Quantum field theory and the two-dimensional Ising model*

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A relationship between the two-point correlation function of the Ising model in the critical region and the $\langle \sin\sqrt{\pi}\phi(0)\sin\sqrt{\pi}\phi(\rho)\rangle$ Green's function of a sine-Gordon field is constructed. This relationship is tested by means of a mass perturbation expansion with an infrared cutoff and shown to agree, at the leading-logarithm approximation and up to the second order, with the exact result of the Ising model.

I. INTRODUCTION

Ever since Onsager's derivation of the free energy,[1] the Ising model in two dimensions has remained a notorious problem in statistical mechanics. In this paper, we would like to exhibit a relationship with a field-theoretical model, and derive a well-known result of the Ising model by this new method.

Long ago, Schultz, Mattis, and Lieb[2] showed the equivalence of the Ising system with a free fermion gas in one dimension. This analogy can be used to compute its behavior in the critical domain characterized by $T \to T_c$, where T_c is the critical temperature, and $\rho \gg a$, where a is the lattice spacing, and ρ is a typical distance of interest. It turns out that at $T = T_c$, the fermions are massless, whereas departures from T_c can be described by the introduction of a mass term. On the other hand, it is well known in statistical mechanics[3] as well as in field theory[4-8] that a free Fermi field in two space-time dimensions can be parametrized as the exponential of a Bose field. In particular, there is an intimate relation between relativistic massive fermions with or without four-fermion coupling and the sine-Gordon (SG) interaction of this Bose field.[5-8] This parametrization enables us to connect the two-point correlation function of the Ising model in the so-called scaling limit with the $\langle \sin\sqrt{\pi}\,\phi(0)\sin\sqrt{\pi}\,\phi(\rho)\rangle$ Green's function of the sine-Gordon field. The behavior of the Ising correlation function in the scaling limit $[\rho \to \infty, \rho(T-T_c)$ fixed$]$ is well known in statistical mechanics. After some pioneering work,[9,10] the exact form of this scaling limit has been derived rigorously in the recent monumental work of Wu *et al.*[11] Here we shall use the mass perturbation developed by Coleman[5] for the SG field to investigate the leading corrections to the Ising correlation function in this scaling limit, when $\rho(T-T_c)$ is small.

In Sec. II, we introduce our notations and review briefly the results of Ref. 2. The third section is devoted to the critical domain. In Sec. III A, we show that expanding the transfer matrix near $T = T_c$ and near the mode $q = 0$, which is valid for the study of long wavelengths (with respect to lattice spacing), leads to the Hamiltonian of a free relativistic Majorana field of mass $m \propto T - T_c$.[12] In Sec. III B, we turn to the study of the two-spin correlation function. We introduce two noninteracting Ising systems, in order to be able to describe them in terms of a complex Fermi field. The latter enables us to use the above-mentioned parametrization of the free Fermi field in terms of a Bose field, the sine-Gordon field ϕ. A close analysis of the short-distance singularities which appear when the lattice spacing a goes to zero, or equivalently, when the distance ρ and the correlation length $1/m \sim (T-T_c)^{-1}$ increase with a fixed value of the ratio $m\rho$, reveals that the leading contribution comes from the $\langle \sin\sqrt{\pi}\,\phi(\rho)\sin\sqrt{\pi}\,\phi(0)\rangle$ Green's function of the SG field: This is done in Sec. III C. Although we believe this result to hold generally, we rely, for the actual computation of this Green's function, on a perturbation expansion, where the $\cos 2\sqrt{\pi}\phi$ interaction of the SG field is considered as a perturbation for the massless field. This mass perturbation is plagued with infrared divergences. A simple example in Sec. III D allows us to understand the mechanism of a natural infrared cutoff. However, this method does not enable us to go beyond the leading-logarithmic correction in $(m\rho \ln m\rho)^n$ to the behavior at the critical temperature. Nevertheless, the two first leading logarithms are found to agree with the exact result of Refs. 10 and 11. Finally, Sec. IV contains some comments on higher-order correlation functions and our concluding remarks.

Recently several authors[12-15] have investigated the field-theoretical formulation of the Ising problem along lines similar to ours. Ferrell[12] introduced a doubling of the number of degrees of freedom of the Ising model, in order to write the transfer matrix as the Hamiltonian of a Dirac field;

then he computed the logarithmic singularity of the specific heat. More recently, Luther and Peschel have used a relationship between the Baxter model[16] in two dimensions and the Luttinger[17] model in one dimension to calculate the critical exponents of a Baxter-type model: In the special case when the Baxter model reduces to the superposition of two noninteracting Ising lattices, their method is very similar, although not identical, to ours. Berg and Schroer have shown that the formulation of the Ising model in terms of Majorana fields may be used to study the correlation function. This was completed quite recently by Bander and one of the authors, who were able to prove that a definite Green's function of a massive Majorana field in an external field may be computed exactly and coincides with one of the various exact expressions obtained by Wu *et al.*[11] This is not in contradiction to the present work but rather shows fruitful connections between various models. On the other hand, the relationship between the Ising system and the sine-Gordon theory may be interesting for the latter.

II. NOTATIONS: REVIEW OF THE RESULTS OF SCHULTZ, MATTIS, AND LIEB

In this section, we introduce our notations and review the results[2] of Schultz, Mattis, and Lieb, for the sake of consistency in our paper. We refer the reader to the original paper of these authors and we shall use the same notations whenever possible.

A. The Ising model: notations

The partition function of a two-dimensional $N \times N$ square lattice is

$$Z = \sum_{\sigma_i=\pm 1} \exp\left(-\frac{E(\sigma)}{kT}\right) = \sum_{\sigma_i=\pm 1} \exp\left(K\sum_{(ij)} \sigma_i\sigma_j\right), \qquad (2.1)$$

where the energy $E(\sigma)$ is a sum of nearest-neighbor contributions. The free energy per site is defined as

$$F(K) = \lim_{N\to\infty} \frac{1}{N^2} \ln Z \qquad (2.2)$$

and the correlation functions

$$\langle \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2n}}\rangle = Z^{-1} \sum_{\sigma_i=\pm 1} \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{2n}} \times \exp\left(K\sum_{(ij)}\sigma_i\sigma_j\right). \qquad (2.3)$$

Onsager[1] was able to compute the function $F(K)$, showing in particular that its second derivative is singular at the transition point K_c given by

$$\sinh 2K_c = 1, \quad K_c = 0.4407\ldots . \qquad (2.4)$$

It is the purpose of the critical theory to investigate the singularity of $F(K)$ and the behavior of the correlation functions in the vicinity of $K = K_c$.

B. Transfer matrix formalism: Schultz-Mattis-Lieb results

The $2^N \times 2^N$ transfer matrix V connects configurations of successive rows on the lattice in such a way that for appropriate boundary conditions

$$Z = \mathrm{tr} V^N. \qquad (2.5)$$

V may be expressed in terms of spin operators along a row, which are represented by Pauli matrices $\vec{\sigma}_n$ n denotes the site index along this row and the original spin variables σ_n correspond to the operators σ_n^x. It turns out that it is very convenient to introduce fermion operators by the well-known Jordan-Wigner transformation.

$$c_r = \exp\left(i\pi \sum_1^{r-1} \sigma_s^+\sigma_s^-\right)\sigma_r^-, \quad c_r^\dagger = \exp\left(i\pi \sum_1^{r-1} \sigma_s^+\sigma_s^-\right)\sigma_r^+, \qquad (2.6)$$

where

$$\sigma^\pm = \tfrac{1}{2}(\sigma^x \pm i\sigma^y). \qquad (2.7)$$

Their Fourier transforms η_q are defined by

$$c_r = N^{-1/2} \exp(-i\pi/4) \sum_q e^{iqr}\eta_q, \qquad (2.8)$$

where the momentum q takes discrete values of the form

$$q = \frac{2\pi}{N}p, \quad p = 0, \pm 1, \ldots, \pm\frac{N-2}{2}, \frac{N}{2}.$$

The c's and the η's satisfy canonical fermion anticommutation relations

$$\{c_r, c_{r'}\} = \{c_r^\dagger, c_{r'}^\dagger\} = 0, \quad \{c_r, c_{r'}^\dagger\} = \delta_{r,r'}. \qquad (2.9)$$

In terms of the η's, the transfer matrix V reads

$$V = (2\sinh 2k)^{N/2} W, \qquad (2.10)$$

$$W = \exp\left\{-\sum_q \epsilon_q[\cos 2\phi_q(\eta_q^\dagger\eta_q - \tfrac{1}{2}) - \tfrac{1}{2}\sin 2\phi_q(\eta_q\eta_{-q} + \eta_{-q}^\dagger\eta_q^\dagger)]\right\}, \qquad (2.11)$$

where K^*, ϵ_q, and ϕ_q are defined by

$$\tanh K^* = \exp(-2K), \qquad (2.12)$$

$$\cosh\epsilon_q = \cosh 2(K - K^*) + (1-\cos q)\sinh 2K^* \sinh 2K, \tag{2.13}$$

with the condition $\epsilon_q \geq 0$, and

$$\begin{aligned}\sinh\epsilon_q \cos 2\phi_q &= \sin^2 q \sinh 2K^* + \cos^2 q \cosh 2K \sinh 2K^* - \cos q \sinh 2K \cosh 2K^*,\\ \sinh\epsilon_q \sin 2\phi_q &= \sin q \cos q \sin 2K^*(1-\cosh 2K) + \sin q \sinh 2K \cosh 2K^*.\end{aligned} \tag{2.14}$$

Noting that these definitions imply that

$$\phi_q = -\phi_{-q}, \tag{2.15}$$

we introduce finally the operators ξ_q,

$$\xi_q = \eta_q \cos\phi_q + \eta^\dagger_{-q}\sin\phi_q, \tag{2.16}$$

which diagonalize W:

$$W = \exp(-H) = \exp\left[-\sum_q \epsilon_q(\xi^\dagger_q\xi_q - \tfrac{1}{2})\right]. \tag{2.17}$$

H describes an assembly of free fermions of "energy" ϵ_q. Only the largest eigenvalue of W contributes in the limit $N\to\infty$ provided it is nondegenerate, which is the case for $T\neq T_c$. In turn this means that only the ξ-vacuum state survives in this limit as it is the lowest eigenstate of H. Henceforth angular brackets will denote averages in this state.

As long as $K \neq K_c$, then $K\neq K^*$ and $\epsilon_c \geq 2|K-K^*|$, which means that the vacuum is an isolated point in the spectrum. As $K\to K_c$ the energy gap goes to zero and the infrared region becomes predominant in the calculation of physical quantities.

We shall concentrate our attention here on the two-point correlation along the same row. As we expect an isotropic behavior close to the critical point,[18] this is not a serious limitation in this case.

Let $r<r'$ refer to points on the same row; the thermal average $\langle\sigma_r\sigma_{r'}\rangle$ is given in terms of a (ξ) vacuum expectation value as

$$\begin{aligned}\langle\sigma_r\sigma_{r'}\rangle &= \left\langle (c^\dagger_r + c_r)\exp\left(i\pi\sum_{r}^{r'-1} c^\dagger_s c_s\right)(c^\dagger_{r'}+c_{r'})\right\rangle\\ &= \left\langle (c^\dagger_r - c_r)\exp\left(i\pi\sum_{r+1}^{r'-1} c^\dagger_s c_s\right)(c^\dagger_{r'}+c_{r'})\right\rangle.\end{aligned} \tag{2.18}$$

Owing to the anticommutativity of the c's, the right-hand side operators are Hermitian. Even though the underlying dynamics has been brought to the simple form (2.17), which means that the vacuum has a simple structure, the computation of correlations seems to involve complicated expressions in terms of fermion degrees of freedom.

III. CRITICAL REGION

A. Expansion near $T = T_c$

Close to $T=T_c$, singularities appear involving long-range correlations. The discrete nature of the lattice is washed out in the large-distance behavior of correlation functions, and continuous Euclidean invariance is restored in this limit.[18] We set

$$K = K_c - \tfrac{1}{4}m_0, \tag{3.1}$$

where $m_0 \ll 1$, and from (2.12) to leading order

$$K^* = K_c + \tfrac{1}{4}m_0. \tag{3.2}$$

Here $m_0>0$ ($m_0<0$) corresponds to $T>T_c$ ($T<T_c$). Only long wavelengths with respect to the lattice spacing, i.e., small q, are of interest. Thus it seems legitimate to expand ϵ_q near $q=0$. From (2.13)

$$\epsilon_q^{\,2} = m_0^{\,2} + q^2. \tag{3.3}$$

This is the relativistic dispersion relation with the identification of $|m_0|$ with the fermion mass. From the relations (2.14) we also find in this approximation

$$\cos 2\phi_q = \frac{m_0}{(m_0^{\,2}+q^2)^{1/2}}, \qquad \sin 2\phi_q = \frac{q}{(m_0^{\,2}+q^2)^{1/2}}, \tag{3.4}$$

which allows us to write W in the form

$$W = \exp(-H) = \exp\left\{-\sum_{q>0}\left[m_0(\eta^\dagger_q\eta_q + \eta^\dagger_{-q}\eta_{-q} - 1) - q(\eta^\dagger_{-q}\eta^\dagger_q + \eta_q\eta_{-q})\right]\right\}. \tag{3.5}$$

We now define the fermion fields ψ_1 and ψ_2 by

$$c_r = \frac{e^{-i\pi/4}}{\sqrt{2}}(\psi_2 + i\psi_1), \tag{3.6}$$

or equivalently

$$\begin{aligned} i\psi_1(r) &= (2N)^{-1/2}\sum_q(\eta_q - \eta^\dagger_{-q})e^{iqr}\\ &= (2N)^{-1/2}\sum_q(\cos\phi_q - \sin\phi_q)(\xi_q e^{iqr} - \xi^\dagger_q e^{-iqr}),\\ \psi_2(r) &= (2N)^{-1/2}\sum_q(\eta_q + \eta^\dagger_{-q})e^{iqr}\\ &= (2N)^{-1/2}\sum_q(\cos\phi_q + \sin\phi_q)(\xi_q e^{iqr} + \xi^\dagger_q e^{-iqr}).\end{aligned} \tag{3.7}$$

ψ satisfies ordinary anticommutation relations:

$$\{\psi_\alpha(r), \psi_\beta(r')\} = \delta_{\alpha\beta}\delta_{rr'}. \tag{3.8}$$

In the critical region, one can use Eq. (3.4) and write

$$\begin{bmatrix}\psi_1(r)\\ \psi_2(r)\end{bmatrix} = N^{-1/2}\sum_q [\xi_q e^{iqr}u(q) + \xi_q^\dagger e^{-iqr}u^*(q)], \tag{3.9}$$

where

$$u(q) = [2\omega(\omega+q)]^{-1/2}\begin{bmatrix}-im_0\\ \omega+q\end{bmatrix}, \quad \omega \equiv (m_0{}^2+q^2)^{1/2}. \tag{3.10}$$

The Hermitian fields ψ_1 and ψ_2 are the two components of a massive Majorana field, if we allow for Minkowskian time development:

$$\psi(r,t) = e^{iHt}\psi(r)e^{-iHt}. \tag{3.11}$$

Indeed, if we introduce the following γ matrices:

$$\gamma^0 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \gamma^{0\dagger} = \sigma_1, \quad \gamma^1 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = -\gamma^{1\dagger} = i\sigma_2, \quad \gamma^5 = \gamma^{5\dagger} = \gamma^0\gamma^1 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = -\sigma_3,$$
$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad g^{00} = -g^{11} = 1, \quad g^{01} = g^{10} = 0, \tag{3.12}$$

and if we consider the argument r in (3.9) as a continuous variable, the field ψ satisfies the Dirac equation with an unconventional mass term:

$$i\gamma^\mu \frac{\partial}{\partial x^\mu}\psi(x^0 = t,\ x^1 = ra) = -im\gamma^5\psi(x) \tag{3.13}$$

or more explicitly:

$$(\partial_0 - \partial_1)\psi_1 = -m\psi_2, \qquad (\partial_0 + \partial_1)\psi_2 = m\psi_1. \tag{3.13'}$$

(We have changed m_0 into a dimensioned parameter $m = a^{-1}m_0$.) Of course, at the critical temperature, this Majorana field becomes massless.[14]

B. Two-point correlation function in the critical region

To express the two-point correlation function in terms of ψ we observe that since

$$e^{i\pi c^\dagger c} = (1 - 2c^\dagger c) = (c^\dagger + c)(c^\dagger - c), \tag{3.14}$$

one can rewrite (2.18) as

$$\langle\sigma_r\sigma_{r'}\rangle = \langle(c_r^\dagger - c_r)(c_{r+1}^\dagger + c_{r+1})(c_{r+1}^\dagger - c_{r+1})(c_{r+2}^\dagger + c_{r+2})\cdots(c_{r'-1}^\dagger - c_{r'-1})(c_{r'}^\dagger + c_{r'})\rangle$$
$$= \langle\psi_r^-\psi_{r+1}^+\psi_{r+1}^-\cdots\psi_{r'-1}^-\psi_{r'}^+\rangle, \tag{3.15}$$

where from (3.6)

$$c^\dagger + c = \psi_1 + \psi_2 \equiv \psi^+,$$
$$c^\dagger - c = i(\psi_2 - \psi_1) \equiv \psi^-. \tag{3.16}$$

The field ψ^+ is Hermitian and ψ^- is anti-Hermitian. For arbitrary n, m, they satisfy

$$\{\psi_n^-, \psi_m^+\} = 0, \quad \psi_n^{+2} = -\psi_n^{-2} = 1,$$
$$\{\psi_n^+, \psi_m^+\} = -\{\psi_n^-, \psi_m^-\} = 2\delta_{nm}. \tag{3.17}$$

We now introduce a second set of Majorana fields χ which anticommute with the fields ψ, and we calculate the *square* of the correlation function as given by

$$\langle\sigma_r\sigma_{r'}\rangle^2 = \langle\psi_r^-\psi_{r+1}^+\cdots\psi_{r'}^+\rangle\langle\chi_r^-\chi_{r+1}^+\cdots\chi_{r'}^+\rangle. \tag{3.18}$$

The reason for introducing this second set of Majorana fields is the following: This trick will enable us to construct a complex Dirac field and then to use the machinery of the boson representation of a Dirac field.[4-8] In terms of the Bose field, calculations will be greatly simplified.

If J and K denote the combinations

$$J = i\psi^+\chi^+ \text{ and } K = i\psi^-\chi^-, \tag{3.19}$$

they satisfy

$$J = J^\dagger, \quad K = K^\dagger, \quad J^2 = K^2 = 1, \quad [J, K] = 0, \tag{3.20}$$

hence

$$J = \exp\left[-i\frac{\pi}{2}(J-1)\right], \quad K = \exp\left[i\frac{\pi}{2}(K-1)\right]. \tag{3.21}$$

This allows us to write

$$\begin{aligned}\langle\sigma_r\sigma_{r'}\rangle^2 &= \langle(\psi_r^-\chi_r^-)(\chi_{r+1}^+\psi_{r+1}^+)(\psi_{r+1}^-\chi_{r+1}^-)\cdots(\psi_{r'-1}^-\chi_{r'-1}^-)(\chi_{r'}^+\psi_{r'}^+)\rangle\\ &= \langle(i\psi_r^-\chi_r^-)(i\psi_{r+1}^+\chi_{r+1}^+)(i\psi_{r+1}^-\chi_{r+1}^-)\cdots(i\psi_{r'-1}^-\chi_{r'-1}^-)(i\psi_{r'}^+\chi_{r'}^+)\rangle\\ &= \left\langle(i\psi_r^-\chi_r^-)\exp\left[-i\frac{\pi}{2}\sum_{n=r+1}^{r'-1} i(\psi_n^+\chi_n^+-\psi_n^-\chi_n^-)\right](i\psi_{r'}^+\chi_{r'}^+)\right\rangle. \qquad (3.22)\end{aligned}$$

Let D be the complex Dirac field

$$D=\frac{1}{\sqrt{2a}}(\psi+i\chi), \qquad (3.23)$$

$$\{D_\alpha(r),D_\beta^\dagger(r')\}=\frac{1}{a}\delta_{\alpha\beta}\delta_{rr'} \qquad (3.24)$$

(the other anticommutators vanish). D has been given the dimension $(\text{mass})^{1/2}$, and when $a\to 0$ we recover the canonical anticommutator. We recognize that the argument of the exponential in the right-hand side of Eq. (3.22) is the time component of the current J:

$$\begin{aligned}J^0&=\bar{D}\gamma^0 D\\ &=\frac{i}{a}(\psi_1\chi_1+\psi_2\chi_2)\\ &=\frac{i}{2a}(\psi_n^+\chi_n^+-\psi_n^-\chi_n^-). \qquad (3.25)\end{aligned}$$

In the continuum limit, the current J^μ is conserved: $\partial_\mu J^\mu=0$, which implies that the equation

$$J^\mu=\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi \qquad (3.26)$$

can be solved. As is well known,[5-7] this field ϕ is nothing other than the sine-Gordon field corresponding to the Dirac field D. When the separation $|r-r'|$ is large (i.e., distance $|x-x'|=a|r-r'|$ much larger than the lattice spacing a), we approximate the discrete sum in Eq. (3.22) by an integral. With the convention $\epsilon^{01}=-1$, we have $J^0=-(1/\sqrt{\pi})\partial_1\phi$ and

$$\frac{\pi}{2}\sum_{n=r+1}^{r'-1}(\psi_n^+\chi_n^+-\psi_n^-\chi_n^-)=-ia\pi\sum_{r+1}^{r'-1}J^0(n) \approx i\sqrt{\pi}\int_{(r+1)a}^{(r'-1)a}d\xi\,\partial_1\phi(\xi). \qquad (3.27)$$

The correlation function becomes

$$\langle\sigma(x=ra)\sigma(x'=r'a)\rangle^2=\langle i\psi^-\chi^-(x)\exp\{i\sqrt{\pi}[\phi(x'-a)-\phi(x+a)]\}[i\psi^+\chi^+(x')]\rangle. \qquad (3.28)$$

Notice that the sign of the argument of the exponential in (3.21) and (3.28) is arbitrary: Our results should not depend on this arbitrariness. The educated reader may also wonder whether any singularities are encountered in the expressions $\psi^+\phi^+$ or $\psi^-\psi^-$, when use is made of the continuous field theory. In the following section, we address ourselves to this problem, using the boson representation in the manner of Mandelstam for the Dirac field D.

At this point a remark is in order, concerning what we call the continuous limit. As pointed out in the Introduction, it is equivalent, for the dimensionless function $\langle\sigma\sigma\rangle$, to letting the lattice spacing a go to zero or letting both the distance $\rho=|r-r'|a$ and the inverse mass m_0^{-1} increase with a fixed value of the ratio $m_0\rho$; the latter corresponds to the critical regime. We thus let the lattice spacing a go to zero and replace the discrete variables, r,r' by continuous variables. However, we do not strictly set $a=0$, which would cause the occurrence of short-distance singularities, but rather use a^{-1} as a natural ultraviolet cutoff whenever necessary.

C. Boson representation

Mandelstam's parametrization for a one-dimensional massive free Dirac field reads[7]

$$D_\alpha(x)=-\gamma_\alpha^5\left(\frac{c\mu}{2\pi}\right)^{1/2}e^{-\mu R\pi/8} \times N_\mu\left\{\exp\left[-i\sqrt{\pi}\left(\gamma_\alpha^5\phi(x)-\int_{-\infty}^{x}d\xi\,e^{\xi/R}\pi(\xi)\right)\right]\right\}. \qquad (3.29)$$

Here ϕ is the sine-Gordon field satisfying the equation

$$\Box\phi+\frac{2cm\mu}{\sqrt{\pi}}N_\mu\{\sin 2\sqrt{\pi}\,\phi\}=0 \qquad (3.30)$$

and π its conjugate momentum. N_μ refers to the normal-ordering with respect to the operator $\phi^\mp$ defined by[5]

$$\phi^\mp(x)=\frac{1}{2}\left[\phi\pm\frac{i}{(-\Delta+\mu^2)^{1/2}}\pi\right] \qquad (3.31)$$

and μ is an arbitrary mass scale. c is a constant related to Euler's constant and R is a spatial cutoff[7,8]: We will let R go to ∞ at the end of the calculations. When handling the field ϕ, the crucial point to remember is that, owing to the superrenormalizability of the sine-Gordon interaction, the short-distance behavior of the commutators, Green's functions, etc. is the same as in the free field case; for instance, for spacelike separation

$$[\phi^+(x), \phi^-(y)]_{x \approx y} \frac{1}{4\pi}\ln[-c^2\mu^2(x-y)^2]. \quad (3.32)$$

The field D ad defined above satisfies equal-time anticommutation relations

$$\{D_\alpha(x), D_\beta(y)\}_{\substack{x^0=y^0\\ x^1\neq y^1}}=0, \quad \{D_\alpha(x), D_\beta^\dagger(y)\}_{\substack{x^0=y^0\\ x^1\neq y^1}}=0. \quad (3.33)$$

It is possible to define a regularized version of the charge current. We choose Klaiber's definition[4]

$$J^\mu(x)=\tfrac{1}{2}\lim_{\xi\to 0}[\overline{D}(x+\xi)\gamma^\mu D(x)-\gamma^\mu D(x+\xi)\overline{D}(x)] \quad (3.34)$$

rather than Mandelstam's, because this regularization seems more natural in our Ising problem. One is led to the expression (3.26)

$$J^\mu=\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi \quad (3.35)$$

for the current J^μ in terms of the scalar field ϕ and thus to the canonical equal-time commutation relations[7]

$$[J^\mu(x), D(y)]_{x^0=y^0}=-\gamma^0\gamma^\mu D(x)\delta(x-y)$$
$$=-(g^{\mu 0}+\epsilon^{\mu 0}\gamma^5)D(x)\delta(x-y). \quad (3.36)$$

Finally, as shown by Mandelstam, the field D satisfies the field equation

$$i\partial^\mu\gamma_\mu D(x)=[S,\gamma^0 D], \quad (3.37)$$

where

$$S=m\int dx\,\overline{D}(x)\gamma^5 D(x). \quad (3.38)$$

Equation (3.37) is the quantized form of the Dirac equation (3.13) which we are interested in. We thus conclude that Eq. (3.79) is a good parametrization of the field D.

We now return to our Ising problem and use this parametrization in the expression obtained in (3.25) and (3.28). We first remark that the bilinear operators $\psi^+\chi^+, \psi^-\chi^-$ in (3.22) and (3.25) need to be regularized. However, there is a very natural regularization in our problem. Suppose that our second Ising lattice is slightly shifted by a translation ξ. The bilinear products now have the form $\psi^\pm(x+\xi)\chi^\pm(x)$, which can be expressed in terms of the components of the field D:

$$i\psi^\pm(x+\xi)\chi^\pm(x)=\pm\tfrac{1}{2}[D_1^\dagger(x+\xi)D_1(x)-D_1(x+\xi)D_1^\dagger(x)+D_2^\dagger(x+\xi)D_2(x)-D_2(x+\xi)D_2^\dagger(x)]$$
$$+D_1^\dagger(x+\xi)D_2(x)+D_2^\dagger(x+\xi)D_1(x). \quad (3.39)$$

The quantity in the brackets in the right-hand side is the regularized definition of J^0 [cf. Eq. (3.34)], whereas the two last terms $D_1^\dagger D_2$ and $D_2^\dagger D_1$ are regular when $\xi\to 0$. Thus

$$\lim_{\xi\to 0} i\psi^\pm(x+\xi)\chi^\pm(x)=\pm J^0(x)+D_1^\dagger(x)D_2(x)+D_2^\dagger(x)D_1(x). \quad (3.40)$$

Note that this regularization justifies our calculation of Eq. (3.25). We also find that

$$D_1^\dagger D_2(x)=-i\frac{c\mu}{2\pi}N_\mu\exp[-2i\sqrt{\pi}\,\phi(x)],$$
$$D_2^\dagger D_1(x)=i\frac{c\mu}{2\pi}N_\mu\exp[+2i\sqrt{\pi}\,\phi(x)]. \quad (3.41)$$

Equation (3.28) thus becomes

$$\langle\sigma(x=ra)\sigma(x'=r'a)\rangle^2=a^2\left\langle\left[-J^0(x)-\frac{c\mu}{\pi}N_\mu\sin 2\sqrt{\pi}\phi(x)\right]e^{-i\sqrt{\pi}\phi(x+a)}e^{i\sqrt{\pi}\phi(x'-a)}\left[J^0(x')-\frac{c\mu}{\pi}N_\mu\sin 2\sqrt{\pi}\,\phi(x')\right]\right\rangle. \quad (3.42)$$

When we compute this expression using the continuous field theory, two types of singularities may appear. First the exponentials $\exp(\pm i\sqrt{\pi}\,\phi)$ may give divergent self-contractions: We claim that the natural ultraviolet cutoff of our problem is $\Lambda=a^{-1}$. On the other hand, when we let a go to zero, the products of operators in (3.42) become singular. One finds that

$$J^0(x)e^{-i\sqrt{\pi}\phi(x+a)}=(c\mu a)^{1/4}N_\mu\left[\frac{i}{2\pi a}e^{-i\sqrt{\pi}\phi(x)}(1+O(a))+J^0(x)e^{-i\sqrt{\pi}\phi(x)}(1+O(a))\right] \quad (3.43)$$

and

$$\frac{c\mu}{\pi}N_\mu[\sin 2\sqrt{\pi}\,\phi(x)]e^{-i\sqrt{\pi}\phi(x+a)}=\frac{1}{2\pi i a}(c\mu a)^{1/4}N_\mu\left[e^{i\sqrt{\pi}\phi(x)}(1+O(a))+(c\mu a)^2e^{-3i\sqrt{\pi}\phi(x)}(1+O(a))\right] \quad (3.44)$$

with similar expressions for the operators at the point x'. We now insert these results into Eq. (3.42). All ultraviolet divergences have been explicitly exhibited, and we may keep the leading power of a^{-1}:

$$\langle \sigma(x)\sigma(x')\rangle^2 \underset{a\to 0}{\approx} (c\mu a)^{1/2} \frac{1}{\pi^2} \langle N_\mu[\sin\sqrt{\pi}\,\phi(x)]N_\mu[\sin\sqrt{\pi}\,\phi(x')]\rangle . \tag{3.45}$$

This is our major result: In the critical region, the square of the two-point correlation function of the Ising model is the Green's function $\langle \sin\sqrt{\pi}\,\phi(x)\sin\sqrt{\pi}\,\phi(x')\rangle$ of the sine-Gordon field—up to an unknown normalization factor originating from the terms neglected in the original Hamiltonian (3.5). Equation (3.45) can be used to compute the behavior at the critical temperature. At this point, the Dirac field is massless and the field ϕ is free:

$$\langle \sigma(x=ra)\sigma(x'=r'a)\rangle^2 = \frac{1}{2\pi^2}\left|\frac{a}{x-x'}\right|^{1/2} = \frac{1}{2\pi^2}\frac{1}{(r-r')^{1/2}} \tag{3.46}$$

or

$$\langle \sigma_r\sigma_{r'}\rangle|_{T=T_c} \sim \frac{1}{|r-r'|^{1/4}}, \tag{3.47}$$

which is a well-known result of the Ising model.

D. Mass perturbation

We now want to use this relationship between the Ising and sine-Gordon models to investigate the vicinity of the critical temperature. Our calculation is based on a mass perturbation expansion: We assume $m|x-x'|$ to be small, and consider the sine-Gordon interaction as a perturbation. To illustrate the care required to deal with such a perturbation and its infrared singularities, let us first treat a simple exercise.

(1) Consider the free Dirac field D solution of Eq. (3.37); we expand D as

$$D = \frac{1}{(2\pi)^{1/2}}\int dq^1 [a(q)e^{-iqx}u(q) + b^\dagger(q)e^{iqx}v(q)], \tag{3.48}$$

where the spinors u and v,

$$u(q) = \frac{1}{[2q^0(q^0+q^1)]^{1/2}}\begin{pmatrix} -im \\ q^0+q^1\end{pmatrix}, \qquad v(q)=u(q)^* , \tag{3.49}$$

are normalized to $|u_1|^2+|u_2|^2=|v_1|^2+|v_2|^2=1$. The equal-time Wightman function for this field is an average computed in the vacuum state of the a and b operators, and is found to be

$$\langle D_\alpha(x)D_\beta^\dagger(y)\rangle = \frac{1}{2\pi}\int\frac{dq^1}{q^0}e^{iq^1(x^1-y^1)}\begin{pmatrix} q^0-q^1 & -im \\ im & q^0+q^1\end{pmatrix} = \frac{im}{2\pi}\begin{pmatrix} -\frac{\xi}{|\xi|}K_1(m|\xi|) & -K_0(m|\xi|) \\ K_0(m|\xi|) & \frac{\xi}{|\xi|}K_1(m|\xi|)\end{pmatrix}, \tag{3.50}$$

where $\xi = x-y$. As $m\to 0$, we find

$$\langle D_\alpha(x)D_\beta^\dagger(y)\rangle|_{x_0=y_0} = \begin{pmatrix} -\frac{i}{2\pi\xi} & 0 \\ 0 & \frac{i}{2\pi\xi}\end{pmatrix} + \begin{pmatrix} 0 & \frac{im}{2\pi}\ln Cm|\xi| \\ -\frac{im}{2\pi}\ln Cm|\xi| & 0\end{pmatrix} + \cdots . \tag{3.51}$$

The first term is of course the mass-zero value of $\langle D_\alpha D_\beta^\dagger\rangle$ and the correction term is of order $m\ln m$. It would seem that perturbation theory in m is in trouble. However, the origin of the $\ln m$ factor lies in an infrared singularity. If we assume the existence of an infrared cutoff of order m in momentum space, or $1/m$ in configuration space, we should be able to recover from perturbation theory that to lowest order

$$\delta\langle D_\alpha(x)D_\beta^\dagger(y)\rangle = \begin{pmatrix} 0 & +\frac{im}{2\pi}\ln Cm|\xi| \\ -\frac{im}{2\pi}\ln Cm|\xi| & 0\end{pmatrix} . \tag{3.52}$$

Let us see now if this idea works. The interaction-picture Gell-Mann–Low formula tells us that

$$\langle D_\alpha(x)D_\beta^\dagger(y)\rangle = \frac{\langle TD_\alpha^0(x)D_\beta^{0\dagger}(y)\exp[-i\int d^2z\,\mathcal{H}_m(z)]\rangle}{\langle \exp[-i\int d^2z\,\mathcal{H}_m(z)]\rangle}, \tag{3.53}$$

where $\mathcal{H}_m = -im\bar{D}^0\gamma^5 D^0$. On the right-hand side, we use a massless field D^0 and a Minkowski space formula. Since we are computing an equal-time Wightman function which does not distinguish between real and imaginary time, this should not make any difference. The massless propagators are

$$\langle TD_1^0(x)D_1^{0\dagger}(y)\rangle = -\frac{i}{2\pi}\frac{1}{x^1-y^1+(x^0-y^0)(1-i\epsilon)},$$

$$\langle TD_2^0(x)D_2^{0\dagger}(y)\rangle = \frac{i}{2\pi}\frac{1}{x^1-y^1-(x^0-y^0)(1-i\epsilon)}, \tag{3.54}$$

and $\langle TD_1^0(x)\, D_2^{0\dagger}(y)\rangle$ vanishes. Since $\mathcal{H}_m$ satisfies $\langle\mathcal{H}_m\rangle = 0$, we find to lowest order

$$\delta\langle D_\alpha(x)D_\beta^\dagger(y)\rangle|_{x^0=y^0} = -m\int d^2z\langle TD_\alpha^0(x)D_\beta^{0\dagger}(y)\overline{D}^0(z)\gamma^5D^0(z)\rangle\,. \tag{3.55}$$

Applying Wick's theorem we see that this implies

$$\delta\langle D_\alpha(x)D_\alpha^\dagger(y)\rangle|_{x^0=y^0} = 0$$

in agreement with (3.52), while for instance

$$\delta\langle D_1(x)D_2^\dagger(y)\rangle = \frac{m}{(2\pi)^2}\int d^2z\frac{1}{[(x^0-z^0)(1-i\epsilon)+(x^1-z^1)][(z^0-x^0)(1-i\epsilon)-(z^1-y^1)]}\,. \tag{3.56}$$

The z^0 integral is readily evaluated by contour-integral methods, and with $|\xi| = |x^1-y^1|$ we obtain, in configuration space, the logarithmic infrared-divergent integral

$$\delta\langle D_1(x^1,x^0), D_2^\dagger(y^1,x^0)\rangle = -\frac{im}{2\pi}\int_0^\infty\frac{du}{u+|\xi|}\,. \tag{3.57}$$

This is exactly what was expected and we supply an infrared cutoff by replacing the upper limit of integration by $1/Km$, where, of course, the constant K is unknown.

Then

$$\delta\langle D_1(x^1x^0)D_2^\dagger(y^1y^0)\rangle = -\frac{im}{2\pi}\int_0^{1/Km}\frac{du}{u+|\xi|} = \frac{im}{2\pi}\ln Km|\xi|\,. \tag{3.58}$$

Fortunately, we reproduce the exact result given in (3.52), at least at the leading-logarithm approximation. We thus feel confident to apply the same method to the Ising correlation function.

(2) We now consider this Ising function as expressed by

$$\langle\sigma(x)\sigma(x')\rangle^2 = \frac{(\mu a)^{1/2}}{\pi^2}\frac{\left\langle N_\mu[\sin\sqrt{\pi}\,\phi(x)]N_\mu[\sin\sqrt{\pi}\,\phi(x')]\exp\left[\int d^2z_{\rm Eucl}\frac{cm\mu}{\pi}N_\mu\cos2\sqrt{\pi}\,\phi(z)\right]\right\rangle}{\left\langle\exp\left[\int d^2z_{\rm Eucl}\frac{cm\mu}{\pi}N_\mu\cos2\sqrt{\pi}\,\phi(z)\right]\right\rangle}, \tag{3.59}$$

where, from now on, the field ϕ is a free field. As explained by Coleman,[5] the mass parameter μ has to be set equal to zero at the end of the calculation. In Eq. (86), a Wick rotation has been performed. Using an infrared cutoff of order $(K|m|)^{-1}$ in the Euclidean z integrations, we readily calculate the first two corrections.

$$\delta_1(\langle\sigma(x)\sigma(x')\rangle^2) = -\frac{ma^{1/2}}{4\pi^3}|x-x'|^{1/2}\int^{|z|<(K|m|)^{-1}}\frac{d^2z}{|z-x||z-x'|}$$

$$\simeq\frac{ma^{1/2}}{2\pi^2}|x-x'|^{1/2}(\ln K|m||x-x'|+\cdots)\,. \tag{3.60}$$

Thus

$$\frac{\delta_1\langle\sigma(x)\sigma(x')\rangle}{\langle\sigma(x)\sigma(x')\rangle|_{T=T_c}} = \tfrac{1}{2}m|x-x'|\ln(K|m||x-x'|)+\cdots\,. \tag{3.61}$$

In the second-order correction, the vacuum graphs have to be taken into account:

$$\delta_2(\langle\sigma(x)\sigma(x')\rangle^2) = \frac{m^2\mu^2}{2}\frac{(\mu a)^{1/2}}{\pi^4}$$

$$\times\int\int d^2z_1d^2z_2\{\langle N_\mu[\sin\sqrt{\pi}\,\phi(x)]N_\mu[\sin\sqrt{\pi}\,\phi(x')]N_\mu[\cos2\sqrt{\pi}\,\phi(z_1)]N_\mu[\cos2\sqrt{\pi}\,\phi(z_2)]\rangle$$

$$-\langle N_\mu[\sin\sqrt{\pi}\,\phi(x)]N_\mu[\sin\sqrt{\pi}\,\phi(x')]\rangle\langle N_\mu[\cos2\sqrt{\pi}\,\phi(z_1)]N_\mu[\cos2\sqrt{\pi}\,\phi(z_2)]\rangle\}$$

$$= \frac{ma^{1/2}}{8\pi^4|x-x'|^{1/2}}\int\frac{d^2z_1d^2z_2}{(z_1-z_2)^2}\left\{\left[\frac{(x-x'-z_1)^2z_2^2}{(x-x'-z_2)^2z_1^2}\right]^{1/2}-1\right\}$$

$$\simeq\frac{m^2|x-x'|^2}{4}\ln^2K|m(x-x')|\frac{a^{1/2}}{2\pi^2|x-x'|^{1/2}}\,. \tag{3.62}$$

The second-order leading logarithm in $\langle\sigma(x)\sigma(x')\rangle$ is thus

$$\frac{\delta_2\langle\sigma(x)\sigma(x')\rangle}{\langle\sigma(x)\sigma(x')\rangle|_{T=T_c}}=\frac{\delta_2\langle\sigma\sigma\rangle^2}{2\langle\sigma\sigma\rangle^2|_{T=T_c}}-\frac{1}{8}\frac{\delta_1\langle\sigma\sigma\rangle^2}{\langle\sigma\sigma\rangle^2|_{T=T_c}}=0\,. \tag{3.63}$$

From our perturbative calculation using an infrared cutoff, we can only conclude that there is no term of the form $m^2|x-x'|^2\ln^2|m(n-n')|$ in the function

$$\frac{\langle\sigma(x)\sigma(x')\rangle}{\langle\sigma(x)\sigma(x')\rangle|_{T=T_c}}=F(m|x-x'|)\,. \tag{3.64}$$

Notice also that our perturbative calculation should still make sense for negative values of the "mass" parameter m, corresponding to $T<T_c$. These results, as well as the first-order correction (3.61), are in agreement with the exact result obtained by Wu *et al.*[11] These authors write

$$F_{T\gtrless T_c}(t)=1\pm|t|\Omega+\tfrac{1}{16}t^2\pm\tfrac{1}{32}|t|^3\Omega+\cdots\,, \tag{3.65}$$

where $\Omega=\ln(|t|/8)+\gamma$ (γ is Euler's constant).

IV. REMARKS AND CONCLUSION

In the preceding sections, a relationship between the two-point functions of the Ising model and of the sine-Gordon model has been exhibited and successfully tested, using a mass perturbation. What can be said about the higher-order correlation functions? It is easy to convince oneself that the duplication of the Ising lattice and the manipulations performed in Secs. III B and C lead to

$$\langle\sigma_{r_1}\cdots\sigma_{r_{2n}}\rangle=\left\langle\prod_{j=1}^{n}(\psi^-_{r_{2j-1}}\psi^+_{r_{2j-1}+1}\cdots\psi^+_{r_{2j}})\right\rangle \tag{4.1}$$

and

$$\begin{aligned}\langle\sigma(x_1)\cdots\sigma(x_{2n})\rangle^2&=\left\langle\prod_{j=1} i\psi^-\chi^-(x_{2j-1})\exp\{i\sqrt{\pi}\epsilon_j[\phi(x_{2j})-\phi(x_{2j-1})]\}\,i\psi^+\chi^+(x_{2j})\right\rangle\\&=\left(\frac{C\mu a}{\pi^2}\right)^{n/2}\left\langle\prod_{k=1}^{2n}\sin\sqrt{\pi}\,\phi(x_k)\right\rangle+\cdots\,,\end{aligned} \tag{4.2}$$

where it is understood that the $2n$ points $x_1,\ldots,x_{2n}$ are along the same row and are ordered: $x_1<x_2<\cdots<x_{2n}$; indeed this ordering is implicit in Eq. (4.1). In (4.2), the dots stand for terms suppressed by powers of ma or any $a/|x_i-x_j|$. As remarked in Sec. III B, there is some arbitrariness in the exponentiation; this is reflected by the presence of the ϵ's: $\epsilon_i=\pm1$. At the critical temperature, we can use Eq. (4.2) to compute the behavior of any correlation function. For example, consider the four-spin on-line function:

$$\langle\sigma(x)\sigma(y)\sigma(z)\sigma(w)\rangle^2=\frac{a}{8\pi^4}[|x-y|^{-1/2}|x-z|^{1/2}|x-w|^{-1/2}|y-z|^{-1/2}|y-w|^{1/2}|z-w|^{-1/2}+(y\leftrightarrow z)+(z\leftrightarrow w)]. \tag{4.3}$$

The expression obtained by Kadanoff and Ceva[19] coincides with the first term of the right-hand side of Eq. (4.3). Thus our formalism seems to give a *symmetrized* version of Kadanoff and Ceva's result. On the other hand, in any possible regime where all distances increase at the same rate, or where a given distance is much larger than the others, the two expressions give the same behavior: For instance, if $|x-y|=L$ and $|z-w|=L'$ are much larger than $|y-z|=l$ (which is itself much larger than the lattice spacing), the first two terms of (4.3) behave as $al^{-1/2}(L+L')^{-1/2}$ (Kadanoff and Ceva's result), whereas the third term is $\sim al^{1/2}L^{-1}L'^{-1}(L+L')^{1/2}$. Neither for $L\sim L'\gg l\gg a$ nor $L\gg L'\gg l\gg a$ does this term affect the $(a/l)(l/L)^{1/2}$ behavior of the first ones. As a consequence of this discussion, it is clear that the calculation of the corrections near the critical temperature must be done in a definite regime.

At the end of this paper many questions remain open: Is there a deeper connection between the sine-Gordon model and the Ising model (more precisely, the model consisting of two noninteracting Ising lattices)? Our relationship has been built using the transfer matrix formalism, its formulation in terms of fermion operators, and finally the boson representation of these operators. Would it be possible to avoid these intermediary steps and construct a direct relationship? This seems to be a difficult problem.

Finally, if the exact expressions of Refs. 11 and 15 are to be identified with the two-point function of the sine-Gordon theory, we can learn some useful information about the large-distance behavior ($mx \gg 1, x \gg \xi \sim 1/m$) of the latter. Such an investigation is beyond the scope of this paper.

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