PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Quantum mechanics as a generalized stochastic theory in phase space*

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(Received 15 March 1976)

In this paper we formulate a new stochastic description of quantum mechanics in phase space. The theory of phase-space representations of quantum mechanics, initiated by Wigner, Groenewold, and Moyal and systematized recently by Agarwal and Wolf is essentially a single-time theory, in that it deals only with the quantum-mechanical joint distribution functions for position and momentum at a single instant of time. We develop a natural multitime extension of such a single-time theory. We consider a class of multitime phase-space distribution functions such that an arbitrary quantum multitime correlation function can be expressed as a phase-space average of the form encountered in classical stochastic theories. We study the nonclassical features of these multitime distribution functions and show that they may be considered as characterizing a generalized stochastic process in phase space. We demonstrate that the multitime distribution functions that correspond to Hamiltonian evolution of isolated quantum systems satisfy a certain condition that may be regarded as characterizing a generalized Markov process. We also investigate certain special features of the generalized stochastic processes that characterize the evolution of open systems.

I. INTRODUCTION

Probabilistic concepts appear in quantum mechanics in an entirely different way than in classical statistical mechanics. The analysis of Birkhoff and von Nuemann[1] shows that the conceptual foundations of classical statistical theories are the same as those of the classical theory of probability as systematized by Kolmogorov.[2] However, from the point of view of classical probability theory, quantum mechanics seems to manipulate somewhat mysteriously state vectors and operators to end up with probabilities and expectation values. Also, quantum theory in what may be called the "orthodox formulation"[3] does not give any prescription for calculating joint probability distributions for noncommuting observables.

The search for joint probability distributions for position and momentum is closely related to investigations about the possibility of a phase-space formulation of quantum mechanics. The first step in this direction was taken in 1932 by Wigner,[4] who demonstrated the possibility of expressing quantum-mechanical expectation values as averages over phase-space distribution functions. Important contributions were later made by Groenewold[5] and Moyal,[6] and subsequently by several other authors. These investigations led gradually to the realization that a phase-space formulation of quantum theory is generated by a "rule of association,"[7] which is essentially a linear one-to-one mapping of operators into c-number functions.[8] Each rule of association determines a correspondence between quantum states and phase-space distribution functions in such a way that the quantum expectation values can be calculated as phase-space averages. Hence a phase-space formulation of quantum mechanics may be viewed as a certain concrete realization of the abstract Hilbert-space formulation, where the language in which the formulation is framed closely parallels the structure of classical statistical mechanics. However, while the objects representing the states and observables in a phase-space formulation of quantum theory have the appearance of the corresponding objects of classical statistical mechanics, their properties and relationships will not be determined by classical statistical mechanics, except for the requirement that expectation values should be expressible as phase-space averages. These relations will be completely determined by quantum theory, via the rule of association.

The phase-space distribution functions that correspond to quantum-mechanical states in any particular representation are sometimes called "quasiprobabilities," as they do not possess all the attributes of ordinary probabilities. In particular, they are not non-negative in general. This becomes clear from a theorem due to Wigner,[4,9] that "if one imposes the condition that the distribution function yields the usual marginal probabilities, in addition to the requirements such as reality and linear association, then one cannot avoid negative probabilities in general." Wigner's theorem was the first among the several results (see Refs. 10–13), which brought into evidence certain nonclassical features of quantum mechanics, and demonstrated conclusively that quantum mechanics cannot be formulated as ordinary stochastic theory in phase space.

The theory of phase-space representation schemes discussed so far (sometimes referred to as the "single-time theory") deals only with the quantum-mechanical joint distribution function for position and momentum at a given instant of time. It can also be shown that these quantum distribution functions and the corresponding characteristic functions (see Ref. 13 for appropriate definitions and results) have properties that are analogous, to some extent, to those possessed by the corresponding objects of classical probability theory. However, the main problem considered in the present investigation is to formulate quantum evolution as a certain generalized stochastic process in phase space. The results we mentioned earlier deal only with certain "kinematical" aspects of the above problem.

Historically, the first step towards formulating quantum theory as a generalized stochastic process was taken by Stratonovich,[14] who introduced a set of multitime distribution functions in phase space, and showed that they satisfy the classical Markov factorization property. Subsequent investigations of Lax and collaborators[15,16] showed that there exists a general class of such "Markovian multitime distribution functions." However, it can be shown[17] that these multitime distribution functions are not suitable for calculating an arbitrary quantum correlation function as a phase-space average of c-number functions. It thus appears that such multitime distribution functions are considered mainly because they satisfy the factorization relation characteristic of classical Markov processes. There have been discussions on alternative approaches to the description of quantum evolution as a generalized stochastic process in phase space, in particular those of Agarwal and Wolf[8,18] and Bausch, Schlogl, and Stahl,[19,20] both of which are closest in spirit to our approach.

In all the phase-space formulations of quantum theory, the single-time phase-space distribution functions are determined by the requirement that the expectation value of any observable can be calculated as a phase-space average of the c-number representative of the observable. In the same way we require that the multitime distributions are to be determined by the condition that any multitime correlation function of a set of observables can be calculated as a phase-space average of the product of the c-number representatives of the observables, just as in the theory of classical stochastic processes. In other words, the multitime distribution functions $p_r(q_1p_1t_1, \ldots, q_rp_rt_r)$ are determined by the requirement that given any set of observables $\{\hat{g}_i\}$, their multitime correlation functions can be calculated via a relation of the form

$$\langle\hat{g}_1(t_1)\hat{g}_2(t_2)\cdots\hat{g}_r(t_r)\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p_r(q_1p_1t_1, q_2p_2t_2, \ldots, q_rp_rt_r) \times g_1(q_1,p_1)g_2(q_2,p_2)\cdots g_r(q_r,p_r) \times dq_1\cdots dp_r, \tag{1.1}$$

where $\{\hat{g}_i(t_i)\}$ are the time-evolved observables in the Heisenberg picture and $\{g_i(q_i,p_i)\}$ are the phase-space representatives of the corresponding Schrödinger-picture observables $\{\hat{g}_i(0)\}$ (we assume that the observables in both pictures coincide at time $t=0$). Apart from the requirement expressed by Eq. (1.1), the properties and relations of these multitime distribution functions will be determined solely by quantum theory and not by any analogy with classical stochastic processes. We will often refer to these multitime distribution functions as characterizing a certain quantum (or generalized) stochastic process in phase space. Such a statement should be taken to imply that these distribution functions satisfy a certain set of consistency relations (see theorem 2.1 below).

Finally, a few remarks may be made on the relation between the formalism outlined in this paper and the various investigations on the probabilistic foundations of quantum mechanics. There have been several attempts to construct a framework of "quantum probability theory" based on the work of Birkhoff and von Neumann[1] (see Ref. 21b for a detailed discussion of this subject). Recently an operational approach to quantum probability theory has been developed.[21] This approach makes use of von Neumann's theory of successive observations[22] to introduce statistical concepts into quantum theory. In this approach, physically meaningful joint probability distributions can be defined for any collection of observables, provided that in characterizing an observable we also specify the corresponding "measurement transformation" (i.e., the effect of performing a measurement of the observable on the state of the system; see Ref. 21b). A theory of quantum stochastic processes can also be formulated[21b] on the basis of the above framework of "quantum probability theory."

The present investigation (in contrast to those mentioned above), is solely concerned with obtaining a stochastic phase-space formulation of quantum theory in its orthodox formulation.[3] Here the statistical features of the theory are completely characterized by the various multitime correlation functions of observables, now merely specified as self-adjoint operators on a Hilbert space. We introduce multitime phase-space distribution functions (in a manner analogous to the way single-time distribution functions of Wigner and others were introduced) as calculational tools in evaluating multitime correlation functions as phase-space averages; in this repect they resemble the multitime distribution functions of a classical stochastic process. However, since in the evaluation of a correlation function like $\langle \hat{g}_1(t_1)\hat{g}_2(t_2)\rangle$ we do not take into account the influence of the measurement at t_1 on the outcome of the measurement at $t_2 > t_1$, the multitime phase-space distribution function that we introduce are not physically meaningful as the joint probabilities for successive measurements.

We now summarize the main results of our investigations. In Sec. II we consider the case of a closed quantum system undergoing Hamiltonian evolution. Since the evolution is given by a unitary transformation, we can define a family of time-evolved representation operators, which [together with an "orthogonal family," see Eq. (2.7) below] is a phase-space representation scheme. We then express an arbitrary multitime correlation function of a set of observables of a phase-space average of the product of the phase-space representatives of the observables with respect to multitime phase-space distribution functions. These multitime distribution functions turn out to be correlation functions of the time-dependent representation operators. In theorem 2.1, the consistency relations among these multitime distribution functions are enumerated. In particular it is shown that the multitime distribution functions are complex, nonsymmetric, and are related nonlocally to the conditional distribution functions. We define a generalized Markov process in terms of the conditional distribution functions in the same way as in classical theory. We then demonstrate that the multitime distribution functions that correspond to Hamiltonian evolution of isolated quantum systems characterize such a generalized Markov process.

In Sec. III, we obtain a stochastic phase-space description of the evolution of an open system. We show that the multitime correlation functions of a set of observables that, initially, refer to the system alone, can be calculated by using a set of "reduced" multitime distribution functions. Our theorem 3.1 demonstrates that these distribution functions also satisfy the same consistency relations that are obeyed by the multitime distribution functions that correspond to the evolution of closed systems. However, we also emphasize that the dynamics of open systems does not, in general, give rise to a generalized Markov process in phase space.

II. MULTITIME PHASE-SPACE DISTRIBUTION FUNCTIONS

In this section we consider a closed quantum system undergoing Hamiltonian evolution. If $\rho(0)$ is the initial density operator of the system, the evolution can be described in the Schrödinger picture by a one-parameter group of unitary operators $U(t)$, such that

$$\rho(t) = U(t)\rho(0)U^{-1}(t), \tag{2.1}$$

with

$$U(t) = e^{-(i/\hbar)\hat{H}t}, \tag{2.2}$$

where $\hat{H}$ is the Hamiltonian operator. The evolution in the Heisenberg picture which is equivalent to (2.1) can be expressed by the equation

$$A(t) = U^{-1}(t)A(0)U(t), \tag{2.3}$$

valid for each observable A. From (2.1) and (2.3), and the cyclic invariance of the trace operation, it follows that the expectation value of an observable A in the state ρ at the time t can be equivalently expressed as

$$\mathrm{Tr}[\rho(t)A(0)] = \mathrm{Tr}[\rho(0)A(t)]. \tag{2.4}$$

In developing the multitime formalism, it will be useful to begin with a brief outline of the single-time theory. We will only present the basic results necessary for an understanding of our analysis. A detailed discussion of this subject may be found in Refs. 8, 13.

Let $\hat{q}$ and $\hat{p}$ be the canonical position and momentum operators, respectively, that satisfy the commutation relations

$$[\hat{q}, \hat{p}] = i\hbar I,$$

where I is the unit operator. A phase-space representation is obtained by mapping the canonical operators $\hat{q}$ and $\hat{p}$ and any operator function $\hat{g}(\hat{q}, \hat{p})$ that depends on them, onto the phase space,[23]

$$\hat{q} \to q, \quad \hat{p} \to p, \quad \hat{g}(\hat{q}, \hat{p}) \to g(q, p)$$

according to some rule of association. Each mapping of a large class of linear mappings can be characterized by a set of representation operators $\{\Delta(q, p)\}$ parametrized by q and p [see Refs. 8, 13 for several examples of representation operators that satisfy Eqs. (2.5)–(2.7) below]. These representation operators are self-adjoint, i.e.,

$$\Delta^\dagger(q, p) = \Delta(q, p), \tag{2.5}$$

and satisfy the condition

$$\int_{-\infty}^{\infty}\int \Delta(q, p)\,dq\,dp = I, \tag{2.6}$$

where I is the unit operator. Associated with the family of representation operators $\{\Delta(q, p)\}$ is another family of operators $\{\overline{\Delta}(q', p')\}$ which satisfy the relation

$$\mathrm{Tr}[\Delta(q, p)\overline{\Delta}(q', p')] = \delta(q - q')\delta(p - p'), \tag{2.7}$$

where δ is the Dirac δ function. We often refer to the relation (2.7) as an orthogonality relation and also refer to the family of operators $\{\overline{\Delta}(q',p')\}$ as being "orthogonal" to $\{\Delta(q,p)\}$.

In terms of the family of representation operators $\{\Delta(q,p)\}$ and the orthogonal family $\{\overline{\Delta}(q,p)\}$, the transformations from the set of operators on Hilbert space to functions on phase space are given by the formulas

$$g(q,p)=\mathrm{Tr}[\hat{g}(\hat{q},\hat{p})\overline{\Delta}(q,p)], \tag{2.8}$$

$$\hat{g}(\hat{q},\hat{p})=\int\int_{-\infty}^{\infty} g(q,p)\Delta(q,p)dqdp. \tag{2.9}$$

Thus, we see that a family of operators $\{\Delta(q,p)\}$ which satisfy the relations (2.5)–(2.7) along with an orthogonal family $\{\overline{\Delta}(q,p)\}$ defines a phase-space representation scheme for quantum mechanics.

We now extend the above framework of the single-time theory by introducing the set of time-evolved representation operators $\{\Delta(q,p,t)\}$ which enable us to obtain a phase-space description of the dynamics of the system. For the case of a closed quantum system undergoing Hamiltonian evolution, characterized by the Hamiltonian operator $\hat{H}$, the time-evolved representation operators $\{\Delta(q,p,t)\}$ and $\{\overline{\Delta}(q,p,t)\}$ may be defined by means of the relations

$$\Delta(q,p,t)=U^{-1}(t)\Delta(q,p)U(t), \tag{2.10}$$

$$\overline{\Delta}(q,p,t)=U^{-1}(t)\overline{\Delta}(q,p)U(t), \tag{2.11}$$

with $U(t)$ as given by (2.2). As the time-evolved representation operators $\Delta(q,p,t)$ and $\overline{\Delta}(q,p,t)$ are defined in terms of a unitary transformation in (2.10) and (2.11), it is evident that the properties (2.5)–(2.7) of $\Delta(q,p)$ and $\overline{\Delta}(q,p)$ are also valid for $\Delta(q,p,t)$ and $\overline{\Delta}(q,p,t)$:

$$\Delta^\dagger(q,p,t)=\Delta(q,p,t), \tag{2.12}$$

$$\int\int_{-\infty}^{\infty}\Delta(q,p,t)dqdp=I, \tag{2.13}$$

$$\mathrm{Tr}[\Delta(q,p,t)\overline{\Delta}(q',p',t)]=\delta(q-q')\delta(p-p'). \tag{2.14}$$

Thus $\Delta(q,p,t)$ also form a family of representation operators for each t. Also Eqs. (2.3), (2.9), and (2.10) imply that the time-evolved observable $\hat{g}(\hat{q},\hat{p},t)$ is given by the relation

$$\hat{g}(\hat{q},\hat{p},t)=\int\int_{-\infty}^{\infty} g(q,p)\Delta(q,p,t)dqdp. \tag{2.15}$$

In the course of our discussions, it will become clear that the time-dependent representation operators $\Delta(q,p,t)$ and $\overline{\Delta}(q,p,t)$ play a very important role in the phase-space description of the dynamics of the system.

We now employ Eq. (2.15) to calculate the time-correlation functions of the observables $\hat{g}_i(\hat{q},\hat{p},t_i)$. Starting from the relation

$$\langle\hat{g}_1(t_1)\cdots\hat{g}_r(t_r)\rangle=\mathrm{Tr}[\rho\hat{g}_1(t_1)\cdots\hat{g}_r(t_r)]$$

we obtain

$$\begin{aligned}\langle\hat{g}_1(t_1)\cdots\hat{g}_r(t_r)\rangle&=\mathrm{Tr}\Big[\rho\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}g_1(q_1,p_1)\cdots g_r(q_r,p_r)\Delta(q_1,p_1,t_1)\cdots\Delta(q_r,p_r,t_r)dq_1\cdots dp_r\Big]\\&=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}g_1(q_1,p_1)\cdots g_r(q_r,p_r)\mathrm{Tr}[\rho\Delta(q_1,p_1,t_1)\cdots\Delta(q_r,p_r,t_r)]dq_1\ldots dp_r,\end{aligned} \tag{2.16}$$

where

$$g_i(q_i,p_i)=\mathrm{Tr}[\hat{g}_i(0)\overline{\Delta}(q_i,p_i)] \tag{2.17}$$

are the phase-space representatives of the observables $\hat{g}_i$ considered in the Schrödinger picture. A comparison on Eqs. (2.17) and (1.1) immediately yields the multitime distribution functions

$$p_r(q_1p_1t_1,\ldots,q_rp_rt_r)=\mathrm{Tr}[\rho\Delta(q_1,p_1,t_1)\cdots\Delta(q_r,p_r,t_r)]. \tag{2.18}$$

The multitime phase-space distribution functions $p_r(q_1p_1t_1,\ldots,q_rp_rt_r)$ as defined in (2.18), will form the basis of our stochastic formulation of quantum theory in phase space. We will first show that they exhibit many features of whay may be called a "generalized stochastic process."

(1) From the definition (2.18), it is clear that the multitime distribution functions $p_r(q_1p_1t_1,\ldots,q_rp_rt_r)$ are not restricted to assume only real values in general. For the case of the single-time phase-space distribution function, it is known that the reality of the distribution function is due to the fact that the representation operators (and the density operator) are self-adjoint. However, a product of the self-adjoint operators $\Delta(q_i,p_i,t_i)$, is not necessarily self-adjoint as they do not commute in general. Hence, for the multitime distribution functions, we only have the relation

$$p_r^*(q_1p_1t_1,q_2p_2t_2,\ldots,q_{r-1}p_{r-1}t_{r-1},q_rp_rt_r)=p_r(q_rp_rt_r,q_{r-1}p_{r-1}t_{r-1},\ldots,q_2p_2t_2,q_1p_1t_1). \tag{2.19}$$

(2) Another consequence of the fact that $\Delta(q_i, p_i, t_i)$ do not commute among themselves in general, is that the multitime distribution functions $p_r(q_1p_1t_1, \ldots, q_rp_rt_r)$ are not symmetric under a joint permutation of $\{(q_i, p_i)\}$ and $\{t_i\}$. This lack of symmetry of the multitime distribution functions is a direct reflection of the same feature of the time-correlation functions in quantum theory. Since there is no general relation between all the differently time-ordered multitime distribution functions, except that expressed by Eq. (2.19), we will have to consider $(r!/2)$ independent distribution functions p_r for each r, and times $(t_1, \ldots, t_r)$. Also, for a complete specification of a generalized stochastic process, we will have to consider distribution functions such as $p_3(q_1p_1t_1, q_2p_2t_2, q_3p_3t_1)$ which are not encountered in classical theory.[24]

(3) Since the multitime distribution functions are not even real in general, there does not arise any question of their being non-negative. For the single-time distribution functions, it has been shown[13] that the usual non-negativity requirement of classical probability theory has to be replaced by a generalized non-negativity requirement, which is a direct reflection of the fact that the density operator is a positive operator. In order to obtain a generalized non-negativity requirement on the multitime distribution functions, we use the condition

$$\langle \hat{g}_1^\dagger(t_1) \cdots \hat{g}_r^\dagger(t_r)\hat{g}_r(t_r) \cdots \hat{g}_1(t_1)\rangle \geq 0\,, \qquad (2.20)$$

for all $\hat{g}_i(t_i)$, for all r. From (2.20) we obtain the condition that $p_{2r}(q_1p_1t_1, \ldots, q_rp_rt_r, q_r'p_r't_r, \ldots, q_1'p_1't_1)$, is non-negative kernel for each r and for all $\{t_i\}$, i.e.,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_1^*(q_1, p_1) \cdots g_r^*(q_r, p_r) g_r(q_r', p_r') \cdots g_1(q_1', p_1') p_{2r}(q_1p_1t_1, \ldots, q_1'p_1't_1')dq_1dp_1 \cdots dq_1'dp_1' \geq 0 \qquad (2.21)$$

for each r, for all $\{t_i\}$, and for arbitrary functions $g_i(q, p)$.

(4) As the representation operators $\Delta(q_ip_it_i)$ satisfy (2.6), we have the marginal distribution relation

$$\int\int_{-\infty}^{\infty} p_r(q_1p_1t_1, \ldots, q_{i-1}p_{i-1}t_{i-1}, q_ip_it_i, q_{i+1}p_{i+1}t_{i+1}, \ldots, q_rp_rt_r)dq_idp_i = p_{r-1}(q_1p_1t_1, \ldots, q_{i-1}p_{i-1}t_{i-1}, q_{i+1}p_{i+1}t_{i+1}, \ldots) \qquad (2.22)$$

for all r and for each $i \leq r$. In the case of classical stochastic process, it is sufficient to verify the consistency relation (2.22), for the last variable $i = r$ only, as all the other relations for $i < r$ can be derived from the case when $i = r$ with the aid of the symmetry property. However, for quantum stochastic processes, the consistency relations in (2.22) are independent of each other.

(5) Finally, the multitime phase-space distribution functions defined in (2.18) do not satisfy the equal-time relation of classical stochastic processes; i.e., we have in general

$$p_r(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1}, q_rp_rt_{r-1}) \neq p_{r-1}(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1})\delta(q_r - q_{r-1})\delta(p_r - p_{r-1})\,. \qquad (2.23)$$

Relation (2.23) is established once we note that the right-hand side is symmetric under the exchange of (q_r, p_r) and (q_{r-1}, p_{r-1}), whereas the left-hand side is not, since $\Delta(q_rp_rt_{r-1})$ and $\Delta(q_{r-1}p_{r-1}t_{r-1})$ do not commute in general. In fact, instead of the classical relation, we have the following nonlocal relation for quantum stochastic processes proved in the Appendix:

$$p_r(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1}, q_rp_rt_{r-1}) = \{\delta(q_r - q_{r-1})\delta(p_r - p_{r-1})\} \otimes_{r-1} p_{r-1}(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1})\,, \qquad (2.24)$$

where the nonlocal phase-space product $\otimes_{r-1}$ is defined as follows:

$$A(\cdots, q_{r-1}p_{r-1}) \otimes_{r-1} B(\cdots, q_{r-1}p_{r-1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K(q_{r-1}p_{r-1}t_{r-1}; q'_{r-1}p'_{r-1}, q''_{r-1}p''_{r-1}) A(\cdots, q'_{r-1}p'_{r-1}) B(\cdots, q''_{r-1}p''_{r-1}) dq'_{r-1} \cdots dp''_{r-1}\,. \qquad (2.25)$$

In (2.25), we have suppressed the other phase-space variables in $A(\ )$ and $B(\ )$ for convenience, and the kernel $K(\ )$ is given by the formula

$$K(q_{r-1}p_{r-1}t_{r-1}; q'_{r-1}p'_{r-1}, q''_{r-1}p''_{r-1}) = \mathrm{Tr}[\Delta(q_{r-1}, p_{r-1}, t_{r-1})\Delta(q'_{r-1}, p'_{r-1}, t_{r-1})\overline{\Delta}(q''_{r-1}, p''_{r-1}, t_{r-1})]\,. \qquad (2.26)$$

Since the time-dependent representation operators $\Delta(q, p, t)$ and $\overline{\Delta}(q, p, t)$ are given by (2.10) and (2.11), it can be shown from the cyclic invariance of trace, that the kernel $K(q_{r-1}p_{r-1}t_{r-1}, q'_{r-1}p'_{r-1}, q''_{r-1}p''_{r-1})$ is independent of the time t_{r-1}, and can be expressed as

$$K(q_{r-1}p_{r-1}; q'_{r-1}p'_{r-1}, q''_{r-1}p''_{r-1}) = \mathrm{Tr}[\Delta(q_{r-1}, p_{r-1})\Delta(q'_{r-1}, p'_{r-1})\overline{\Delta}(q''_{r-1}, p''_{r-1})]\,. \qquad (2.27)$$

A common feature of all the phase-space representation schemes of quantum theory is that the algebraic relations of quantum theory are transcribed in terms of nonlocal products of the phase-space representatives. For example, in Refs. 8, 13 a nonlocal phase-space product $\otimes$ is introduced, which corresponds to the product of two quantum-mechanical observables. The phase-space product $\otimes_{r-1}$ that is defined by (2.25)–(2.28), is different from the product $\otimes$ introduced in Refs. 8, 13, mainly because of the different representation schemes by which the multitime distribution functions (2.18) and (as we shall see), the conditional distribution functions are obtained from the density operators.

The results we have established so far are summarized in the following proposition.

Theorem 2.1. The multitime phase-space distribution functions $p_r(q_1p_1t_1, \ldots, q_rp_rt_r)$ *of a quantum stochastic process satisfy the following consistency relations:*

(i) $p_r^*(q_1p_1t_1, \ldots, q_rp_rt_r) = p_r(q_rp_rt_r, \ldots, q_1p_1t_1)$,

(ii) $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p_{2r}(q_1p_1t_1, \ldots, q_rp_rt_r, \ldots, q_1'p_1't_1) g_1^*(q_1, p_1) \cdots g_r^*(q_r, p_r) \cdots g_1(q_1'p_1') dq_1 \cdots dp_1' \geq 0$

for arbitrary functions $g_i(q_i, p_i)$;

(iii) $\int_{-\infty}^{\infty}\int p_r(q_1p_1t_1, \ldots, q_{i-1}p_{i-1}t_{i-1}, q_ip_it_i, q_{i+1}p_{i+1}t_{i+1}, \ldots) dq_i dp_i = p_{r-1}(q_1p_1t_1, \ldots q_{i-1}p_{i-1}t_{i-1}, q_{i+1}p_{i+1}t_{i+1}, \ldots)$

for all $i \leq r$;

(iv) $p_r(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1}, q_rp_rt_{r-1}) = \{\delta(q_r - q_{r-1})\delta(p_r - p_{r-1})\} \otimes_{r-1} p_{r-1}(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1})$.

At this stage, we should make it clear that we have only obtained a description of a quantum stochastic process in phase space in terms of a given set of multitime distribution functions which satisfy the conditions (i)–(iv). A complete mathematical characterization of these multitime distribution functions is a nontrivial problem, as may be inferred from the fact that even in the single-time theory a complete characterization of the distribution function, has been accomplsihed only for a few special cases.[25] However, for the purposes of the present investigation it is sufficient to specify a quantum stochastic process in phase space in terms of a set of multitime phase-space distribution functions that satisfy the conditions (i)–(iv) of theorem 2.1.

We now proceed to define the *conditional distribution functions* $w_r(q_rp_rt_r | q_{r-1}p_{r-1}t_{r-1}, \ldots)$ implicitly, by means of the relation

$$p_r(q_1p_1t_1, \ldots, q_rp_rt_r) = w_{r-1}(q_rp_rt_r | q_{r-1}p_{r-1}t_{r-1}, \ldots, q_1p_1t_1) \otimes_{r-1,\ldots,1} p_{r-1}(q_1p_1t_1, \ldots, q_{r-1}p_{r-1}t_{r-1}), \tag{2.28}$$

where the multivariable nonlocal product $\otimes_{r-1,r-2,\ldots,1}$ is defined by the relation

$$A(q_1p_1, \ldots, q_{r-1}p_{r-1}) \otimes_{r-1,\ldots,1} B(q_1p_1, \ldots, q_{r-1}p_{r-1})$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{i=1}^{r-1} K(q_ip_i; q_i'p_i', q_i''p_i'')\right] A(q_1'p_1', \ldots, q_{r-1}'p_{r-1}') B(q_1''p_1'', \ldots, q_{r-1}''p_{r-1}'') dq_1'dp_1' \cdots dq_{r-1}''dp_{r-1}'', \tag{2.29}$$

where the kernel $K(\ )$ is given by (2.27).

In order to enumerate some of the properties of the conditional distribution functions, we require the following properties of the kernel $K(\ )$, which can be readily deduced from the definition (2.27):

$$K^*(qp; q'p', q''p'') = K(qp; q'p', q''p''), \tag{2.30}$$

$$\int_{-\infty}^{\infty}\int K(qp; q'p', q''p'') dq dp = \delta(q'' - q')\delta(p'' - p'), \tag{2.31}$$

$$\int_{-\infty}^{\infty}\int K(qp; q'p', q''p'') dq' dp' = \delta(q - q'')\delta(p - p''). \tag{2.32}$$

The following two relations follow from (2.32):

$$1 \otimes_{r-1,\ldots,1} A(q_1p_1, \ldots, q_{r-1}p_{r-1}) = A(q_1p_1, \ldots, q_{r-1}p_{r-1}), \tag{2.33}$$

$$A(q_{r-1}p_{r-1}) \otimes_{r-1,\ldots,1} B(q_1p_1, \ldots, q_{r-1}p_{r-1}) = A(q_{r-1}p_{r-1}) \otimes_{r-1} B(q_1p_1, \ldots, q_{r-1}p_{r-1}). \tag{2.34}$$

We will now list a few of the properties of the conditional distribution functions $w_{r-1}(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}, \ldots, q_1 p_1 t_1)$. Since these are defined implicitly by Eq. (2.28), we can at once deduce from (2.22), (2.24) and (2.33), (2.34) that they satisfy the constraints

$$\text{(i)} \quad \int\int_{-\infty}^{\infty} w_{r-1}(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}, \ldots, q_1 p_1 t_1) dq_r dp_r = 1 \tag{2.35}$$

and

$$\text{(ii)} \quad w_{r-1}(q_r p_r t | q_{r-1} p_{r-1} t, \ldots, q_1 p_1 t_1) = \delta(q_r - q_{r-1}) \delta(p_r - p_{r-1}) . \tag{2.36}$$

(iii) However, unlike classical conditional distribution functions, w_r are not constrained to be non-negative. This fact that w_r are not necessarily non-negative turns out to be of much importance in a discussion of the stochastic equations for the multitime distribution functions.

(iv) Finally, we can obtain integral relationships connecting the multitime distributions corresponding to different times. On integrating (2.28), and using (2.31) and (2.32), we obtain the formula

$$p_{r-1}(q_1 p_1 t_1, \ldots, q_{r-2} p_{r-2} t_{r-2}, qpt) = \int\int_{-\infty}^{\infty} w_{r-1}(qpt | q_{r-1} p_{r-1} t_{r-1}, \ldots, q_1 p_1 t_1) \otimes_{r-2, \ldots, 1} p_{r-1}(q_1 p_1 t_1, \ldots, q_{r-1} p_{r-1} t_{r-1}) dq_{r-1} dp_{r-1} . \tag{2.37}$$

Carrying out successive integrations, we finally obtain a relation that is valid for classical multitime distribution functions also:

$$p(q, p, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_{r-1}(qpt | q_{r-1} p_{r-1} t_{r-1}, \ldots) p_{r-1}(q_1 p_1 t_1, \ldots, q_{r-1} p_{r-1} t_{r-1}) dq_1 \cdots dp_{r-1} . \tag{2.38}$$

The special significance of Eq. (2.38) lies in the fact that the conditional probability and the multitime distribution function are multiplied locally as in classical probability theory.

Now that we have defined the conditional distribution functions we can arrive at a natural definition of a quantum Markov process as one for which the relation

$$w_{r-1}(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}, \ldots, q_1 p_1 t_1) = w(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}) \tag{2.39}$$

is satisfied for all r. In classical probability theory the conditional probabilities were defined with a specified time ordering which was also utilized in the definition of Markov process. Since, for a quantum stochastic process, all the different time orderings yield different (and independent) multitime and conditional distribution functions, we require that Eq. (2.39) be satisfied irrespective of the time ordering in order that a process be Markovian. Using (2.25) and (2.34), we can obtain an equivalent characterization of a quantum Markov process, that the relation

$$p_r(q_1 p_1 t_1, \ldots, q_r p_r t_r) = w(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}) \otimes_{r-1} w(q_{r-1} p_{r-1} t_{r-1} | q_{r-2} p_{r-2} t_{r-2}) \otimes_{r-2} \cdots \otimes_1 p(q_1 p_1 t_1) , \tag{2.40}$$

should be satisfied for all r, and all times $(t_1, \ldots, t_r)$.

Having arrived at the definition of a quantum Markov process, we now proceed to obtain the central result of this section, which is that the Hamiltonian evolution of closed quantum systems (the case we have considered so far) gives rise to a quantum Markov process in phase space. For this purpose we state first the following relation established in the Appendix, and which depends crucially on the Hamiltonian evolution of the system as described by Eqs. (2.1)–(2.4) and (2.10)–(2.14):

$$p_r(q_1 p_1 t_1, \ldots, q_r p_r t_r) = \{\mathrm{Tr}[\Delta(q_r p_r t_r) \bar{\Delta}(q_{r-1} p_{r-1} t_{r-1})]\} \otimes_{r-1} p_{r-1}(q_1 p_1 t_1, \ldots, q_{r-1} p_{r-1} t_{r-1}) , \tag{2.41}$$

for all r and times $\{t_i\}$. Setting $r = 2$ in (2.41), we can immediately identify the conditional distribution function as

$$w(q_2 p_2 t_2 | q_1 p_1 t_1) = \mathrm{Tr}[\Delta(q_2, p_2, t_2) \bar{\Delta}(q_1, p_1, t_1)] . \tag{2.42}$$

From (2.41) and (2.42) we can readily deduce the relation

$$p_r(q_1 p_1 t_1, \ldots, q_r p_r t_r) = w(q_r p_r t_r | q_{r-1} p_{r-1} t_{r-1}) \otimes_{r-1} \cdots \otimes_1 p(q_1, p_1, t_1) , \tag{2.43}$$

which establishes the following result:

Theorem 2.2. For a closed quantum system undergoing Hamiltonian evolution, the multitime phase-space distribution functions (2.18) characterize a quantum Markov process in phase space.

This result shows that there exists a close similarity between the stochastic phase-space formulation of quantum theory outlined above, and classical statistical mechanics; for, it can be shown that the Hamiltonian evolution of the closed classical system can be formulated as a classical Markov process in phase space.[17] Theorem 2.2 also shows that the definition of a quantum Markov process that we have employed [see Eq. (2.39) or Eq. (2.40)] is the most appropriate one for a discussion of quantum theory in phase space.

III. STOCHASTIC DESCRIPTION OF OPEN SYSTEMS

An open system is one which is coupled to another (usually large) system called the reservoir.[26] If H_S and H_R are the Hilbert spaces suitable for a quantum-mechanical description of the system and of the reservoir, respectively, then for a complete description of the composite system, we have to consider the Hilbert space

$$H = H_S \otimes H_R , \tag{3.1}$$

which is the tensor product of H_S and H_R. The set of all dynamical variables of the composite system (which we shall denote by $S \otimes R$), will be the set of all self-adjoint operators on H.

Corresponding to each state of $S\otimes R$, characterized by a self-adjoint, positive, trace-class ("density") operator ρ on H, we can associate the reduced operators

$$\rho_S = \mathrm{Tr}_R \rho \tag{3.2a}$$

and

$$\rho_R = \mathrm{Tr}_S \rho , \tag{3.2b}$$

where Tr_R and Tr_S denote the partial trace operations. It can be shown that ρ_S and ρ_R are density operators on H_S and H_R, respectively. We shall consider the situation where the composite system $S\otimes R$ is isolated and undergoes Hamiltonian evolution. The total Hamiltonian $\hat{H}$ is assumed to be of the form

$$\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR} , \tag{3.3}$$

where $\hat{H}_S$ and $\hat{H}_R$ are the free Hamiltonians of S and R, respectively, and $\hat{H}_{SR}$ is the interaction Hamiltonian.

Throughout this section, we shall make use of the following notation. The pair (q, p) refers collectively to the set of all phase-space variables of the system S and the pair (Q, P) refers collectively to the set of all phase-space variables of the reservoir R. If $\{\Delta(q,p)\}$ is a family of representation operators for S and $\{\Delta(Q,P)\}$ is a family of representation operators for R, then we can consider operators

$$\Delta(q,p;Q,P) = \Delta(q,p) \otimes \Delta(Q,P) . \tag{3.4}$$

$\{\Delta(q,p;Q,P)\}$ is a family of representation operators for the composite system $S\otimes R$. In Eq. (3.4) the symbol $\otimes$ stands for the tensor product of operators. From the relations (2.11) and (2.12), which are satisfied by both $\Delta(q,p)$ and $\Delta(Q,P)$, we obtain the following equations:

$$\Delta^\dagger(q,p;Q,P) = \Delta(q,p;Q,P) , \tag{3.5}$$

and

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Delta(q,p;Q,P)\, dq\,dp\,dQ\,dP = I , \tag{3.6}$$

where I is the unit operator in $B(H)$. Also, the orthogonal family of operators $\overline{\Delta}(q,p;Q,P)$ can be obtained from the formula

$$\overline{\Delta}(q,p;Q,P) = \overline{\Delta}(q,p) \otimes \overline{\Delta}(Q,P) , \tag{3.7}$$

and we can easily deduce the relation

$$\mathrm{Tr}[\Delta(q,p;Q,P)\overline{\Delta}(q',p';Q',P')] = \delta(q-q')\delta(p-p')\delta(Q-Q')\delta(P-P') . \tag{3.8}$$

Relations (3.5)–(3.8) show that the family of operators $\{\Delta(q,p;Q,P)\}$ gives rise to a phase-space representation of the composite system $S\otimes R$. In fact we can readily deduce relations analogous to (2.5)–(2.9) in the same way as in Sec. II. In addition we have the following equations:

$$\int_{-\infty}^{\infty}\int \Delta(q,p;Q,P)\,dQ\,dP = \Delta(q,p) \otimes I_R , \tag{3.9a}$$

and

$$\int_{-\infty}^{\infty}\int \Delta(q,p;Q,P)\,dq\,dp = I_S \otimes \Delta(Q,P) , \tag{3.9b}$$

where I_R and I_S are unit operators on H_R and H_S, respectively.

The time-dependent representation operators $\Delta(q,p;Q,P,t)$ are defined in the usual way [see Eqs. (2.10), (2.11)], as follows

$$\Delta(q,p;Q,P,t) = U^{-1}(t)\Delta(q,p;Q,P)U(t) , \tag{3.10}$$

where the unitary operator

$$U(t) = e^{-(i/\hbar)\hat{H}t} , \tag{3.11}$$

depends on the total Hamiltonian. The time-dependent representation operators satisfy relations analogous to (2.12)–(2.14). We may also define the reduced (time-dependent) representation operators $\Delta(q,p,t)$ and $\Delta(Q,P,t)$ via the relations

$$\Delta(q,p,t)=\int\int_{-\infty}^{\infty}\Delta(q,p;Q,P,t)dQdP$$
$$=U^{-1}(t)\{\Delta(q,p)\otimes I_R\}U(t), \qquad (3.12a)$$

and

$$\Delta(Q,P,t)=\int\int_{-\infty}^{\infty}\Delta(q,p;Q,P,t)dqdp$$
$$=U^{-1}(t)\{I_S\otimes\Delta(Q,P)\}U(t). \qquad (3.12b)$$

It may be noted that both $\Delta(q,p,t)$ and $\Delta(Q,P,t)$ are operators on the Hilbert space $H=H_S\otimes H_R$.

A stochastic phase-space formulation of the composite system $S\otimes R$ can be obtained by considering the multitime phase-space distribution functions $p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_rp_rQ_rP_rt_r)$, which are defined in the same way as in Eq. (2.18), by the relation

$$p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_rp_rQ_rP_rt_r)$$
$$=\mathrm{Tr}[\rho\Delta(q_1,p_1;Q_1,P_1,t_1)\cdots\Delta(q_r,p_r;Q_r,P_r,t_r)]. \qquad (3.13)$$

Since the system $S\otimes R$ is assumed to be isolated and undergoing Hamiltonian evolution, the multitime phase-space distribution functions (3.13) satisfy all the consistency relations enumerated in theorem 2.1. In addition, they also fulfill the conditions of theorem 2.2 and, therefore, characterize a generalized Markov process in phase space.

Any time-dependent correlation function of the observables of the composite system $S\otimes R$ can be calculated by a relation of the form (2.16); for example, if $\{\hat{g}_i\}$ is an arbitrary set of observables, we have the relation

$$\langle\hat{g}_1(t_1)\cdots\hat{g}_r(t_r)\rangle$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}g_1(q_1,p_1,Q_1,P_1)\cdots g_r(q_r,p_r,Q_r,P_r)p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\cdots,q_rp_rQ_rP_rt_r)dq_1dp_1\cdots dQ_rdP_r, \qquad (3.14)$$

where

$$g_i(q_i,p_i,Q_i,P_i)=\mathrm{Tr}[\hat{g}_i(t_i)\overline{\Delta}(q_i,p_i;Q_i,P_i,t_i)]$$
$$=\mathrm{Tr}[\hat{g}_i(0)\overline{\Delta}(q_i,p_i;Q_i,P_i)]. \qquad (3.15)$$

In particular, we may consider a set of observables $\{\hat{g}_i\}$, which at time $t=0$ refer to the system S alone, i.e., they satisfy the condition

$$\hat{g}_i(0)=\hat{g}_i^S(0)\otimes I_R, \qquad (3.16)$$

where $\hat{g}_i^S(0)$ is an element of $B(H_S)$. For such observables, we also have the relation

$$\mathrm{Tr}[\hat{g}_i(0)\overline{\Delta}(q_i,p_i;Q_i,P_i)]=\mathrm{Tr}_S[\hat{g}_i^S(0)\overline{\Delta}(q_i,p_i)]\mathrm{Tr}_R\overline{\Delta}(Q_i,P_i)$$
$$=\mathrm{Tr}_S[\hat{g}_i^S(0)\overline{\Delta}(q_i,p_i)], \qquad (3.17)$$

where we have made use of Eq. (2.7). Substituting (3.17) in (3.15), we deduce the following relation for the time correlation function of a set of observables $\{\hat{g}_i\}$ *that refer to the system S alone at the initial time,* i.e., satisfy Eq. (3.16):

$$\langle\hat{g}_1(t_1)\cdots\hat{g}_r(t_r)\rangle=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}g_1(q_1,p_1)\cdots g_r(q_r,p_r)p_r^S(q_1p_1t_1,\ldots,q_rp_rt_r)dq_1\cdots dp_r, \qquad (3.18)$$

where

$$g_i(q_i,p_i)=\mathrm{Tr}_S[\hat{g}_i^S(0)\overline{\Delta}(q_i,p_i)] \qquad (3.19)$$

and

$$p_r^S(q_1p_1t_1,\ldots,q_rp_rt\,)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_rp_rQ_rP_rt_r)dQ_1dP_1\cdots dQ_rdP_r. \qquad (3.20)$$

By using the relation (3.12a) which defines the reduced representation operators $\Delta(q,p,t)$, the distribution functions p_r^S can also be expressed in the following form:

$$p_r^S(q_1p_1t_1,\ldots,q_rp_rt_r)$$
$$=\mathrm{Tr}[\rho\Delta(q_1,p_1,t_1)\cdots\Delta(q_r,p_r,t_r)]. \qquad (3.21)$$

Equation (3.18) demonstrates that all the time correlation functions of observables that refer to the system S alone initially, can be evaluated as phase-space averages with respect to the "reduced multitime distribution functions" p_r^S given by Eqs. (3.20) and (3.21). Therefore the multitime distribution functions p_r^S completely characterize all the statistical features of the evolution of the open subsystems S. We therefore take Eqs. (3.18)–(3.21) as the basis for a stochastic formulation of open quantum systems in phase space. In the rest of this section, we shall investigate some of the properties of the multitime distribution functions (3.20), and the generalized stochastic process they characterize.

The single-time distribution functions $p^S(q,p,t)$ is given by the relation

$$p^S(q,p,t)=\mathrm{Tr}[\rho\Delta(q,p,t)]\,. \tag{3.22}$$

Using Eq. (3.12a), we can express (3.22) in the following way:

$$\begin{aligned}p^S(q,p,t)&=\mathrm{Tr}[\rho U^{-1}(t)\{\Delta(q;p)\otimes I_R\}U(t)]\\&=\mathrm{Tr}[\rho(t)\{\Delta(q,p)\otimes I_R\}]\,,\end{aligned} \tag{3.23}$$

where

$$\rho(t)=U(t)\rho U^{-1}(t)\,.$$

From (3.23) we obtain the relation

$$\begin{aligned}p^S(q,p,t)&=\mathrm{Tr}_S[\Delta(q,p)\mathrm{Tr}_R\{\rho(t)\}]\\&=\mathrm{Tr}_S[\Delta(q,p)\rho_S(t)]\,.\end{aligned} \tag{3.24}$$

Equation (3.24) shows that the reduced single-time distribution function is nothing but the phase-space distribution function corresponding to the reduced density operator of the system $\rho_S(t)$. The special feature of Eq. (3.24) is that all the quantities on the right-hand side refer to the Hilbert space H_S only. It might also be mentioned that in most of the discussions found in the literature,[26,27] on master equations for open systems, the result expressed by our Eq. (3.24) is treated as an assumption.

The reduced distribution functions p_r^S of order $r>1$ cannot be expressed in terms of the reduced density operator $\rho_S(t)$ alone, as they also depend on the correlations between the open subsystem S and the reservoir R. We now proceed to show that these distribution functions satisfy all the consistency relations enumerated in theorem 2.1. This result is derived using the fact that the composite system $S\otimes R$ is a closed system, undergoing Hamiltonian evolution, and hence the multitime distribution functions $p_r^{S\otimes R}$ satisfy the results contained in the two theorems 2.1 and 2.2.

Integrating the relations analogous to (2.19) and (2.22) for $p_r^{S\otimes R}$ with respect to the variables $\{Q,P\}$ of the reservoir, we obtain the following formulas:

$$p_r^{S*}(q_1p_1t_1,\ldots,q_rp_rt_r)=p_r^S(q_rp_rt_r,\ldots,q_1p_1t_1) \tag{3.25}$$

and

$$\int\!\!\int_{-\infty}^{\infty} p_r^S(q_1p_1t_1,\ldots,q_{i-1}p_{i-1}t_{i-1},q_ip_it_i,q_{i+1}p_{i+1}t_{i+1},\ldots)dq_idp_i=p_{r-1}^S(q_1p_1t_1,\ldots,q_{i-1}p_{i-1}t_{i-1},q_{i+1}p_{i+1}t_{i+1},\ldots)\,. \tag{3.26}$$

The multitime distribution functions $p_r^{S\otimes R}$ also satisfy the following positivity condition analogous to (2.21), that must hold with any choice of the function $g_i(q_i,p_i;Q_i,P_i)$:

$$\begin{aligned}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g_1^*(q_1,p_1;Q_1,P_1)\cdots g_r^*(q_r,p_r;Q_r,P_r)g_1(q_1',p_1';Q_1',P_1')\cdots g_r(q_r',p_r';Q_r',P_r')\\ \times p_{2r}^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_1'p_1'Q_1'P_1't_1)dq_1dp_1\cdots dQ_1'dP_1'\geq 0\,.\end{aligned} \tag{3.27}$$

In particular, if $g_i(q_i,p_i;Q_i,P_i)$ are functions of q_i,p_i alone then we obtain the following positivity relation for the reduced distribution functions p_r^S:

$$\begin{aligned}\int\!\!\int_{-\infty}^{\infty} g_1^*(q_1,p_1)\cdots g_r^*(q_r,p_r)g_r(q_r',p_r')\cdots g_1(q_1',p_1')\\ \times p_{2r}^S(q_1p_1t_1,\ldots,q_rp_rt_r,q_r'p_r't_r,\ldots,q_1'p_1't_1)dq_1dp_1\cdots dq_1'dp_1'\geq 0\,.\end{aligned} \tag{3.28}$$

We will now derive the equal-time relation for the distribution functions p_r^S. For that, we again start with the following relation for the distribution functions $p_r^{S\otimes R}$ that is analogous to (2.24):

$$p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_{r-1}p_{r-1}Q_{r-1}P_{r-1}t_{r-1},q_rp_rQ_rP_rt_{r-1})$$
$$=\{\delta(q_r-q_{r-1})\delta(p_r-p_{r-1})\delta(Q_r-Q_{r-1})\delta(P_r-P_{r-1})\}\otimes_{r-1}p_{r-1}^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q_{r-1}p_{r-1}Q_{r-1}P_{r-1}t_{r-1})$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\delta(q_r-q'_{r-1})\delta(p_r-p'_{r-1})\delta(Q_r-Q'_{r-1})\delta(P_r-P'_{r-1})$$
$$\times\operatorname{Tr}[\Delta(q_{r-1},p_{r-1};Q_{r-1},P_{r-1})\Delta(q'_{r-1},p'_{r-1};Q'_{r-1},P'_{r-1})\bar{\Delta}(q''_{r-1},p''_{r-1};Q''_{r-1},P''_{r-1})]$$
$$\times p_{r-1}^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q''_{r-1}p''_{r-1}Q''_{r-1}P''_{r-1}t_{r-1})dq'_{r-1}dp'_{r-1}\cdots dQ''_{r-1}dP''_{r-1}$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\operatorname{Tr}[\Delta(q_{r-1},p_{r-1};Q_{r-1},P_{r-1})\Delta(q_r,p_r;Q_r,P_r)\bar{\Delta}(q''_{r-1},p''_{r-1};Q''_{r-1},P''_{r-1})]$$
$$\times p_r^{S\otimes R}(q_1p_1Q_1P_1t_1,\ldots,q''_{r-1}p''_{r-1}Q''_{r-1}P''_{r-1}t_{r-1})dq''_{r-1}dp''_{r-1}dQ''_{r-1}dP''_{r-1}. \quad (3.29)$$

In simplifying the right-hand side of (3.29), we have used the explicit form of the kernel defining the product $\otimes_{r-1}$. Using the definitions (3.4) and (3.7) of the representation operators $\Delta(q,p;Q,P)$ and $\bar{\Delta}(q,p;Q,P)$, we obtain the relation

$$\operatorname{Tr}[\Delta(q_{r-1},p_{r-1};Q_{r-1},P_{r-1})\Delta(q_r,p_r;Q_r,P_r)\bar{\Delta}(q''_{r-1},p''_{r-1};Q''_{r-1},P''_{r-1})]$$
$$=\operatorname{Tr}_S[\Delta(q_{r-1},p_{r-1})\Delta(q_r,p_r)\bar{\Delta}(q''_{r-1},p''_{r-1})]\operatorname{Tr}_R[\Delta(Q_{r-1},P_{r-1})\Delta(Q_r,P_r)\bar{\Delta}(Q''_{r-1},P''_{r-1})]. \quad (3.30)$$

Let us integrate both sides of Eq. (3.29) with respect to all the variables $\{Q_i,P_i\}$, and substitute from Eq. (3.30) and use the equation $\operatorname{Tr}_R\bar{\Delta}(Q,P)=1$, which follows from (2.7). We then obtain the relation

$$p_r^S(q_1p_1t_1,\ldots,q_{r-1}p_{r-1}t_{r-1},q_rp_rt_{r-1})=\{\delta(q_r-q_{r-1})\delta(p_r-p_{r-1})\}\otimes_{r-1}p_{r-1}^S(q_1p_1t_1,\ldots,q_{r-1}p_{r-1}t_{r-1}), \quad (3.31)$$

where the product $\otimes_{r-1}$ is defined in the usual way in terms of the kernel

$$K(q_{r-1}p_{r-1};q'_{r-1}p'_{r-1},q''_{r-1}p''_{r-1})=\operatorname{Tr}_S[\Delta(q_{r-1},p_{r-1})\Delta(q'_{r-1},p'_{r-1})\bar{\Delta}(q''_{r-1},p''_{r-1})]. \quad (3.32)$$

Having obtained relations (3.25), (3.26), (3.28), and (3.31), we have established the following result:

Theorem 3.1. The reduced multitime phase-space distribution functions p_r^S satisfy the consistency relations (i)–(iv) of theorem 2.1.

This theorem shows clearly that the consistency relations (i)–(iv) are characteristic of all generalized stochastic processes, be they generated by evolution of isolated systems or of open systems. However, the proof of theorem 2.2 cannot be carried over for the case of multitime distribution functions p_r^S of an open system. The non-Markovian character of the generalized stochastic processes induced by the evolution of open systems is also reflected by the fact that the stochastic equations for reduced distribution functions such as $p^S(q,p,t)$ (usually referred to as "master equations" in physics literature), turn out to be integrodifferential equations. The fact that the reduced distribution functions of a Markov process do not in general characterize another Markov process is also well known in classical probability theory.

ACKNOWLEDGMENT

The author is deeply indebted to Professor E. Wolf for his invaluable suggestions regarding the preparation of this manuscript and several stimulating discussions.

APPENDIX: PROOF OF RELATIONS (2.24) and (2.41)

In this Appendix we shall deduce the relations (2.24) and (2.41) which are essential in the proof of theorems 2.1 and 2.2. The relation (2.41) establishes the generalized Markov property of the multitime distribution functions (2.18), generated by the Hamiltonian evolution of a closed quantum system.

We start with the following expression for the multitime distribution functions $p_r(q_1p_1t_1,\ldots,q_rp_rt_r)$:

$$p_r(q_1p_1t_1,\ldots,q_rp_rt_r)=\operatorname{Tr}[\rho\Delta(q_1,p_1,t_1)\cdots\Delta(q_r,p_r,t_r)] \quad (A1)$$

We shall use the following relations for the time-dependent representation operators $\{\Delta(q,p,t)\}$ and the orthogonal family $\{\bar{\Delta}(q,p,t)\}$, which follow easily from their definition [cf. (2.10) and (2.11)]:

$$\operatorname{Tr}[\Delta(q,p,t)\bar{\Delta}(q',p',t)]=\delta(q-q')\delta(p-p'), \quad (A2)$$

and

$$\operatorname{Tr}(AB)=\int\int_{-\infty}^{\infty}\operatorname{Tr}[A\Delta(q,p,t)]\times\operatorname{Tr}[B\bar{\Delta}(q,p,t)]dq\,dp. \quad (A3)$$

Using the equation (A3) for $t=t_{r-1}$, Eq. (A1) can be expressed in the form

$$p_r(q_1p_1t_1,\ldots,q_rp_rt_r)=\int_{-\infty}^{\infty}\!\!\int \mathrm{Tr}[\Delta(q_{r-1},p_{r-1},t_{r-1})\Delta(q_r,p_r,t_r)\Delta(q''_{r-1},p''_{r-1},t_{r-1})]$$
$$\times\,\mathrm{Tr}[\rho\Delta(q_1,p_1,t_1)\cdots\Delta(q_{r-2},p_{r-2},t_{r-2})\Delta(q''_{r-1},p''_{r-1},t_{r-1})]dq''_{r-1}dp''_{r-1}. \quad \text{(A4)}$$

From (2.8) and (2.15) the following relation can be obtained:

$$\Delta(q_r,p_r,t_r)=\int_{-\infty}^{\infty}\!\!\int \Delta(q'_{r-1},p'_{r-1},t_{r-1})\mathrm{Tr}[\Delta(q_r,p_r,t_r)\overline{\Delta}(q'_{r-1},p'_{r-1},t_{r-1})]dq'_{r-1}dp'_{r-1}. \quad \text{(A5)}$$

Substituting (A5) in (A4), we obtain the relation

$$p_r(q_1p_1t_1,\ldots,q_rp_rt_r)$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{Tr}[\Delta(q_{r-1},p_{r-1},t_{r-1})\Delta(q'_{r-1},p'_{r-1},t_{r-1})\overline{\Delta}(q''_{r-1},p''_{r-1},t_{r-1})]$$
$$\times\,\mathrm{Tr}[\Delta(q_r,p_r,t_r)\overline{\Delta}(q'_{r-1},p'_{r-1},t_{r-1})]p_{r-1}(q_1p_1t_1,\ldots,q''_{r-1}p''_{r-1}t_{r-1})dq'_{r-1}dp'_{r-1}dq''_{r-1}dp''_{r-1}. \quad \text{(A6)}$$

If we recall the formula for the kernel $K(q_{r-1}p_{r-1}t_{r-1};q'_{r-1}p'_{r-1},q''_{r-1}p''_{r-1})$ of the product $\otimes_{r-1}$ [see (2.26)], then Eq. (A6) can be expressed in the form

$$p_r(q_1p_1t_1,\ldots,q_rp_rt_r)=\mathrm{Tr}[\Delta(q_r,p_r,t_r)\overline{\Delta}(q_{r-1},p_{r-1},t_{r-1})]\otimes_{r-1}p_{r-1}(q_1p_1t_1,\ldots,q_{r-1}p_{r-1}t_{r-1}). \quad \text{(A7)}$$

Equation (A7) is the same as the Eq. (2.41). If we set $t_r=t_{r-1}$, and use (A2), we obtain the equal-time relation (2.24), viz.,

$$p_r(q_1p_1t_1,\ldots,q_{r-1}p_{r-1}t_{r-1},q_rp_rt_{r-1})=\{\delta(q_r-q_{r-1})\delta(p_r-p_{r-1})\}\otimes_{r-1}p_{r-1}(q_1p_1t_1,\ldots,q_{r-1}p_{r-1}t_{r-1}).$$

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*Research supported in part by the Army Research Office.

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ferent times by using different suffixes.

[24]It may be noted that, for a classical stochastic process, such a distribution function can be expressed in terms of the multitime distribution functions which are usually encountered, via the relation $p_3(q_1p_1t_1, q_2p_2t_2, q_3p_3t_1) = p_2(q_1p_1t_1, q_2p_2t_2)\delta(q_3-q_1) \times\delta(p_3-p_1)$.

[25]In the single-time theory, the problem of obtaining a complete mathematical characterization of the distribution function has been investigated only for some special representation schemes such as those generated by the Wigner-Weyl, the normal, and the antinormal rules of association. See, for example, J. R. Klauder and E. C. G. Sudarshan, *Fundamentals in Quantum Optics* (Benjamin, New York, 1969).

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