Radiation reaction in nonrelativistic quantum electrodynamics*

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We derive the Heisenberg operator equation of motion for a nonrelativistic point electron coupled to the quantized electromagnetic field, including radiation reaction. The derivation proceeds in close analogy with the classical theory of extended charges (with the Compton wavelength formally playing the role of a size parameter), and we give a systematic treatment of the classical problem, showing explicitly from the equation of motion that the classical theory shows no runaway solutions or preacceleration when the electron size exceeds the classical electron radius. In the quantum-mechanical case, we show that the electrostatic self-energy of a point electron is zero and that, for values of the fine-structure constant $\alpha \lesssim 1$, the equation of motion admits neither runaway solutions nor noncausal motion. Furthermore, the correspondence limit of the solutions to the quantum-mechanical equation of motion agrees with that of the Lorentz-Dirac theory in the classical regime, but without the imposition of additional conditions and with no possibility of observable noncausality. Thus, a consistent picture of a classical point electron emerges in the correspondence limit of the quantum-mechanical theory.

I. INTRODUCTION

Despite the unparalled success of quantum electrodynamics as a predictive algorithm, several fundamental questions concerning the structure of the theory remain unresolved. For example, how can one construct the classical limit of quantum electrodynamics? Do the problems of runaway solutions and preacceleration, which emerge in the classical theory of radiation reaction, appear in quantum electrodynamics as well? While such questions were central in much of the discussion[1] preceding the work of Feynman, Schwinger, and Tomonaga,[2] the very success of renormalized perturbation theory has resulted in there being relatively little attention devoted since then to these problems of internal consistency. Nevertheless, in order to understand properly the structure and correspondence limit of quantum electrodynamics, it is important that such questions be answered. Here, we use the Heisenberg equations of motion to study the interconnected problems of electrostatic self-energy, runaway solutions, and preacceleration for a nonrelativistic electron coupled to the quantized electromagnetic field. Although this model does not display the full complexity of relativistic quantum electrodynamics, the corresponding classical theory is already beset by the problems of internal consistency which we address. We begin by outlining these problems more explicitly.

In the classical Lorentz-Dirac theory of radiation reaction a nonrelativistic point electron, interacting with its self-field and subject to an external force $\vec{F}(t)$, obeys the equation of motion[3-5]

$$m_0\ddot{\vec{R}}(t)=\vec{F}(t)-\delta m\ddot{\vec{R}}(t)+\frac{2e^2}{3c^3}\dddot{\vec{R}}(t), \tag{1.1}$$

where δm is the electrostatic self-energy of the electron. This theory of radiation reaction suffers from a number of defects besides the fact that $\delta m=\infty$ for a point electron, a fact which can after all be ignored by working with the physical mass $m=m_0+\delta m$, according to the philosophy of renormalization. These difficulties are apparent in the general solution to Eq. (1.1), which is

$$\ddot{\vec{R}}(t)=e^{t/\tau}\left[\ddot{\vec{R}}(0)-\frac{1}{m\tau}\int_0^t dt' e^{-t'/\tau}\vec{F}(t')\right], \tag{1.2}$$

with $\tau\equiv 2e^2/3mc^3$. For convenience, let us assume that the force begins to act at time $t=0$ and operates for a finite time interval. Clearly, the particle accelerates before the force is turned on (noncausal acceleration) and continues to accelerate exponentially even after the force is turned off (runaway solution). Of course, since Eq. (1.1) contains the third time derivative of $\vec{R}(t)$, the acceleration at time $t=0$ enters into Eq. (1.2) as an initial condition. Therefore, if we insist that the motion be completely determined by the usual Newtonian initial conditions [i.e., $\vec{R}(0)$ and $\dot{\vec{R}}(0)$], we must impose a further condition on $\ddot{\vec{R}}(0)$, and this can in fact be done in such a way as to eliminate one of the above defects. For example, the choice $\ddot{\vec{R}}(0)=0$ eliminates the noncausality but does not remedy the unreasonable runaway behavior. The more conventional choice[5] is

$$\ddot{\vec{R}}(0)=\frac{1}{m\tau}\int_0^\infty dt' e^{-t'/\tau}\vec{F}(t'), \tag{1.3}$$

in which case[6]

$$\ddot{\vec{R}}(t)=\frac{1}{m}\int_0^\infty ds\, e^{-s}\vec{F}(t+s\tau)\,. \tag{1.4}$$

The noncausal nature of this solution is obvious, since the particle samples the force for a time τ into the future.

While these defects mar the internal consistency of classical electrodynamics, it is nevertheless true that Eq. (1.4) correctly describes classical radiation damping, insofar as it has been tested. Consequently, the view is generally adopted that, since preacceleration occurs over such a short time scale ($\tau\sim10^{-23}$ seconds for an electron), the noncausal effects belong in the domain of quantum theory, which is where one has to look for a resolution of the problem.

It is a very reasonable proposition that runaway solutions do not occur in quantum theory, since one would not expect a Heisenberg-picture operator $O(t)$ to display an exponentially growing dependence on time if its time development were given by

$$O(t)=e^{iHt}O(0)e^{-iHt}, \tag{1.5}$$

with e^{iHt} unitary. Nevertheless, to date no rigorous proof of the absence of runaways in nonrelativistic quantum electrodynamics has been given, although several theorems of a kind that would be pertinent to such a proof have been established.[7]

Unfortunately, a rigorous proof is not the subject of the present paper either. Instead, starting from the operator form of Maxwell's equations and the Lorentz force equation which follows from the standard Hamiltonian[8] governing the interaction of a nonrelativistic electron with a quantized electromagnetic field, we derive[9] a quantum-mechanical operator equation of motion for a point electron which reduces to Eq. (1.1) in the correspondence limit (i.e., $\hbar\to0$) and show that (i) the electrostatic self-energy of a point electron is zero in quantum mechanics, (ii) the equation of motion does not admit runaway solutions, (iii) the correspondence limit of the *solution* of the quantum-mechanical equation likewise does not display runaways, and (iv) the solutions do not display noncausality. These calculations thus show how nonrelativistic quantum electrodynamics manages to suppress the runaways, conforming to requirements which presumably follow from the general principles of quantum mechanics, and in addition possesses a physically reasonable correspondence limit.

A detailed outline of our paper follows. The results in the quantum-mechanical case are best understood by comparison to the classical results for the radiation reaction of an extended charge. This problem is treated systematically in Sec. II. Section II A reviews the standard derivation of the equation of motion of a spherically symmetric extended charge interacting with its self-fields. In Sec. II B we specialize for simplicity to the case of a spherical shell of charge and show that the equation of motion can be written as a linear differential-difference equation. The solutions of this equation are analyzed for a free electron (no external force) in Sec. II C. In addition to the constant-velocity solution, the equation admits an infinite number of damped-oscillatory solutions, and the physical interpretation of these solutions to the homogeneous equation is discussed. Furthermore, when the charge radius L is less than the classical electron radius $c\tau$, there is one exponentially growing or runaway solution. In Sec. II D, it is shown that there is no preacceleration if the condition $L>c\tau$ is satisfied. In other words, we show directly from the equation of motion that the classical theory is consistent (i.e., no runaways and no noncausality) so long as $L>c\tau$. Although our treatment is in some ways more systematic than others available in the literature, some of the results have been obtained previously by other authors.[10] Nevertheless, the detailed discussion of the classical theory serves both as a guide to the quantum-mechanical calculation and as a reference point for our later discussion of the classical limit of the solutions to the quantum-mechanical equation.

In Sec. III we turn to quantum mechanics and derive an operator equation of motion for an electron subject to a force of radiation reaction. The derivation, presented in Secs. III A and III B, closely parallels the classical Abraham-Lorentz approach, in which Eq. (1.1) (or the corresponding equation for an extended charge) is obtained by eliminating the self-fields from the Lorentz force equation. The resulting quantum-mechanical equation of motion is similar in form to the classical equation, but the structure-dependent coefficients in the equation are now given by power series in λ^2/L^2 ($\lambda=\hbar/mc$ is the Compton wavelength). These structure coefficients are evaluated in Sec. III C, and several interesting conclusions emerge from a study of the self-energy term: (i) If one lets $\lambda\to0$ and then goes to the point limit, the classical divergent result for the electrostatic self-energy δm is obtained, whereas if one goes to the point limit keeping λ finite, then $\delta m=0$. (ii) If the particle is assumed to have a static extended-charge distribution, the maximum value for the self-energy is $\delta m\sim\alpha m_0$, occurring for $L\sim\lambda$. (iii) The electrostatic self-energy can even be negative if $0<L\lesssim\lambda$. Finally, the remaining coefficients are evaluated in the point limit, leaving us with an equation resembling the classical equation for an ex-

tended charge and with the Compton wavelength formally playing the role of a size parameter.

In Sec. IV we study the solutions of the quantum-mechanical equation for the case of no external forces. Taking matrix elements of the equation between stationary states of the compound system comprised of particle plus radiation field, and writing $\langle m|\vec{\mathrm{R}}(t)|n\rangle = \vec{\mathrm{R}}_{mn}(t) = e^{iE_{mn}t/\hbar}\vec{\mathrm{R}}_{mn}(0)$, the equation is converted into a power series in the variable $\eta = \beta\lambda/c$, where $\beta = iE_{mn}/\hbar$. Inside its circle of convergence $|\eta| < \frac{1}{2}$, the series can be summed, and we determine the roots of the equation as a function of the fine-structure constant α. Our conclusion is that, for the physical value of α, there are no roots inside the circle of convergence (except for the trivial root $\eta = 0$), implying that there are no runaways. However, for large α (interpreted as the strong-coupling case if the value of $\hbar$ is fixed, or as the semiclassical regime if e^2 is fixed and $\hbar \to 0$), there is a runaway solution.

Preacceleration is discussed in Sec. V, where we consider a c-number time-dependent force. We must assume that the force changes little in the time required for a light signal to cross an electron Compton wavelength (this condition is closely related to the previously stated convergence condition $|\eta| < \frac{1}{2}$). It is shown that there is no observable noncausality in the quantum-mechanical theory, and that this theory yields the usual radiation damping effects in the correspondence limit. Recall that we are dealing here with the correspondence limit of the *point*-charge theory and that the point limit lies outside the domain of a consistent classical theory. In other words, our calculations show how a consistent classical theory of point charges, including radiation reaction, can be formulated as the correspondence limit of the quantum theory.

Section VI is the concluding section and contains a few comments on the result of the preceding sections and a list of questions that remain to be answered.

It is to be stressed that at no point in this work do we employ the dipole approximation or, indeed, any other approximation to the Hamiltonian aside from the nonrelativistic treatment of the electron. This is one of the reasons our results differ from those of previous workers, such as van Kampen,[11] Wildermuth and Baumann,[12] Norton and Watson,[13] and Coleman and Norton,[14] each of whom use the dipole approximation and each of whom find runaways in the models they study. A further point to be stressed is that our results have been obtained without using a perturbation expansion in α. Their relationship to results that might be obtained using standard perturbation theory remains to be understood.

II. RADIATION REACTION OF A CLASSICAL EXTENDED CHARGE

A. Equation of motion

In this section we treat the radiation reaction of an extended charge which we characterize by a spherically symmetric static charge density $e\rho(\vec{x}, t) = e\rho(\vec{x} - \vec{\mathrm{R}}(t))$, with e the total charge (ρ normalized to unity) and $\vec{\mathrm{R}}(t)$ the coordinate of the mean position of the charge. First we recall the familiar derivation of the equation of motion of an extended charge subject to its electromagnetic self-force and to an external force $\vec{\mathrm{F}}(t)$. The particle motion will be treated nonrelativistically, meaning that terms of order $|\dot{\vec{\mathrm{R}}}(t)/c|^2$ or higher will be systematically dropped from the equation of motion.

The starting point of the derivation is the Lorentz force equation

$$m_0\ddot{\vec{\mathrm{R}}}(t) = \vec{\mathrm{F}}(t) + \int d\vec{x}\left[\rho(\vec{x}, t)\vec{\mathrm{E}}(\vec{x}, t) + \frac{1}{c}\vec{\mathrm{j}}(\vec{x}, t)\times\vec{\mathrm{B}}(\vec{x}, t)\right], \quad (2.1)$$

where $\vec{\mathrm{j}}(\vec{x}, t) = e\rho(\vec{x} - \vec{\mathrm{R}}(t))\dot{\vec{\mathrm{R}}}(t)$, m_0 is the mechanical mass of the particle, and $\vec{\mathrm{F}}(t)$ is an external force. The self-force is given by the second term on the right-hand side of Eq. (2.1) when the electric and magnetic fields are taken as the self-fields produced by the extended charge. The next step is to eliminate the self-fields from the equation of motion by introducing the retarded solutions to Maxwell's equation (Lorentz gauge):

$$\begin{aligned}
\vec{\mathrm{E}}(\vec{x}, t) &= -\vec{\nabla}\phi(\vec{x}, t) - \frac{1}{c}\frac{\partial\vec{\mathrm{A}}(\vec{x}, t)}{\partial t}, \\
\vec{\mathrm{B}}(\vec{x}, t) &= \vec{\nabla}\times\vec{\mathrm{A}}(\vec{x}, t), \\
\vec{\mathrm{A}}(\vec{x}, t) &= \vec{\mathrm{A}}_{\mathrm{in}}(\vec{x}, t) + \frac{1}{c}\int d\vec{x}'\,\frac{[\vec{\mathrm{j}}(\vec{x}', t')]_{\mathrm{ret}}}{|\vec{x} - \vec{x}'|}, \\
\phi(\vec{x}, t) &= \phi_{\mathrm{in}}(\vec{x}, t) + e\int d\vec{x}'\,\frac{[\rho(\vec{x}', t')]_{\mathrm{ret}}}{|\vec{x} - \vec{x}'|},
\end{aligned} \quad (2.2)$$

where $t' = t - |\vec{x} - \vec{x}'|/c$ is the retarded time. Incoming radiation is described by $\vec{\mathrm{A}}_{\mathrm{in}}(\vec{x}, t)$ and $\phi_{\mathrm{in}}(\vec{x}, t)$ and, since we want to concentrate on radiation reaction, it is appropriate classically to set $\vec{\mathrm{A}}_{\mathrm{in}}(\vec{x}, t) = 0 = \phi_{\mathrm{in}}(\vec{x}, t)$. Therefore, inserting Eq. (2.2) into Eq. (2.1) results in an equation depending only upon the particle coordinate and the external force. The charge or current density evaluated at the retarded time can be expressed as a Taylor series about the time t; for example,

$$\vec{\mathrm{j}}(\vec{x}', t')_{\mathrm{ret}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{|\vec{x} - \vec{x}'|}{c}\right)^n\frac{\partial^n}{\partial t^n}\vec{\mathrm{j}}(\vec{x}', t). \quad (2.3)$$

The continuity equation, spherical symmetry, and angular averaging can be used to simplify the resulting expression (see, for example, the textbook by Jackson[5] for the details). Finally, the equation of motion is

$$m_0\ddot{\vec{R}}(t)=\vec{F}(t)-\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}}, \quad (2.4)$$

where

$$\gamma_n=\int\int d\vec{x}d\vec{x}'\rho(\vec{x},t)|\vec{x}-\vec{x}'|^{n-1}\rho(\vec{x}',t). \quad (2.5)$$

Note that the structure-dependent coefficients γ_n are proportional to L^{n-1}, where L is the effective charge radius, and that Eq. (2.4) reduces to Eq. (1.1) in the point limit. In deriving Eq. (2.4) we have kept only those terms linear in the time derivatives of $\vec{R}(t)$, all of which come from the electric term in Eq. (2.1). In addition to these terms, there are terms of order $|\dot{\vec{R}}/c|^2$ times the linear terms, arising from both the electric and magnetic self-fields; these have been neglected since we are considering the motion of a nonrelativistic electron.

B. Equation for a spherical shell

To study the motion of an extended charge, it is useful to write Eq. (2.4) in closed form. This is easily done for simple charge distributions because the structure coefficients γ_n can be explicitly evaluated and the series summed.

We consider a spherical charged shell of radius L, $e\rho(\vec{x}-\vec{R}(t))=(e/4\pi L^2)\delta(|\vec{x}-\vec{R}(t)|-L)$, and an elementary calculation shows that

$$\gamma_n=2e^2(2L)^{n-1}/(n+1). \quad (2.6)$$

This can now be substituted into Eq. (2.4) and the series summed, with the result

$$\ddot{\vec{R}}(t)=\frac{\vec{F}(t)}{m}+\left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)\times\left(e^{-(2L/c)(d/dt)}+\frac{2L}{c}\frac{d}{dt}-1\right)\dot{\vec{R}}(t), \quad (2.7)$$

where $m\equiv m_0+2e^2/3c^2L$ and $\tau\equiv 2e^2/3mc^3$. This can be simplified further by using the fact that $\exp[-(2L/c)(d/dt)]$ is a time-delay operator; i.e.,

$$e^{-(2L/c)(d/dt)}\dot{\vec{R}}(t)=\dot{\vec{R}}(t-2L/c). \quad (2.8)$$

This equation is true at least when the operator on the left-hand side is defined by its series expansion, while $\dot{\vec{R}}(t-2L/c)$ is defined as a Taylor series about its value at time t. Therefore, if $c\tau\neq L$, we have

$$\ddot{\vec{R}}(t)=\frac{\vec{F}(t)}{m(1-c\tau/L)}+\xi[\dot{\vec{R}}(t-2L/c)-\dot{\vec{R}}(t)], \quad (2.9)$$

where $\xi=(c/2L)(c\tau/L)/(1-c\tau/L)$. This linear differential-difference equation determines the motion of a spherical shell of charge, and we will investigate its solution in the following two sections.

C. Motion in the absence of external forces: The question of runaway solutions

We first discuss the motion of a shell of charge in the absence of external forces, $\vec{F}(t)=0$. Therefore, we consider the homogeneous equation

$$\ddot{\vec{R}}(t)=\xi[\dot{\vec{R}}(t-2L/c)-\dot{\vec{R}}(t)], \quad (2.10)$$

and a simple argument indicates that the character of the solutions depends markedly on whether $L>c\tau$ (i.e., $\xi>0$) or $L<c\tau$ (i.e., $\xi<0$). Suppose the equation has a monotonically increasing solution $\ddot{\vec{R}}(t)$. Then $\ddot{\vec{R}}(t)>0$ and the quantity inside the square brackets in Eq. (2.10) is negative, implying that a monotonically increasing runaway solution can occur only for $\xi<0$.

For a more systematic discussion, consider solutions to Eq. (2.10) of the form

$$\dot{\vec{R}}(t)=\vec{A}e^{\beta t/\tau}, \quad (2.11)$$

where $\vec{A}$ is a constant vector and β is a (generally) complex constant. Upon substitution, we find a transcendental equation for β

$$\beta/\tau=\xi(e^{-(2L/c)(\beta/\tau)}-1). \quad (2.12)$$

Corresponding to each root β of this equation, there is a solution to the homogeneous equation, Eq. (2.10), and a general solution is a linear superposition of these with arbitrary coefficients. A runaway solution will occur if any of the roots β has a positive real part.

1. $L>c\tau$

Introduce the dimensionless variables $\eta=2L\beta/c\tau$ and $g=1/(L/c\tau-1)$, and then let $\eta\equiv\mu+i\nu$ (with μ,ν real). Equation (2.12) now reads

$$\mu=-g(1-e^{-\mu}\cos\nu), \quad (2.13a)$$

$$\nu=-ge^{-\mu}\sin\nu. \quad (2.13b)$$

Since g is positive for $L>c\tau$, it is clear that Eq. (2.13a) can have solutions only for *negative* values of μ, implying that there are *no* runaway solutions for $L>c\tau$. Nevertheless, Eq. (2.13) does have an infinite number of complex roots (with $\mu<0$), and these can be located graphically; they occur in complex-conjugate pairs. The trajectories of the three lowest roots β are plotted in Fig. 1 as a function of $\rho\equiv L/c\tau$. The physical significance of these damped-oscillatory solutions will be commented upon later in the section.

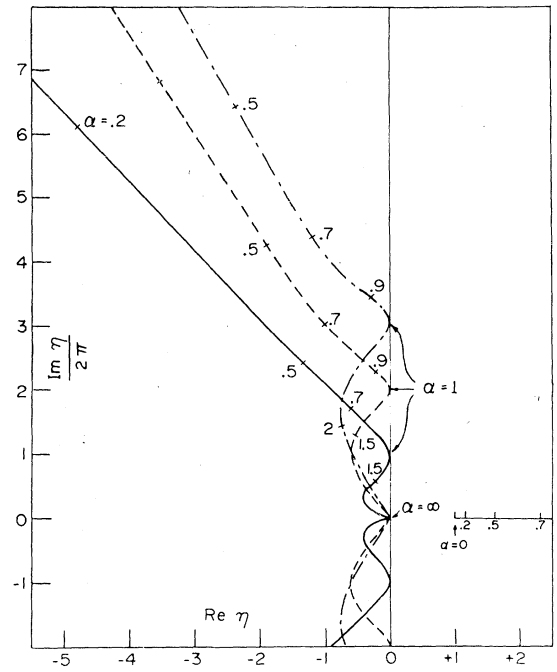

FIG. 1. Complex trajectories (as a function of $\rho = L/c\tau$) of the solutions to Eq. (2.12). There are an infinite number of such trajectories. Shown here are the three trajectories having the longest period for fixed ρ. For $\rho > 1$, there are damped ($\mathrm{Re}\,\beta < 0$) oscillatory solutions. When $\rho < 1$, the runaway root "appears" on the positive real axis. As $\rho \to 0$, all trajectories go to $-\infty$ except for the runaway mode, which approaches $\beta = +1$. The authors thank Herbert Levine for carrying out the numerical calculations which are summarized in this figure.

2. $L = c\tau$

For the case $L = c\tau$, the equation of motion is simply

$$\ddot{\vec{R}}(t - 2\tau) - \ddot{\vec{R}}(t) = 0 . \tag{2.14}$$

The solutions to this equation are clearly periodic; trying a solution of the form of Eq. (2.11), we find

$$\beta = 2\pi i n, \quad n = 0, 1, 2, \ldots . \tag{2.15}$$

The solutions in this case display neither runaway behavior nor damping.

3. $L < c\tau$

For the case $L < c\tau$, Eq. (2.12) becomes

$$\eta = |g|(1 - e^{-\eta}), \tag{2.16}$$

where $|g| = 1/(1 - L/c\tau) > 1$. Note that the function $y(\eta) \equiv |g|(1 - e^{-\eta})$ has the slope $(dy/d\eta)_{\eta=0} = |g| > 1$, making it clear that there is always one positive root. In other words, Eq. (2.16) displays a runaway solution whenever $0 \leq L < c\tau$. It is perhaps worth commenting that these runaway solutions occur for finite values of the charge radius and are not merely artifacts of the point limit.

In addition to this solution, there are again an infinite number of complex roots, corresponding to damped-oscillatory solutions. The trajectories of the three lowest complex roots and of the "runaway" root are plotted in Fig. 1 as a function of $\rho \equiv L/c\tau$. These simple calculations show how the complete series in Eq. (2.4) manages to suppress the runaways for a sufficiently extended charge.

It is interesting to translate the condition that $L > c\tau$ for no runaways into a condition on the particle's mechanical mass. Since $m = m_0 + 2e^2/3c^2L = m_0 + mc\tau/L$, the condition that there be no runaway solutions is just

$$m_0 > 0 . \tag{2.17}$$

Several authors, working in a number of different contexts, have pointed this out before.[10] Conversely, runaway solutions do occur only when the charged particle is given a negative mechanical mass.

D. Physical interpretation of the solutions

We have just found that the general solution of the homogeneous equation has the form

$$\dot{\vec{R}}(t) = \dot{\vec{R}}(0) + \sum_{n=1}^{\infty} e^{-|\beta_r^{(n)}|\,t/\tau}(\vec{A}_n e^{i\beta_I^{(n)} t/\tau} + \vec{A}_n^* e^{-i\beta_I^{(n)} t/\tau}), \tag{2.18a}$$

for $L > c\tau$, and

$$\dot{\vec{R}}(t) = \dot{\vec{R}}(0) + \vec{B} e^{\beta t/\tau} + \sum_{n=1}^{\infty} e^{-|\beta_r^{(n)}|\,t/\tau}(\vec{B}_n e^{i\beta_I^{(n)} t/\tau} + \vec{B}_n^* e^{-i\beta_I^{(n)} t/\tau}) \tag{2.18b}$$

for $L < c\tau$. Recall that the value of $\beta^{(n)} \equiv \beta_r^{(n)} + i\beta_I^{(n)}$ depends upon the parameter $L/c\tau$. For the case $L > c\tau$, the asymptotic motion of the particle is given simply by

$$\dot{\vec{R}}(t) \underset{t\to\infty}{\longrightarrow} \text{constant}, \tag{2.19}$$

showing that the details of the particle's structure affect only the rate and manner in which the asymptotic behavior is approached.

The physical interpretation of the solutions to the homogeneous equation remains to be discussed. It is clear that in order to specify the solution (2.18) we need to know all the time derivatives of the position vector or, equivalently, knowledge of the particle's history over a finite time interval. This is a standard property of solutions to differential-difference equations of retarded type.[15] The

point is that we began with a set of coupled equations, Eqs. (2.1) and (2.2), for the interacting system of particle plus radiation fields and, in obtaining Eq. (2.10), eliminated the field degrees of freedom. We always have the option of returning to the original set of equations and specifying at some definite time the radiation field plus the Newtonian initial conditions for the particle. But, after eliminating the field variables, we must specify all time derivatives of the particle position vector in order to supply information equivalent to specifying the initial conditions of the particle and the radiation fields. In effect, the damped-oscillatory solutions to the homogeneous equation tell us how the interacting system returns to "equilibrium" (which essentially means constant velocity for the particle) in the time after a force has acted on the particle.

In our treatment of runaways, we have employed a linear equation of motion derived under the assumption that nonlinear terms, which are of order $\ddot{\vec{R}}^2/c^2$ or higher times the terms retained, could be neglected. It may be objected that this procedure can be justified only if the analysis of the linear terms does not reveal runaway behavior. Thus, for $L<c\tau$, when the linear equation has a runaway solution, the nonlinear terms could become large and conceivably modify radically the character of the solution of the linear equation. On the other hand, if one does work with the correct relativistic form of the Lorentz-Dirac equation, one again finds that a point charge displays runaway behavior. Thus it would not seem likely that one can look to the nonlinear terms in the equation of motion to suppress the runaways which occur for $L<c\tau$.

E. Motion with time-dependent external forces: The question of preacceleration

We can now investigate the solution to the inhomogeneous equation

$$\ddot{\vec{R}}(t)=\frac{\vec{F}(t)}{m}+\left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)\left[\dot{\vec{R}}\left(t-\frac{2L}{c}\right)-\dot{\vec{R}}(t)+\frac{2L}{c}\ddot{\vec{R}}(t)\right], \tag{2.20}$$

corresponding to motion in a time-dependent external force field.[16] This equation is conveniently solved by introducing the Fourier transforms

$$\ddot{\vec{R}}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dt\,\ddot{\vec{R}}(t)e^{-i\omega t},\quad \vec{F}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dt\,\vec{F}(t)e^{-i\omega t}. \tag{2.21}$$

Then, we have

$$\ddot{\vec{R}}(\omega)=\frac{\vec{F}(\omega)/m}{1+(i/\omega)(c/2L)(c\tau/L)[e^{-i\omega 2L/c}+i\omega 2L/c-1]}, \tag{2.22}$$

and the result can be rewritten in terms of the response function $G(t-t')$:

$$m\ddot{\vec{R}}(t)=\int_{-\infty}^{+\infty}dt'G(t-t')\vec{F}(t'), \tag{2.23}$$

with

$$G(t-t')=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{1+(i/\omega)(c/2L)(c\tau/L)[e^{-i\omega 2L/c}+i\omega 2L/c-1]}. \tag{2.24}$$

The behavior, causal or noncausal, of the motion is obviously reflected in the properties of $G(t-t')$.

In the point-charge limit ($L\to 0$), Eq. (2.24) reduces to the expression

$$G(t-t')=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{1-i\omega\tau}=\begin{cases}0 & \text{if } (t-t')>0\\ \frac{1}{\tau}e^{-(t'-t)/\tau} & \text{if } (t-t')<0.\end{cases} \tag{2.25}$$

Therefore, the response function shows noncausal behavior, originating in the fact that the singularity in the integrand of Eq. (2.25) lies in the lower half ω plane ($\omega=-i/\tau$). This result is, of course, exactly that given in Eq. (1.4) during the discussion of the Lorentz-Dirac theory.

For the extended charge, we must ask for the zeros of the full denominator given in Eq. (2.22), namely

$$\omega+i\left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)\left(e^{-i\omega 2L/c}+i\omega\frac{2L}{c}-1\right)=0. \tag{2.26}$$

With the substitution $\beta=i\omega\tau$ and some elementary rearrangement, Eq. (2.26) is seen to be the same as Eq. (2.12) and the condition that all zeros of Eq. (2.26) lie in the upper half ω plane the same

as the condition for no runaways. Therefore, we can immediately adopt the results from Sec. II C and conclude that the response function is causal if $L > c\tau$ but noncausal if $L < c\tau$. We can be more specific and explicitly display the form of the response function. Let ω_n denote the upper half ω plane zeros of Eq. (2.26), implying that this expression can be rewritten as

$$\left\{\left(1-\frac{c\tau}{L}\right)\omega_n + i\left(\frac{c}{2L}\right)\left(\frac{c\tau}{L}\right)\left(e^{-i\omega_n 2L/c}-1\right)\right\} + (\omega-\omega_n)\left\{1+\frac{2L}{c}\left(1-\frac{c\tau}{L}\right)i\omega_n\right\}+\cdots, \quad (2.27)$$

where the expression inside the first curly brackets is identically zero and the expression inside the second curly brackets represents the residue. Then, for $(t-t')>0$, we have

$$G(t-t') = \sum_{n=1}^{\infty} \frac{i\omega_n e^{i\omega_n(t-t')}}{1+(2L/c)(1-c\tau/L)i\omega_n}. \quad (2.28)$$

In addition, Eq. (2.26) has the lower half ω plane zero at $\omega = -i\beta/\tau$ for the case $L < c\tau$, giving the response function for $(t-t')<0$ as

$$G(t-t'<0) = \begin{cases} 0 & \text{for } L > c\tau \\ -\dfrac{(\beta/\tau)e^{-\beta(t'-t)/\tau}}{1+(2L/c)(1-c\tau/L)\beta/\tau} & \text{for } L < c\tau. \end{cases} \quad (2.29)$$

This again shows that the defect in the classical theory is not simply a manifestation of the point limit. Also, one must recall that a general solution to Eq. (2.20) consists of the particular solution Eq. (2.23) plus the general solution to the homogeneous equation.

Summarizing, we have found that including the effect of radiation reaction on a charged spherical shell results neither in runaway behavior nor in preacceleration if the charge radius of the shell $L > c\tau$, while the opposite conclusions follow if $L < c\tau$. This implies that the classical theory of charged particles, including the force of radiation reaction, is an internally consistent theory so long as the mechanical mass of the particle is positive. The previously mentioned defects, namely runaway behavior and preacceleration, appear only when the mechanical mass becomes negative in the attempt to work with the observed mass and at the same time to reach the point-charge limit. This limit lies outside the domain of the classical theory.

III. QUANTUM-MECHANICAL TREATMENT OF RADIATION REACTION

A. Equation of motion

Proceeding in close analogy with the Abraham-Lorentz treatment of radiation reaction described in Sec. II, we propose to demonstrate how nonrelativistic quantum electrodynamics suppresses the runaway solutions to the equation of motion of a *point* electron, while preserving the physically correct correspondence limit. Our procedure will be to derive from the standard Hamiltonian the (Heisenberg-picture) operator equation of motion for a nonrelativistic charged particle interacting with a quantized electromagnetic field. Eliminating the self-fields in favor of the retarded solution to Maxwell's equations, we shall finally arrive at an equation similar in form to the classical equation of motion (2.4). A detailed derivation of the quantum-mechanical equation will occupy the remainder of this section.

We shall work with the Hamiltonian[8]

$$H = \frac{1}{2m_0}\left[\vec{P} - \frac{e}{c}\vec{A}(\vec{R})\right]^2 + \frac{1}{8\pi}\int d\vec{r}\{\vec{E}^2(\vec{r},t) + [\vec{\nabla}\times\vec{A}(\vec{r},t)]^2\}, \quad (3.1)$$

where

$$\vec{A}(\vec{R}) = \int d\vec{r}\,\rho(\vec{r}-\vec{R}(t))\vec{A}(\vec{r},t)$$

and

$$\vec{E} = \vec{E}_{\text{long}} + \vec{E}_{\text{trans}}.$$

This describes a nonrelativistic charged particle of mechanical mass m_0 and (spherically symmetric) charge distribution[17] ρ [defined so that $\int d\vec{r}\,\rho(\vec{r}-\vec{R}) = 1$] interacting with the electromagnetic field, computed in the Coulomb gauge. $\vec{P}(t)$ and $\vec{R}(t)$ are, respectively, the Heisenberg-picture momentum and position operators of the particle.

Proceeding in standard fashion, we can write down the Heisenberg equations of motion and arrive at the operator form of the Lorentz force equation

$$\frac{d}{dt}(m_0\dot{\vec{R}}) = e\vec{E}(\vec{R}) + \frac{e}{2c}(\dot{\vec{R}}\times\vec{H} - \vec{H}\times\dot{\vec{R}}), \quad (3.2a)$$

where

$$\vec{H}(\vec{R}) = \vec{\nabla}\times\vec{A}(\vec{R}), \quad \vec{E}(\vec{R}) = -\vec{\nabla}\tilde{\phi}(\vec{R}) - \frac{1}{c}\frac{\partial\vec{A}(\vec{R})}{\partial t},$$

and the usual operator field equations for the potentials (here written in the Lorentz gauge)

$$-\Box\vec{A}(\vec{r},t) = 4\pi e\vec{j}(\vec{r},t), \quad -\Box\phi(\vec{r},t) = 4\pi e\rho(\vec{r},t). \quad (3.2b)$$

As in the classical case, our goal now is to use the operator field equations to eliminate the self-fields from the equation of motion of the particle. While the Coulomb gauge is the most convenient for setting up the canonical formalism, these calculations (i.e., the evaluation of the electric and

magnetic fields in the force equation) are best carried out in the Lorentz gauge, in which the retarded solutions to Eq. (3.2b) take the form

$$\vec{A}(\vec{r},t)=\vec{A}_{in}(\vec{r},t)+\frac{e}{c}\int d\vec{r}'\frac{\vec{j}(\vec{r}',t'_{ret})}{|\vec{r}-\vec{r}'|}$$

and (3.3)

$$\phi(\vec{r},t)=\phi_{in}(\vec{r},t)+e\int d\vec{r}'\frac{\rho(\vec{r}',t'_{ret})}{|\vec{r}-\vec{r}'|},$$

where $t'_{ret}=t-(|\vec{r}-\vec{r}'|/c)$ and the single-particle current density is the symmetrized product of the density and velocity operators

$$\vec{j}(\vec{r},t)=\tfrac{1}{2}[\rho(\vec{r}-\vec{R}(t)),\dot{\vec{R}}(t)]_+. \tag{3.4}$$

Any operator evaluated at the retarded time t'_{ret} can be expanded formally about its value at the time t as

$$O(t'_{ret})=e^{+iH(t'_{ret}-t)}O(t)e^{-iH(t'_{ret}-t)}$$
$$=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\frac{|\vec{r}-\vec{r}'|^n}{c^n}(\mathrm{ad}^nH)O(t), \tag{3.5}$$

where

$$(\mathrm{ad}H)O=[H,O]_-,\quad (\mathrm{ad}^2H)O=[H,[H,O]_-]_-,\ \text{etc.}$$

Using Eqs. (3.3) and (3.5) in the Lorentz force equation, and dropping the magnetic and nonlinear terms as before, we arrive at a quantum-mechanical equation of motion which differs from the classical one only by symmetrization and by the replacement of time derivatives with commutators [i.e., $d/dt\to i(\mathrm{ad}H)$]:

$$m_0\ddot{\vec{R}}(t)=\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!\,c^n}\int\int d\vec{x}\,d\vec{x}'\tfrac{1}{2}[\rho(\vec{x}-\vec{R}(t))|\vec{x}-\vec{x}'|^{n-1},(\mathrm{ad}^{n+1}H)\vec{j}(\vec{x}',t)]_+ + e\vec{E}_{in}(\vec{R}(t),t). \tag{3.6}$$

This equation represents the starting point for our study of radiation reaction, and we begin by evaluating the required nested commutators.

B. Evaluation of the commutators

In evaluating the multiple commutators $(\mathrm{ad}^n_{\cdot}H)\vec{j}(\vec{x},t)$, we shall greatly simplify the computation by retaining only those terms which are a product of the charge density and some time derivative of the position operator. This is analogous to the classical treatment of Sec. II A and corresponds to dropping terms of order $(\dot{\vec{R}}^2/c^2)$ or higher, consistent with the nonrelativistic treatment of the particle motion employed throughout this paper. With this simplification, the equation of motion will be identical in form to Eq. (2.4), except that the coefficients now include quantum-mechanical corrections.

We use a prime to indicate that only terms linear in $\vec{R}$ or its time derivatives are retained and define a "right-ordering" as that ordering in which the operator d^nR/dt^n stands to the right of $\rho(\vec{r}-\vec{R})$ (note that $d^n\vec{R}/dt^n$ and ρ do not commute). This ordering allows us to evaluate the integrals over the charge distributions as simple structure-dependent constants:

$$\int\int d\vec{x}\,d\vec{x}'\rho(\vec{x}-\vec{R}(t))|\vec{x}-\vec{x}'|^n\rho(\vec{x}'-\vec{R}(t))\frac{d^m\vec{R}}{dt^m}$$
$$=\left[\int\int d\vec{x}\,d\vec{x}'\rho(x)|\vec{x}-\vec{x}'|^n\rho(x')\right]\frac{d^m\vec{R}}{dt^m}. \tag{3.7}$$

Actually, since Eq. (3.6) is symmetrized, we see that both the left- and right-ordered linear contributions to the multiple commutator are needed. In all that follows, the left-ordering can be obtained from the right-ordering simply by reversing the order of all operators and by letting $\hbar\to-\hbar$; consequently, only the right-ordered expressions will be given explicitly.

We start by writing down the right-ordered form of the current itself:

$$\vec{j}(\vec{r},t)=\tfrac{1}{2}[\rho(\vec{r}-\vec{R}),\dot{\vec{R}}]_+$$
$$=\rho(\vec{r}-\vec{R})\dot{\vec{R}}-\tfrac{1}{2}[\rho(\vec{r}-\vec{R}),\dot{\vec{R}}]_-$$
$$=\rho(\vec{r}-\vec{R})\dot{\vec{R}}-\frac{i\hbar}{2m_0}\vec{\nabla}_{\vec{R}}\rho(\vec{r}-\vec{R})$$
$$=\rho(\vec{r}-\vec{R})\dot{\vec{R}}+\frac{i\hbar}{2m_0}\vec{\nabla}_{\vec{r}}\rho(\vec{r}-\vec{R}). \tag{3.8}$$

The next step is to commute this with the Hamiltonian to obtain $(\mathrm{ad}H)\vec{j}(\vec{r},t)$:

$$[H,\vec{j}(\vec{r},t)]_-=\frac{1}{i}\left(\rho\ddot{\vec{R}}+\frac{\partial\rho}{\partial t}\dot{\vec{R}}+\frac{i\hbar}{2m_0}\vec{\nabla}\frac{\partial\rho}{\partial t}\right). \tag{3.9}$$

Using the operator current-conservation equation, we can replace the time derivative of the density by the divergence of the current:

$$\frac{\partial\rho}{\partial t}=-\vec{\nabla}\cdot\vec{j}, \tag{3.10}$$

and then again replace $\vec{j}$ with Eq. (3.8). The algebra is straightforward and gives the result

$$[H,\vec{j}(\vec{r},t)]_-=\frac{1}{i}\rho\ddot{\vec{R}}-\frac{1}{i}(\vec{\nabla}\rho\cdot\dot{\vec{R}})\dot{\vec{R}}-\frac{\hbar}{2m_0}\vec{\nabla}(\vec{\nabla}\rho)\cdot\dot{\vec{R}}$$
$$-\frac{\hbar}{2m_0}\vec{\nabla}^2\rho\dot{\vec{R}}-\frac{i\hbar^2}{4m_0^2}\vec{\nabla}\vec{\nabla}^2\rho. \tag{3.11}$$

The second term on the right-hand side of Eq. (3.11) is a nonlinear term of the kind dropped in the classical calculation as well as here. The last three terms, however, arise only in the quantum-mechanical case and it is terms such as these that will generate quantum-mechanical corrections to the Lorentz-Dirac equation.

Next we evaluate $(\mathrm{ad}^2H)\vec{j}(\vec{r},t)$ by commuting H with Eq. (3.11). Again using the continuity equation, we obtain

$$(\mathrm{ad}^2H)\vec{j}(\vec{r},t)=[H,[H,\vec{j}]_-]_- = -\rho\dddot{\vec{R}}+2(\vec{\nabla}\rho\cdot\dot{\vec{R}})\ddot{\vec{R}}+(\vec{\nabla}\rho\cdot\ddot{\vec{R}})\dot{\vec{R}}-\vec{\nabla}(\vec{\nabla}\rho\cdot\dot{\vec{R}})\cdot\dot{\vec{R}}\dot{\vec{R}}+\frac{i\hbar}{2m_0}\vec{\nabla}^2\rho\ddot{\vec{R}}+\frac{i\hbar}{2m_0}\vec{\nabla}(\vec{\nabla}\rho\cdot\ddot{\vec{R}})$$
$$-\frac{i\hbar}{m_0}[\vec{\nabla}(\vec{\nabla}^2\rho)\cdot\dot{\vec{R}}]\dot{\vec{R}}-\frac{i\hbar}{2m_0}\vec{\nabla}[\vec{\nabla}(\vec{\nabla}\rho\cdot\dot{\vec{R}})\cdot\dot{\vec{R}}]+\frac{\hbar^2}{4m_0^2}\vec{\nabla}^2\vec{\nabla}^2\rho\dot{\vec{R}}$$
$$+\frac{\hbar^2}{2m_0^2}\vec{\nabla}[\vec{\nabla}(\vec{\nabla}^2\rho)\cdot\dot{\vec{R}}]+\frac{i\hbar^3}{8m_0^3}\vec{\nabla}\vec{\nabla}^2\vec{\nabla}^2\rho\,.$$

It is clear that the higher-order commutators rapidly become more complicated. Nevertheless, confining ourselves to terms linear in the position operator or its time derivatives, we can work out the necessary combinatoric factors to find the following expression for $(\mathrm{ad}^nH)\vec{j}$:

$$[(\mathrm{ad}^nH)\vec{j}(\vec{r},t)]' = (-i)^n\sum_{m=0}^{n}\left(\frac{-i\hbar}{2m_0}\right)^m\binom{n}{m}(\vec{\nabla}^2)^m\rho\frac{d^{n+1-m}\vec{R}}{dt^{n+1-m}}$$
$$+(-i)^n\sum_{m=1}^{n}\left(\frac{-i\hbar}{2m_0}\right)^m\binom{n}{m-1}\vec{\nabla}(\vec{\nabla}^2)^{m-1}\left(\vec{\nabla}\rho\cdot\frac{d^{n+1-m}\vec{R}}{dt^{n+1-m}}\right)-i\left(\frac{-\hbar}{2m_0}\right)^{n+1}\vec{\nabla}(\vec{\nabla}^2)^n\rho\,. \tag{3.13}$$

When this expression and the appropriate left-ordered expression are inserted into Eq. (3.6), two simplifications can be made. First, the last term in Eq. (3.13) vanishes upon angular integration because of the assumed spherical symmetry of the charge distribution. Second, in symmetrizing we reverse the order of the operators and let $\hbar\to-\hbar$, with the result that all terms containing odd powers of $\hbar$ drop out of the equation of motion. Therefore, Eq. (3.6) now reads

$$m_0\ddot{\vec{R}}(t)=\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\sum_{k=0}^{(n+1)/2}\frac{(-1)^{n+1}}{n!\,c^{n-2k}}\left(\frac{-\lambda^2}{4}\right)^k\binom{n+1}{2k}\left[\int\int d\vec{x}\,d\vec{x}'\rho(x)\,|\vec{x}-\vec{x}'\,|^{n-1}(\vec{\nabla}_{\vec{x}'}{}^2)^{2k}\rho(x')\right]\frac{d^{n+2-2k}\vec{R}}{dt^{n+2-2k}}$$
$$+\frac{2e^2}{3c^2}\sum_{n=1}^{\infty}\sum_{k=1}^{(n+1)/2}\frac{(-1)^{n+1}}{n!\,c^{n-2k}}\left(\frac{-\lambda^2}{4}\right)^k\binom{n+1}{2k-1}\left[\int\int d\vec{x}\,d\vec{x}'\rho(x)\,|\vec{x}-\vec{x}'\,|^{n-1}(\vec{\nabla}_{\vec{x}'}{}^2)^{2k-1}\vec{\nabla}_{\vec{x}'}(\vec{\nabla}_{\vec{x}'})_i\rho(x')\right]\frac{d^{n+2-2k}\vec{R}_i}{dt^{n+2-2k}}$$
$$+e\vec{E}_{\mathrm{in}}(\vec{R}(t),t)\,, \tag{3.14}$$

where $\lambda=\hbar/m_0c$ is the Compton wavelength and

$$\left\{\frac{m}{2}\right\}=\begin{cases}m/2 & \text{if } m \text{ is even}\\ (m-1)/2 & \text{if } m \text{ is odd}\,.\end{cases}$$

In the second term of the equation, spherical symmetry allows the replacement $(\vec{\nabla}_{\vec{x}'})_j(\vec{\nabla}_{\vec{x}'})_i\to\frac{1}{3}\delta_{ij}\vec{\nabla}_{\vec{x}'}{}^2$. Finally, we make a change of summation indices $\sum_n\sum_k\to\sum_l\sum_k$, where $l\equiv n-2k$. Equation (3.6) then takes the form

$$m_0\ddot{\vec{R}}(t)=e\vec{E}_{\mathrm{in}}(\vec{R}(t),t)-\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}A_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}}\,, \tag{3.15}$$

where

$$A_n=\left(1+\frac{\lambda}{3(n+2)}\frac{\partial}{\partial\lambda}\right)\left(1+\frac{\lambda}{n+1}\frac{\partial}{\partial\lambda}\right)\Omega_n \tag{3.16}$$

and

$$\Omega_n=\sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{-\lambda^2}{4}\right)^k\int\int d\vec{x}d\vec{x}'\rho(x)\,|\vec{x}-\vec{x}'\,|^{n-1+2k}(\vec{\nabla}_{\vec{x}'}{}^2)^{2k}\rho(x')\,.$$

Equation (3.15) is formally the same as Eq. (2.4), the difference being that each coefficient A_n is expressed as a power series in λ^2/L^2, where L is a length characteristic of the size of the particle. The leading term in the expansion [i.e., the $k=0$ term in Eq. (3.16)] gives the classical coefficient γ_n [Eq. (2.5)], guaranteeing that the quantum-mechanical equation has the proper correspondence limit.

Our prime concern in the rest of this paper will be to analyze Eqs. (3.15) and (3.16). In particular, our intention is to study the *point-charge limit* of the quantum-mechanical equations (requiring that we first sum the series for each coefficient A_n) and demonstrate that the theory, in contrast to the classical theory of point charges, is internally consistent.

C. Evaluation of the coefficients A_n

We now present the evaluation of the structure-dependent coefficients A_n which enter into the quantum-mechanical equation of motion, Eq. (3.15). Our main goal is to evaluate these in the point limit and it may be useful to anticipate the results using dimensional arguments. Recall that the analogous classical coefficients γ_n have the behavior L^{n-1} and that the quantum-mechanical coefficients A_n can be expressed as the classical coefficient multiplied by an infinite power series in λ^2/L^2. The important points are that there is a new length parameter, λ, in the quantum theory and it enters these equations only in the square. Consequently, we can expect on simple dimensional grounds that, in the point limit, the odd coefficients A_{2m+1} behave like $(\lambda^2)^m$ while the even coefficients A_{2m}, for $m>0$, vanish as in the classical case. These results will be proved below. The self-energy term A_0 is, of course, not so easily anticipated, since the analogous classical term diverges in the point limit. We start by analyzing this term in more detail.

1. *Electrostatic self-energy*

The expression for A_0 is more easily manipulated if we work with the Fourier transform of the charge distribution,

$$\tilde{\rho}(k)=\int d\vec{x}e^{i\vec{k}\cdot\vec{x}}\rho(x)\,. \tag{3.17}$$

Then, from Eqs. (3.15) and (3.16), we have

$$\delta m=\frac{2e^2}{3c^2}A_0=\frac{2e^2}{3c^2}\left(1+\frac{\lambda}{6}\frac{\partial}{\partial\lambda}\right)\left(1+\lambda\frac{\partial}{\partial\lambda}\right)\Omega_0\,, \tag{3.18}$$

where

$$\Omega_0=\int\frac{d\vec{k}}{(2\pi)^3}\tilde{\rho}(k)^2\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left(\frac{\lambda k^2}{2}\right)^{2l}\int\frac{d\vec{r}}{r}e^{i\vec{k}\cdot\vec{r}}r^{2l}\,. \tag{3.19}$$

Carrying out the $\vec{r}$ integration leads to an improper integral over k, which we regularize by taking the Cauchy principal value, giving the final result

$$\Omega_0=\frac{2}{\pi}\mathrm{P}\int_0^{\infty}dk\,\frac{\tilde{\rho}(k)^2}{1-\lambda^2k^2/4}\,, \tag{3.20}$$

where P denotes the Cauchy principal value.

That this expression reduces to the classical expression when $\lambda=0$ is obvious, but there are several other points worth mentioning. First, if we now go to the point-charge limit $\tilde{\rho}(k)=1$, *keeping* λ *fixed*, we find

$$\Omega_0(L=0)=\frac{2}{\pi}\mathrm{P}\int_0^{\infty}dk\,\frac{1}{1-\lambda^2k^2/4}=0\,. \tag{3.21}$$

Thus, from Eq. (3.18), one finds that the electrostatic self-energy of a point charge is zero in nonrelativistic quantum electrodynamics. Note, however, that any attempt at a "semiclassical" approximation to the self-energy [such as expanding the integrand in Eq. (3.20) in powers of $\hbar$] would still lead to a divergent result.

While Eq. (3.21) represents the main result of this section, it is also interesting to consider Ω_0 for an extended charge. The form factor $\tilde{\rho}(k)$ is "cutoff" at $k\sim L^{-1}$, implying that

$$\begin{aligned}\Omega_0&\sim L^{-1}\quad\text{for } L\gg\lambda\\ &\sim L/\lambda^2\quad\text{for } L\ll\lambda\,.\end{aligned} \tag{3.22}$$

Moreover, the maximum value for the electrostatic self-energy is attained when $L\sim\lambda$, in which case

$$\delta m\sim e^2/\lambda c^2=\alpha m\,. \tag{3.23}$$

This seems physically reasonable and excludes the possibility of a purely electromagnetic origin for the electron mass within the framework of nonrelativistic quantum electrodynamics. The particle's mass and form factor, if any, must arise from interactions outside the scope of this theory.[18] Finally, we note that the electrostatic self-energy actually becomes negative for $0<L\ll\lambda$. All of the these points can be concretely illustrated by evaluating Eq. (3.20) for the simple case of a Yukawa-type charge distribution

$$\tilde{\rho}(k)=(1+k^2L^2)^{-1}\,. \tag{3.24}$$

Then we find

$$\delta m=(\tfrac{2}{3}\alpha m)\xi\left[\frac{\xi^6+8\xi^4-3\xi^2-2}{(1+\xi^2)^4}\right],\quad \xi=2L/\lambda\,. \tag{3.25}$$

This is plotted in Fig. 2.

Additional insight into these perhaps surprising results is obtained by transforming Eq. (3.20) back to coordinate space, which gives

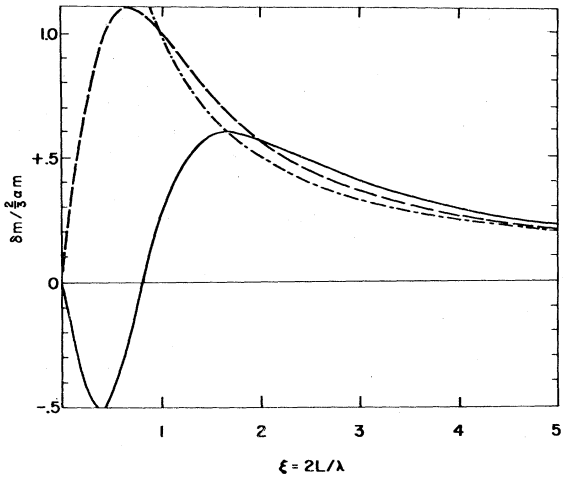

FIG. 2. Electron self-energy with a Yukawa form factor. The solid and dashed curves correspond to Eqs. (3.20) and (3.25), respectively. The dot-dash curve is the classical result ξ^{-1}.

$$\Omega_0 = \int \frac{d\vec{x}\, d\vec{x}'}{|\vec{x}-\vec{x}'|}\rho(x)\left(1+\frac{\lambda^2}{4}\vec{\nabla}_{\vec{x}'}^{\,2}\right)^{-1}\rho(x'), \qquad (3.26)$$

where the integral operator $(1+\lambda^2\vec{\nabla}^2/4)^{-1}$ is defined by

$$(1+\lambda^2\vec{\nabla}^2/4)^{-1}\rho(x) = \int dy\, S_\lambda(x-y)\rho(y) \equiv \rho_{\rm eff}(x), \qquad (3.27)$$

with

$$S_\lambda(r) = \mathrm{P}\int \frac{d\vec{k}}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{r}}}{1-\lambda^2k^2/4} = -\cos(2r/\lambda)/(\pi\lambda^2 r).$$

In this formulation, we see that all the physics involved in the interaction of the charged particle with its quantized self-field is summarized in the "spreading function" $S_\lambda(r)$, which in Eq. (3.27) generates an effective charge distribution "smeared out" over a Compton wavelength. Equivalently, we can write

$$\Omega_0 = \int d\vec{x}\,\rho(x)\phi_{\rm eff}(x), \qquad (3.28)$$

with

$$\phi_{\rm eff}(x) = \int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}\rho_{\rm eff}(x'),$$

so that $\phi_{\rm eff}(x)$ is the effective scalar self-potential which generates the self-energy. For a uniform charge distribution of radius L, we find

$$\phi_{\rm eff}(r>L) = \frac{1}{r}\left[1+\frac{3\lambda^2}{4L^2}\left(\cos\frac{2L}{\lambda}-\frac{\lambda}{2L}\sin\frac{2L}{\lambda}\right)\cos\frac{2r}{\lambda}\right],$$

$$\phi_{\rm eff}(r<L) = \frac{3}{2L}\left(1-\frac{r^2}{3L^2}\right) + \frac{3\lambda^2}{4L^2}\left[1-\left(\cos\frac{2L}{\lambda}+\frac{2L}{\lambda}\sin\frac{2L}{\lambda}\right)j_0\left(\frac{2r}{\lambda}\right)\right]. \qquad (3.29)$$

This expression for $\phi_{\rm eff}(r)$ is shown in Fig. 3 for several values of λ/L, and it is clear that in the limit $\lambda/L \to \infty$ the effective potential vanishes inside the charge radius.

In concluding this subsection we wish to emphasize two points. First, we reiterate that what has been calculated here is the *electrostatic* self-energy of the electron. The *transverse* self-energy, which in second-order perturbation theory has the same linear divergence as the classical electrostatic self-energy, is an effect of order $\vec{R}^2/c^2$. In a relativistic calculation these contributions to the self-energy are treated on an equal footing; however, in a nonrelativistic treatment such as ours these two contributions to the self-energy are physically distinct, and in systematically neglecting terms of order $\vec{R}^2/c^2$ we are discarding the transverse self-energy. Thus our calculation should be understood as showing that the infinite electrostatic self-energy which occurs in the nonrelativistic *classical* calculation has no counterpart in the quantum theory.

We also want to stress the importance of going to the point limit while keeping $\hbar$ finite. In Sec. IV, we shall consider the correspondence limit $(\hbar \to 0)$ of the solutions of the quantum-mechanical equations of motion for a point charge, the important idea being that the order of going to the point

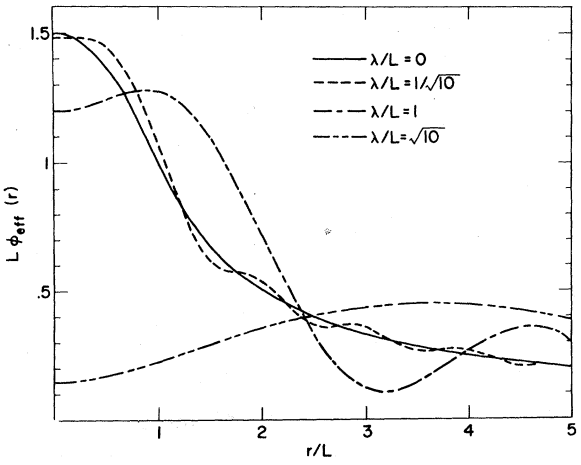

FIG. 3. The effective scalar self-potential for a uniform charge distribution, given by Eq. (3.29).

and correspondence limits is crucial in developing a consistent theory.[19] The classical theory of point charges will prove to be meaningful only when interpreted as the correspondence limit of the quantum mechanical theory.

2. The remaining even coefficients

The remaining even coefficients A_{2m}, for $m>0$, can be treated in a manner very similar to that outlined above. The resulting expression is

$$\Omega_{2m}=\frac{1}{(2m)!}\frac{\partial^{2m}}{\partial\lambda^{2m}}\times\left[\lambda^{2m}\int\int d\vec{x}\,d\vec{x}'\rho(x)\left|\vec{x}-\vec{x}'\right|^{2m-1}\rho_{\text{eff}}(x')\right]. \tag{3.30}$$

In the point limit, the integral inside the large square brackets is easily evaluated, giving

$$\int\int d\vec{x}\,d\vec{x}'\rho(x)\left|\vec{x}-\vec{x}'\right|^{2m-1}\rho_{\text{eff}}(x')=\lim_{L\to 0}\int dr S_\lambda(r)r^{2m+1}=0, \tag{3.31}$$

where the integral is carried out using a convergence factor. Therefore, all the even terms in the equation of motion vanish in the point limit.

3. The odd coefficients

Evaluation of the odd coefficients A_{2m+1} proceeds somewhat differently from that of the even terms, as anticipated in the qualitative remarks at the beginning of this section. Formally, whereas we encountered a principal-value singularity in considering the even coefficients, we now have a δ-function singularity and the evaluation is, in fact, much simpler.

Straightforward manipulation of Eq. (2.16) gives

$$\Omega_{2m+1}=\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left(\frac{\lambda}{2}\right)^{2l}\int_{-\infty}^{\infty}dk\,k\tilde{\rho}(k)^2(k^2)^{2l}I(k), \tag{3.32}$$

where

$$I(k)=\lim_{\tau\to 0}\frac{1}{\pi}\int_0^\infty dx\,x^{2m+2l+1}\sin kx\,e^{-\tau x}=(-1)^{m+l+1}\left(\frac{\partial}{\partial k}\right)^{2m+2l+1}\delta(k). \tag{3.33}$$

This is most easily evaluated in the point limit by partial integration, and we can summarize our results for the point-charge case as

$$\Omega_n=\begin{cases}(-1)^{(n-1)/2}(2n-1)!!\lambda^{n-1} & \text{for } n \text{ odd}\\ 0 & \text{for } n \text{ even}.\end{cases} \tag{3.34}$$

Inserting this into Eq. (3.16), we have the quantum-mechanical equation of motion for a point charge

$$m_0\ddot{\vec{R}}(t)=e\vec{E}_{\text{in}}(\vec{R}(t),t)-\frac{2e^2}{3c^2}\sum_{n\ \text{odd}}\frac{1}{n!c^n}A_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}}, \tag{3.35}$$

with

$$A_n=\begin{cases}(-1)^{(n-1)/2}\dfrac{2n(4n+5)}{3(n+1)(n+2)}(2n-1)!!\lambda^{n-1}, & n \text{ odd}\\ 0, & n \text{ even}.\end{cases}$$

This equation is similar to the classical equation for a *finite*-sized charged particle, with λ formally playing the role of a size parameter [compare Eqs. (2.4)–(2.6)]. We shall analyze this equation using a procedure similar to that employed in the classical case.

IV. THE QUESTION OF RUNAWAYS

A. Motion in the absence of external forces

In this section we investigate the solutions of Eq. (3.35) by taking matrix elements of the equation between exact stationary states $|m\rangle$ of the Hamiltonian (3.1). We assume that among these states there are ones for which the matrix elements $\vec{E}_{\text{in}}(\vec{R}(t),t)$ are negligible, and since we first want to consider a free electron, we confine ourselves to these states. This is possible owing to the linearity of the equation of motion. We also assume there are no other external forces acting on the particle.

Between stationary states, we have

$$\langle m|\dot{\vec{R}}(t)|n\rangle=\dot{\vec{R}}(t)_{mn}=e^{iE_{mn}t/\hbar}\dot{\vec{R}}(0)_{mn}, \tag{4.1}$$

where $E_{mn}=E_m-E_n$. If there are runaway solutions to Eq. (3.35), there must be states $|m\rangle$ and $|n\rangle$ for which $\dot{\vec{R}}(0)_{mn}\neq 0$ and for which $\beta\equiv iE_{mn}/\hbar$ has a positive real part. Taking the required matrix elements of Eq. (3.35) we obtain a power series in the quantity $\eta\equiv(\beta\lambda/c)$:

$$1=\tfrac{2}{3}\alpha\eta\sum_{n=1\ (\text{odd})}^{\infty}\frac{(2n-1)!!}{n!}(-1)^{(n-1)/2}\times\left[\frac{1}{3}\left(\frac{2n}{n+1}\right)\left(\frac{4n+5}{n+2}\right)\right]\eta^{n-1}\equiv\tfrac{2}{3}\alpha f(\eta), \tag{4.2}$$

where we have factored out the root $\beta=0$. The series (4.2) converges for $|\eta|<\frac{1}{2}$, and inside its circle of convergence it can be summed, giving

$$f(\eta) = -\left(\frac{4i}{3}\right)[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}]$$
$$-\frac{7}{3\eta}[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - \tfrac{2}{7}]$$
$$+\frac{2i}{3\eta^2}[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2}]. \qquad (4.3)$$

Note that in the limit $\hbar \to 0$, we have

$$\tfrac{2}{3}\alpha f(\eta) \underset{\hbar\to 0}{\longrightarrow} \beta\tau, \qquad (4.4)$$

which is just the classical result leading to a runaway solution. The task now is to search for any roots of Eq. (4.2) which lie inside the circle of convergence of the series.

A first simple investigation involves searching for the real roots of Eq. (4.2). (Recall that the classical runaway root was purely real.) We plot $\frac{2}{3}\alpha f(\eta)$ as a function of η in Fig. 4, and it is clear that there is no real root for the physical value of α. However, when α is large enough, of order unity, there is a runaway solution, implying a significant difference between the weak- and strong-coupling limits of the theory. We shall return to this point later.

B. Simplified form for $f(\eta)$ (Ref. 20)

In searching for the complex roots of Eq. (4.2), it is useful to consider first the simplified form

$$\tilde{f}(\eta) = -\left(\frac{4i}{3}\right)[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}]. \qquad (4.5)$$

All the essential results are already contained in this truncated form of the function $f(\eta)$.

Now we can solve for the roots of Eq. (4.2), with $\tilde{f}(\eta)$ replacing $f(\eta)$. Defining $g = 8\alpha/9$ and $\xi = 2\eta$, we obtain

$$\xi^2 = (2g^4 - 2g^2 - 1) \pm 2g^3(g^2-2)^{1/2}. \qquad (4.6)$$

Therefore, if $g^2 < 2$, we have

$$\xi^2 = (2g^4 - 2g^2 - 1) \pm i2g^3(2-g^2)^{1/2},$$

so that

$$|\xi|^2 = [(2g^4 - 2g^2 - 1)^2 + 4g^6(2-g^2)]^{1/2}$$
$$= (1+4g^2)^{1/2} > 1. \qquad (4.7)$$

Consequently, if $g^2 < 2$ ($\alpha < \frac{9}{8}\sqrt{2} \approx 1.59$), all roots lie outside the circle of convergence of the series (4.2).

Now suppose that $g^2 > 2$, in which case

$$\xi^2 = (2g^4 - 2g^2 - 1) \pm 2g^3(g^2-2)^{1/2}. \qquad (4.8)$$

After we take the square root, Eq. (4.8) defines four roots, but only two of these are solutions of the original equation. These are the roots which in the limit $g \to \infty$ have the behavior

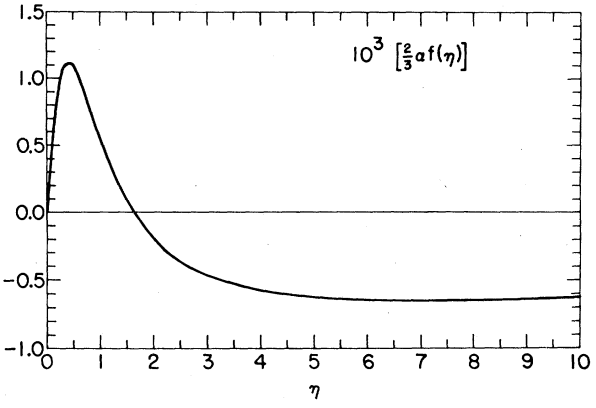

FIG. 4. Right-hand side of Eq. (4.2) for real η and the physical value of α. Note that $f(\eta)$ is an odd function of η and has the asymptotic behavior $\eta^{-1/2}$.

$$\xi_1 \to g^{-1}, \quad \xi_2 \to 2g^2. \qquad (4.9)$$

The limit $g \to \infty$ can be viewed as the classical limit ($\hbar \to 0$, with e^2 fixed). The root ξ_2 moves off to infinity in this limit, while $\xi_1 = g^{-1}$ corresponds to the classical runaway solution [$\xi = g^{-1}$ gives $\beta = \frac{3}{8}(1/\tau)$, the factor of $\frac{3}{8}$ being the result of using the truncated function $\tilde{f}(\eta)$]. For large but finite values of g, one finds

$$\beta = \frac{1}{2}\left(\frac{c}{\lambda}\right)\xi_1$$
$$= \frac{3}{8}\frac{1}{\tau}[1 + (\text{numerical coefficient})\hbar^2 + \cdots]. \qquad (4.10)$$

So, we obtain a small-$\hbar$ expansion about the classical runaway root $\beta = \frac{3}{8}(1/\tau)$.

The large- and small-α regimes are separated in that there is a critical value of α

$$\alpha_{\text{crit}} = \tfrac{9}{8}\sqrt{2}, \qquad (4.11)$$

such that

$$\left.\frac{d\eta}{d\alpha}\right|_{\alpha\to\alpha_{\text{crit}}} \to \infty. \qquad (4.12)$$

Thus the growth rate η depends on α as if there were a "first-order phase transition" at $\alpha = \alpha_{\text{crit}}$. This means the radius of convergence of (4.10) cannot include the physical value of α, and thus that one cannot meaningfully study the question of runaways in quantum mechanics using a semiclassical approach.

One can also ask at what value of g we first get a root inside the circle of convergence of the original series. This is a simple calculation and it gives

$$\alpha < \tfrac{9}{8}(1+\sqrt{2})^{1/2} \approx 1.75 \qquad (4.13)$$

as the condition for no runaways. Comparing (4.11) with (4.13) we also see that the roots at α_{crit} occur for $|\eta| > \frac{1}{2}$, and consequently they do not occur within the range of validity of the nonrelativistic theory.

We note that the convergence condition for the series (4.2), $|\eta| < \frac{1}{2}$, corresponds to the physical condition

$$E_{mn} < \tfrac{1}{2}mc^2 . \tag{4.14}$$

Our nonrelativistic formalism is certainly consistent only within this limited domain, but within this domain there are no runaway solutions for the physical value of the fine-structure constant.

It is interesting that the criterion (4.14) is generated by the equations themselves. Even though the particle is treated nonrelativistically, the parameter c enters through the coupling to the electromagnetic field. It should be borne in mind that the limitation is one on the eigenstates of particle plus field between which one can consistently evaluate matrix elements.

C. Exact form for $f(\eta)$

Finally we investigate the roots of Eq. (4.2) using the exact form of $f(\eta)$ [Eq. (4.3)]. All the results obtained in the preceding subsection are recovered here as well, with only minor quantitative changes in the value of α for which runaways appear and in α_{crit}.

In the strong-coupling regime, the roots are real and can be read off Fig. 4. It remains to be shown that the complex roots in the weak-coupling regime lie outside the circle of convergence. Using the truncated form of $f(\eta)$, we saw that the roots approached $\pm i/2$ as $\alpha \to 0$ [Eq. (4.6)]; similarly, when the exact form of $f(\eta)$ is used, one finds that the roots have the small-α behavior

$$\eta(\alpha) = \pm i[\tfrac{1}{2} + (\tfrac{40}{27})^{1/2}\alpha + \cdots] + \alpha^2[(\tfrac{22}{47})(\tfrac{112}{47})(\tfrac{10}{3})^{1/2} + \cdots] . \tag{4.15}$$

Therefore the roots lie outside the radius of convergence and follow trajectories similar to those found using $f(\eta)$.

We conclude this section with a brief review of our main results to this point. First, the electrostatic self-energy is always finite in nonrelativistic quantum electrodynamics, attaining a maximum value $\sim\alpha m$ for a charge radius $L \sim \lambda$, and going to zero in the point-charge limit. Analysis of the quantum-mechanical equation shows that there are no runaways in the nonrelativistic theory for $\alpha \lesssim 1$. However, for $\alpha = e^2/\hbar c$ large (interpreted either as the strong-coupling or semiclassical limit), the theory does display runaway behavior. For the physical value of α, the only root is $\beta = 0$, giving $\vec{R}(t)_{mn} = \vec{R}(0)_{mn}$. Clearly, the correspondence limit of the solution to the quantum-mechanical free-particle equation (with the inclusion of the radiation reaction) is just the physically correct uniform-velocity solution. Finally, note that while the quantum-mechanical equation (3.35) bears a *formal* resemblance to Eq. (2.4) for a classical extended charge, there is no quantum-mechanical counterpart to the damped-oscillatory solutions to the homogeneous equation found, for finite-charge radius, in the classical case. This provides additional confirmation for the interpretation of Eq. (3.35) as that of a point particle.

V. MOTION WITH EXTERNAL FORCES

In the presence of a c-number time-dependent external force $F(t)$, Eq. (3.35) becomes

$$m_0\ddot{\vec{R}}(t) = \vec{F}(t) - \frac{2e^2}{3c^2}\sum_{n\ \mathrm{odd}} \frac{A_n}{n!\,c^n}\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}} , \tag{5.1}$$

if we again disregard the in-field. Solving (5.1) for the acceleration, we find

$$m_0\ddot{\vec{R}}(\omega) = \frac{\vec{F}(\omega)}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)} , \tag{5.2}$$

where $f(\eta)$ is defined in Eq. (4.3) and the Fourier transforms are defined by

$$\ddot{\vec{R}}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt\, e^{-i\omega t}\ddot{\vec{R}}(t) . \tag{5.3}$$

We proceed in close analogy with the classical analysis of Sec. II, but for consistency we must require that $\vec{F}(\omega)$ vanish for $|\omega| > c/2\lambda$. This condition is closely related to the one discussed previously [$E_{mn} < \frac{1}{2}mc^2$, Eq. (4.14)], and means that the applied force changes slowly in the time required for light to cross an electron Compton wavelength. We now introduce the response function $G(t-t')$, defined as

$$m_0\ddot{\vec{R}}(t) = \int_{-\infty}^{+\infty} dt' G(t-t')\vec{F}(t') . \tag{5.4}$$

Comparison of Eq. (5.4) to (5.1) shows that $G(t-t')$ is given by

$$G(t-t') = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}G(\omega)e^{i\omega(t-t')} = \int_{-c/2\lambda}^{c/2\lambda}\frac{d\omega}{2\pi}\frac{e^{i\omega(t-t')}}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)} \tag{5.5}$$

in the quantum-mechanical case, whereas for a classical point charge one finds [Eq. (2.25)]

$$G_{c1}(t-t') \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) e^{i\omega(t-t')}$$
$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{1-i\omega\tau}$$
$$= \frac{1}{\tau} e^{-(t'-t)/\tau} \theta(t'-t). \quad (5.6)$$

The function $G_{c1}(t-t')$ displays noncausal behavior, which arises because $G(\omega)$ has a pole in the lower half ω plane. As we have seen, this is equivalent to having a runaway solution.

In the quantum-mechanical case, the situation is quite different. First, for the physical value of α, there are no runaway solutions and hence $G(\omega)$ has no poles in the lower half ω plane, within the circle of convergence of the series defining $f(i\omega\lambda/c)$. Furthermore, this model is internally consistent only with the frequency cutoff $|\omega| < c/2\lambda$, implying that the quantum-mechanical response function is spread about the origin ($t = t'$) with a minimum width given by the characteristic time $\Delta t \sim \hbar/E_{mn} \sim \lambda/c$. This time scale is set jointly by the dynamics [through Eq. (3.35)] and the uncertainty principle and is very large compared to the time scale associated with classical preacceleration

$$\tau \sim \alpha(\lambda/c) \ll \Delta t, \quad (5.7)$$

for the physical value of α. This implies that the solutions of the quantum-mechanical equation (5.1) display no observable violations of causality.

These results are very interesting in connection with attempts to construct a consistent classical theory of point charges. Our philosophy is that the point charge (with finite mass) is outside the domain of classical electrodynamics and is bound to lead to a theory with runaway solutions and noncausal behavior. However, the classical point-charge trajectories *can* be thought of as the correspondence limit of the *solutions* of the quantum-mechanical equation. Starting from the quantum-mechanical equations (5.4), (5.5) we see that if the force is cut off at a frequency small compared to c/λ, then the correspondence limit can be obtained by expanding the denominator of the response function [Eq. (5.5)]:

$$m_0 \ddot{\vec{R}}(t) = \vec{F}(t) + \tau \dot{\vec{F}}(t) + \tau^2 \ddot{\vec{F}}(t) + \cdots \quad (5.8)$$
$$= \int_0^{\infty} ds\, e^{-s} \vec{F}(t + s\tau). \quad (5.9)$$

Thus, *in the classical domain of validity*, Eqs. (5.4), (5.5) give the same results as the Lorentz-Dirac equation with the runaway solution removed by fiat [Eq. (1.4)]. In other words, the correspondence limit of the quantum-mechanical equation gives the usual radiative damping effects in the correspondence limit (e.g., the radiative linewidth of a low-frequency oscillator) while eliminating the possibility of observable noncausal behavior.

VI. CONCLUDING REMARKS

We have seen that classical electrodynamics is internally consistent in describing the motion of extended charged particles ($L > c\tau$). However, the point-charge limit is plagued by the well-known problems of runaway solutions and preacceleration and is therefore outside the theory's domain of consistency. On the other hand, we find that in quantum mechanics the point-charge theory *is* consistent (for $\alpha \lesssim 1$), displaying neither runaway behavior nor observable noncausality. Furthermore, the correspondence limit of the *solution* of the quantum-mechanical equation agrees with that of the Lorentz-Dirac equation *in the classical regime* [Eqs. (5.8), (5.9)]. Thus, a consistent picture of a classical point electron emerges only as the correspondence limit of a quantum-mechanical point electron and not as the point limit of a classical extended charge.[19]

Many questions remain unanswered. The extension of our results into the relativistic regime is clearly important. Another point concerns the fact that we found that nonrelativistic quantum electrodynamics does begin to run into trouble with runaways and preacceleration when $\alpha \gtrsim 1$. This suggests an upper bound on α. Is this bound real, or does the restriction disappear when more physics, such as pair creation, is included in the theory? This brings us to a final point. Our results are certainly not overtly perturbative in α. One would like to understand the relationship between our approach to these problems, which is based on the Heisenberg equations of motion,[21] and standard perturbation theory. One approach to these questions is suggested by some work of Trubatch,[22] who discussed the runaway problem for the polaron. He showed that runaways result from the standard application of the dipole approximation to the polaron Hamiltonian, but that they do not occur if the Hamiltonian is renormalized before making this approximation. If a similar approach could be implemented in nonrelativistic quantum electrodynamics, considerable insight into the structure of the theory might be gained.

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