}{n^2-1}$	$\frac{-1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$
$P_{R'}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{n}$	$-\frac{1}{n}$

Equations (5.15) and (5.16) thus each decompose into seven independent diagonal equations:

$$\begin{bmatrix} P_I \\ P_R \\ P_{R'} \\ P_{SA} \\ P_{AS} \\ P_{AA} \\ P_{SS} \end{bmatrix}^{(ab|cd)} A^c\cdot A^d = \begin{bmatrix} \frac{3}{2}n(\alpha+1)P_I \\ \frac{3}{4}n(\alpha+1)P_R \\ -\frac{3}{2}nP_{R'} \\ 0 \\ 0 \\ \frac{3}{2}(\alpha+1) \\ -\frac{3}{2}(\alpha+1) \end{bmatrix}^{(ab|cd)} :A^c\cdot A^d:L \,, \tag{5.17}$$

$$\begin{bmatrix} P_I \\ P_R \\ P_{R'} \\ P_{SA} \\ P_{AS} \\ P_{AA} \\ P_{SS} \end{bmatrix}^{(ab|cd)} (A_\mu^c A_\nu^d - \tfrac{1}{4} g_{\mu\nu} A^c\cdot A^d) = \begin{bmatrix} \frac{1}{2}n(\alpha+1)P_I \\ \frac{1}{4}n(\alpha+1)P_R \\ -\frac{1}{2}nP_{R'} \\ 0 \\ 0 \\ \frac{1}{2}(\alpha+1)P_{AA} \\ -\frac{1}{2}(\alpha+1)P_{SS} \end{bmatrix}^{(ab|cd)} :(A_\mu^c A_\nu^d - \tfrac{1}{4} g_{\mu\nu} A^c\cdot A^d):L\,. \tag{5.18}$$

Equations (5.17) and (5.18) are actually 14 independent operator-product expansions, and the independent projections behave according to the result of 14 uncoupled RG equations. Thus we have

$$A^a\cdot A^b = \tfrac{1}{4}\sum_l P_l(ab\,|\,cd):A^c\cdot A^d:[\ln(x^2)^{-1/2}]^{d_l}\,, \tag{5.19}$$

where

$$\begin{aligned} d_I &= (2\bar b)^{-1}\tfrac{3}{2}n(\alpha_c+1)\,,\\ d_R &= (2\bar b)^{-1}\tfrac{3}{4}n(\alpha_c+1)\,,\\ d_{R'} &= (2\bar b)^{-1}(-\tfrac{3}{2})n\,,\\ d_{SA} &= 0\,,\\ d_{AS} &= 0\,,\\ d_{AA} &= (2\bar b)^{-1}\tfrac{3}{2}(\alpha_c+1)\,,\\ d_{SS} &= (2\bar b)^{-1}(-\tfrac{3}{2})(\alpha_c+1)\,, \end{aligned} \tag{5.20}$$

and

$$A_\mu^a A_\nu^b - \tfrac{1}{4}g_{\mu\nu}A^a\cdot A^b = \tfrac{1}{4}\sum_l P_l(ab\,|\,cd):(A_\mu^c A_\nu^d - \tfrac{1}{4}g_{\mu\nu}A^c\cdot A^d): \times[\ln(x^2)^{-1/2}]^{D_l}\,, \tag{5.21}$$

where

$$\begin{aligned} D_I &= (2\bar b)^{-1}\tfrac{1}{2}n(\alpha_c+1)\,,\\ D_R &= (2\bar b)^{-1}\tfrac{1}{4}n(\alpha_c+1)\,,\\ D_{R'} &= (2\bar b)^{-1}(-\tfrac{1}{2})n\,,\\ D_{SA} &= 0\,,\\ D_{AS} &= 0\,,\\ D_{AA} &= (2\bar b)^{-1}\tfrac{1}{2}(\alpha_c+1)\,,\\ D_{SS} &= (2\bar b)^{-1}(-\tfrac{1}{2})(\alpha_c+1)\,. \end{aligned} \tag{5.22}$$

For the cases where d_l or D_l vanish, the main contribution is no longer given by the RG, and the finite coefficient is not calculable in this framework.[17,21] By inspection, the largest singularity occurs in $P_I A^c\cdot A^d$ and this is the dominant singularity in the $A_\mu^a A_\nu^b$ expansion (5.2):

$$A_\mu^a(x)A_\nu^b(0) \underset{x\to 0}{\sim} [\ln(x^2)^{-1/2}]^{(3/2)n(\alpha_c+1)/2\bar b} \times\frac{1}{n^2-1}\delta^{ab}g_{\mu\nu}:A\cdot A:\,. \tag{5.23}$$

VI. DISCUSSION

Equations (5.19)–(5.22) are the main results of this paper. They completely specify the bilinear singularity structure of the product $A_\mu^a(x)A_\nu^b(0)$ for $x\to 0$ and they precisely define the finite local composite operators $:A_\mu^a(x)A_\nu^b(0):$. Together with the vacuum expectation value and the result (4.16) for the linear singularity structure, we have the complete and explicit OPE

$$\begin{aligned} A_\mu^a(x)A_\nu^b(0) - \langle 0|A_\mu^a(x)A_\nu^b(0)|0\rangle &\\ + \text{const}\times E_2(x)f^{abc}(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c)&\\ + (\text{bilinear terms})\,.& \end{aligned} \tag{6.1}$$

These results were derived using RG methods in AF field theories. The various dimension-2 operators mix upon renormalization as in (2.1), giving rise to a mixed OPE as in (2.22). The OPE coefficient functions $E_i^{(n)}(x)$ [Eq. (3.20)] and the renormalization-constant matrix $\tilde Z_{jk}^{(n)}(t)$ [Eq. (3.26)] satisfy the same RG equation [(3.24), (3.27)]. Because of the AF, the solutions (3.38) are given in terms of the lowest-order results (3.34). For the SU(2) gauge group, the explicit mixing relation is given in (4.1) and the explicit lowest-order evaluation (4.6) gives the γ matrix, which can be put in the triangular form (4.9). Using (3.38), the OPE can then immediately be put in the general form (2.22). For the general SU(n) gauge group, it is more convenient to diagonalize the renormalization procedure, as in Eqs. (2.4)–(2.6). Because the operators that mix are related by exact Lorentz or SU(n) group rotations, the diagonalization can be explicitly performed. The original lowest-order result (5.2) was thus written in terms of projection operators, Eqs. (5.15) and (5.16), and the SU(n) crossing relations then gave the diagonalized relations (5.17) and (5.18). The then uncoupled RG equations gave the results (5.19)–(5.22).

It was, of course, the simplicity of the operators (1.2)–(1.4) under consideration which enabled the explicit diagonalization to be performed in this case. In a more general problem, the mixing matrix in (3.38) cannot be completely determined by our methods. In this respect, our analysis simply illustrates the general problem in a completely soluble situation.

There is at least one case in which our results are of practical, rather than illustrative, importance. This is for an investigation of the further symmetries of quantum field theories which arise as a *consequence* of renormalization, in the sense of Ref. 4. In AF theories, the short-distance behaviors, and in particular the renormalized field equations, are exactly known. It was previously shown how, because of the exact vanishing of appropriate combinations of renormalization constants, the equations can be invariant to a larger symmetry group than are the classical field equations.[4] The results of this paper can be used to investigate this possibility in more detail for NAGT's.

Consider the renormalized YM field equations which follow from the Lagrangian (1.5). These equations will all be formally invariant to the "Abelian" gauge transformations

$$A_\mu^a(x) \to A_\mu^a(x) + \frac{1}{g}\partial_\mu \Lambda^a(x), \quad \psi(x) \to \psi(x), \quad C_i(x) \to C_i(x), \tag{6.2}$$

for arbitrary c-number functions $\Lambda^a(x)$, provided the quantities

$$\frac{Z_1}{Z_3}, \quad \frac{Z_1}{Z_3}A_\mu^a(x), \quad \frac{Z_1}{Z_3}\partial_\mu A_\nu^a(x), \tag{6.3}$$

and

$$\left(\frac{Z_1}{Z_3}\right)^2 A_\mu^a(x) A_\nu^b(x) \tag{6.4}$$

all vanish.[4] Because of the vanishing Z_1/Z_3 in *all* AF NAGT's [cf. Eq. (3.12)], and the finiteness of the field operators $1, A_\mu^a, \partial_\mu A_\nu^a$, the products (6.3) do indeed vanish.[4] We are now in a position to determine when (6.4) also vanishes.

The contribution of the c-number term in (6.1) is not relevant because vacuum subtractions are always assumed.[25] The linear $\partial_\mu A_\nu$ term in (6.1) contributes to (6.4) the product of $Z_1{}^2/Z_3{}^{5/2} = (Z_1/Z_3)(Z_1/Z_3{}^{3/2})$ and a finite field operator. Since each of Z_1/Z_3 and $Z_1/Z_3{}^{3/2} = g_0/g$ vanishes, this product also vanishes. It is therefore sufficient to consider the bilinear contributions to (6.4).

For the SU(n) gauge group, using (3.12), and using the result (5.23) for the leading singularity, we find

$$\left(\frac{Z_1}{Z_3}\right)^2 A_\mu^a(x) A_\nu^b(x) \to [\ln(x^2)^{-1/2}]^{n(\alpha_c - 3)/4\bar{b}} R_{\mu\nu}^{ab}. \tag{6.5}$$

This approaches zero provided

$$\alpha_c - 3 < 0. \tag{6.6}$$

For $C_2/C_1 > \frac{13}{8}$, $\alpha_c = 0$ and so (6.6) is satisfied. For $C_2/C_1 < \frac{13}{8}$, $\alpha_c = \frac{13}{3} - \frac{8}{3}(C_2/C_1)$, and so $\alpha_c - 3 = \frac{4}{3} - \frac{8}{3}C_2/C_1$, which is negative provided $C_2/C_1 > \frac{1}{2}$. Thus (6.6) is equivalent in general to

$$\frac{C_2}{C_1} > \frac{1}{2}. \tag{6.7}$$

So, within the AF range $\frac{11}{4} > C_2/C_1 \geq 0$, (6.6) holds for $\frac{11}{4} > C_2/C_1 > \frac{1}{2}$ and fails for $\frac{1}{2} > C_2/C_1 \geq 0$.

For SU(n), $C_1 = n/2$, and every fundamental representation of fermions contributes an amount $\frac{1}{4}$ to C_2, or

$$C_2 = m/4 \tag{6.8}$$

if there are m fundamental fermion representations. Thus (6.7) is equivalent to

$$m > n. \tag{6.9}$$

For the popular color SU(3) gauge group with three flavors ($\mathfrak{N}, \mathcal{P}, \lambda$), $n = 3$ and $m = 3$ so that (6.9) is not satisfied. On the other hand, for the same gauge group with four flavors ($\mathfrak{N}, \mathcal{P}, \lambda, c$), $m = 4$, and so (6.9) is satisfied. Experimentally,[26] it appears that $m \geq 4$ so that (6.9) and (6.7) are satisfied in nature,[27] and consequently (6.5) vanishes and one has Abelian gauge invariance.[28]

What about models with $C_2/C_1 < \frac{1}{2}$? Although such models are not believed to be of physical interest, it would for completeness be desirable to determine the status of the zero-momentum theorems there also. These models are not strictly R invariant because (6.4) does not vanish. However, since (6.3) continues to vanish, the transformations (6.2) induce a simple change in the field equations, and this may be enough to deduce the theorems. This is presently under investigation.

A source of optimism in this regard may be found in the AF[29] $g\phi^4$ theory with negative renormalized coupling constant. This model is not strictly R invariant because of the mass terms. However, the effects of these terms can be determined completely because of the AF, and the result is that the zero-momentum theorems are still valid. This means that the effective potential can vanish. The details will be given elsewhere.[25]

The renormalization-group methods employed in this paper have been straightforward but tedious. It would clearly be desirable to find more efficient ways to determine the desired exact information about short-distance behavior in AF theories. An attack on this problem has been made using quasicanonical[3] methods and has been reported on elsewhere.[30]

In conclusion, let us recall the roles of the essential ingredients in our analysis. It was the AF of the NAGT's which led to the solutions (3.38) of the RG equations with $\gamma_0^{(n)}$ explicitly known from the lowest-order Feynman diagram evaluations. It was the SU(n) group invariance which led to the explicit diagonalization which effectively determines $\underline{M}^{(n)}$ and thus led to the exact behavior of $\underline{Z}^{(n)}$. In both cases, the solubility of the problem was a consequence of an invariance property of the theory, renormalization convention invariance in the former case and SU(n) transformation invariance in the latter.

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