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Functional equations and Green's functions for augmented scalar fields

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Certain noncanonical self-coupled scalar quantum field theories, previously formulated by means of functional integration, are herein recast into the form of functional differential equations for the Green's functional. From these expressions the set of coupled equations relating the Green's functions is obtained. The new equations are compared with those of the conventional formulation, and are proposed as alternatives, especially for nonrenormalizable models when the conventional equations fail.

I. INTRODUCTION

In an earlier paper[1] an alternative, noncanonical ("augmented") formulation of scalar field quantization was proposed that was particularly intended, but not exclusively so, for nonrenormalizable fields. In the familiar but formal language of Euclidean-space functional integration the relevant expression for a quartic self-interacting scalar field in n space-time dimensions was taken as ($dx \equiv d^n x$)

$$S'(h) \equiv \mathfrak{N}' \int \exp\left(i \int h\Phi\, dx - \int \left\{\tfrac{1}{2}[(\nabla\Phi)^2 + (m^2+X^2)\Phi^2] + \lambda\Phi^4\right\} dx\right) \mathfrak{D}\Phi\, \mathfrak{D}X\,, \tag{1}$$

where, formally,

$$\mathfrak{D}\Phi \equiv \prod_x d\Phi(x), \quad \mathfrak{D}X \equiv \prod_x dX(x)\,, \tag{2}$$

h represents an external "source" field, and $\mathfrak{N}'$ is chosen so that $S'(0)=1$. This formulation differs from the conventional one in that the action is augmented by the term $\frac{1}{2}\int X^2\Phi^2 dx$ and X is treated as an additional field variable. While this change in no way affects the classical equations of motion,[1] the extra term significantly changes the quantum theory. A difference persists, in particular, even for $\lambda = 0$ which gives rise to a (non-Gaussian) pseudofree theory that is fundamentally different from the conventional (Gaussian) free theory. Arguments in favor of such an alternative approach to scalar field quantization were presented in Ref. 1.

The auxiliary field X may easily be integrated out to yield the formal expression

$$S'(h) = \mathfrak{N}' \int \exp\left(i \int h\Phi\, dx - \int \left\{\tfrac{1}{2}[(\nabla\Phi)^2 + m^2\Phi^2] + \lambda\Phi^4\right\} dx\right) \mathfrak{D}'\Phi\,, \tag{3}$$

where $\mathfrak{N}'$ is adjusted so that $S'(0)=1$ and

$$\mathfrak{D}'\Phi \equiv \prod_x d\Phi(x) \Big/ |\Phi(x)|\,. \tag{4}$$

The formal nature of these expressions is nowhere clearer than in this expression for $\mathfrak{D}'\Phi$, which is formally nonintegrable for each x at $\Phi = 0$. In Ref. 1 this situation was rectified in the course of giving meaning to (3) by means of a lattice-space formulation in which the space-time continuum is replaced by a uniform lattice of points of elementary cell volume Δ. In this prescription the field measure (4) is taken as

$$\prod_k d\Phi_k \Big/ |\Phi_k|^{1-2b\Delta}\,, \tag{5}$$

where k (an n-fold index) labels a lattice point, and the dangerous exponent of unity is replaced by $(1-2b\Delta)$. Here, b denotes a necessary but arbitrary positive constant with dimensions $(\text{length})^{-n}$. The same parameter b also enters the proper formulation of local field powers for these models. In Ref. 1 units were generally chosen such that $b=1$.

The lattice-space approach was discussed at some length in Ref. 1, and it is our purpose here to reformulate the quantum theory of augmented models in another fashion amenable to general study. The formulation considered here is analogous to the well-known functional differential characterization of the Green's functional, i.e., the generating functional for the time-ordered vacuum-expectation-value Green's functions. Coupled Green's function equations follow directly from this study. To conform with standard practice we derive these relations in Minkowski space-time rather than Euclidean space-time. The conventional treatment of functional differential equations for the Green's functional is well studied in a formal fashion and summarized in many field theory texts.[2] For pedagogical reasons we carry out our analysis at a comparably formal level.

It may seem curious to develop differential equations for the augmented Green's functional when the formal functional integral for its expression came first, but this approach has the decided advantage of dealing with the *result* of such an integration, thereby sidestepping the need to introduce strange "convergence factors" [as in (5)] so as to represent that result by quadrature. In other words, the regularization that (5) represents and which is needed as a step in giving meaning to the functional integral (3) is replaced by a regularization of the functional differential equation for the Green's functional, and that regularization is of a more familiar kind (and is thus more palatable).

The functional differential equation for the augmented models is treated in Sec. II, and the coupled Green's function equations are presented in Sec. III. Our results are summarized in Sec. IV.

II. DERIVATION AND ANALYSIS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

In order to appreciate the equation obtained later for the augmented model we first rederive the standard functional differential equation for the conventional models. The conventional Green's functional is formally given by

$$Z(h) \equiv \mathfrak{N} \int \exp\Big(i \int h\phi\, dx + i \int \{\tfrac{1}{2}[(\partial_\mu \phi)^2 - m^2\phi^2] - \lambda\phi^4\}\, dx\Big)\mathfrak{D}\phi\,, \tag{6}$$

normalized so that $Z(0)=1$, and where $\mathfrak{D}\phi$ is as given in (2) and has the formal property (translation invariance) that

$$\mathfrak{D}(\phi+\Lambda) = \mathfrak{D}\phi \tag{7}$$

for arbitrary $\Lambda(x)$. Consequently, it follows that an expression of the form $\int F\{\phi+\Lambda\}\,\mathfrak{D}\phi$ is independent of Λ and thus

$$\frac{\delta}{i\delta\Lambda(x)} \int F\{\phi+\Lambda\}\,\mathfrak{D}\phi\Big|_{\Lambda\equiv 0} \equiv 0\,. \tag{8}$$

Applied to (6) this identity leads to the formal differential equation

$$\left[h(x) - K_x\left(\frac{\delta}{i\delta h(x)}\right) - 4\lambda\left(\frac{\delta}{i\delta h(x)}\right)^3\right] Z(h) = 0\,, \tag{9}$$

where

$$K_x \equiv \Box_x + m^2\,. \tag{10}$$

This is the standard functional differential equation for this model.[2]

For the augmented model, on the other hand, the Green's functional $Z'(h)$ is formally given as

$$Z'(h) \equiv \mathfrak{N}' \int \exp\Big(i \int h\phi\, dx + i \int \{\tfrac{1}{2}[(\partial_\mu \phi)^2 - m^2\phi^2] - \lambda\phi^4\}\, dx\Big)\mathfrak{D}'\phi\,, \tag{11}$$

normalized so that $Z'(0)\equiv 1$, and where $\mathfrak{D}'\phi$ is as given in (4) and has the formal property (scale invariance) that

$$\mathfrak{D}'S\phi = \mathfrak{D}'\phi \tag{12}$$

for arbitrary $S(x)>0$. Consequently, it follows that an expression of the form $\int F'\{S\phi\}\,\mathfrak{D}'\phi$ is independent of S and thus

$$\frac{\delta}{i\delta S(x)} \int F'\{S\phi\}\,\mathfrak{D}'\phi\Big|_{S=1} \equiv 0\,. \tag{13}$$

Applied to (11) this identity leads to the formal differential equation

$$\left[h(x)\left(\frac{\delta}{i\delta h(x)}\right) - \left(\frac{\delta}{i\delta h(x)}\right) K_x\left(\frac{\delta}{i\delta h(x)}\right) - 4\lambda\left(\frac{\delta}{i\delta h(x)}\right)^4\right] Z'(h) = 0\,, \tag{14}$$

which is a relation for the augmented model analogous to (9). Of course, each of these equations is highly formal, and their proper formulation needs suitable regularization and renormalization.

Note that by operating on (9) with $\delta/i\delta h(x)$ we find the relation

$$\left[\left(\frac{\delta}{i\delta h(x)}\right) h(x) - \left(\frac{\delta}{i\delta h(x)}\right) K_x\left(\frac{\delta}{i\delta h(x)}\right) - 4\lambda\left(\frac{\delta}{i\delta h(x)}\right)^4\right] Z(h) = 0\,. \tag{15}$$

The difference between (14) and (15) lies in the first term, and the difference in the formal operators is "just" $i\delta(0)$. This seemingly small difference actually has rather profound consequences, which have to do with prescriptions for operator products, renormalization, etc.

Free and pseudofree models

When $\lambda=0$, (9) and (14) relate to the free theory and the pseudofree theory, respectively. In this case the solution $Z_0(h)$ of (9) is given by

$$Z_0(h) = \exp\left[i\tfrac{1}{2}\int h(x)(K_x - i0)^{-1}h(x)dx\right], \tag{16}$$

which characterizes the conventional free theory. For the pseudofree model the Green's functional $Z_0'(h)$ satisfies

$$\left[h(x)\left(\frac{\delta}{i\delta h(x)}\right) - \left(\frac{\delta}{i\delta h(x)}\right) K_x\left(\frac{\delta}{i\delta h(x)}\right)\right] Z_0'(h) = 0\,. \tag{17}$$

Unfortunately, no specific expression for the solution $Z_0'(h)$ is known, but it is nevertheless a functional of basic importance. To see this one need only recall the formal relation of the conventional approach given by

$$Z(h)=N\exp\left[-i\lambda\int\left(\frac{\delta}{i\delta h(x)}\right)^4 dx\right]Z_0(h)\,, \qquad (18)$$

where N restores the normalization, that connects the interacting and free models, and generates on expansion in λ the conventional perturbation series.[2] In like manner there arises the analogous formal relation

$$Z'(h)=N'\exp\left[-i\lambda\int\left(\frac{\delta}{i\delta h(x)}\right)^4 dx\right]Z_0'(h)\,, \qquad (19)$$

with N' chosen for normalization, that connects the interacting and pseudofree models. Evidently, expansion in λ of (19) generates some sort of perturbation series for the interacting augmented model, but the nature of this perturbation series is not immediately evident. In the conventional case it is the Gaussian character of $Z_0(h)$ that ultimately leads to local products defined as Wick powers, to a Feynman diagrammatic representation of the perturbation series, and to the usual renormalization prescriptions. In the augmented case $Z_0'(h)$ is non-Gaussian and so the conventional rules are unlikely to be applicable. Nevertheless, the proper rules to define local field products so as to build the interaction are implicitly contained in the pseudofree functional $Z_0'(h)$ (because it implicitly defines a field operator representation and consequently its local powers). The determination of the prescription for local products is an important by-product of the study of the equation for the pseudofree Green's functional.

Independent-value models

Independent-value models—which may be formally characterized as covariant models stripped of all space-time gradients—are relevant to the present discussion (at least mathematically) and serve to illustrate the kind of renormalizations that arise. Elsewhere[3] we have provided a careful and divergent-free formulation of independent-value models; here it is instructive to treat them heuristically and intuitively, but still basically in a correct fashion.

For the independent-value models the only change from a formal point of view that is neccessary to make contact with our previous discussion is the replacement of $K_x\equiv\Box_x+m^2$ simply by m^2. In this case, it is known that there is no solution for the conventional formulation in (9) unless $\lambda\equiv 0$.[1] On the other hand, valid nontrivial solutions exist for the augmented formulation based on (14) for $\lambda\geq 0$,[1] and it suffices for present purposes to examine the case $\lambda=0$, as in (17). With K_x replaced by m^2—or more generally and properly by $m_0{}^2$, a parameter to be determined—(17) becomes

$$\left[h(x)\left(\frac{\delta}{i\delta h(x)}\right)-m_0{}^2\left(\frac{\delta}{i\delta h(x)}\right)^2\right]\overline{Z}_0'(h)=0\,. \qquad (20)$$

This relation is formal as it stands and requires renormalization; this is evident from the fact that symmetry demands a solution in the form

$$\overline{Z}_0'(h)=\exp\left\{i\int W_0[h(x)]dx\right\}, \qquad (21)$$

with the condition that $W_0[0]\equiv 0$. To make sense of this equation the second term in (20) needs a multiplicative renormalization to counteract the infinite factor $\delta(0)$ that would otherwise arise. This multiplicative renormalization—in other words, the definition of the local square of the field in the present case—has two consequences. The formal mass renormalization $m_0{}^2=m^2/\delta(0)$ is strictly speaking unacceptable on dimensional grounds if m_0 and m both have mass dimension one; instead, the relation $m_0{}^2=bm^2/\delta(0)$ must be used where b is an arbitrary positive constant of dimension $(\text{length})^{-n}$, the arbitrariness of b just reflecting the arbitrariness in the finite scale that exists after an infinite rescaling. This is the same parameter that was already introduced in relation to (5). (Of course, one could always arrange to set $b=1$.) The second feature of an infinite multiplicative renormalization is that only truncated (connected) functions enter the differential equation; the nontruncated parts are scaled to zero by the renormalization. One further point in the renormalization is needed in the form of subtracting the vacuum expectation (like normal-ordering) of the mass term since (20) is superficially inconsistent at $h(x)\equiv 0$. Such an extra term actually has its origin in the S dependence of $\mathfrak{N}'$ that should be taken into account in deriving (14) from (11) and (13). Thus we are led to the more proper relation for (20) given by

$$\left[h(x)\left(\frac{\delta}{i\delta h(x)}\right)-\frac{bm^2}{\delta(0)}:\left(\frac{\delta}{i\delta h(x)}\right)^2:\right]\overline{Z}_0'(h)=0\,. \qquad (22)$$

Here : : denotes subtraction of the vacuum expectation, and is interpreted as

$$:\left(\frac{\delta}{i\delta h(x)}\right)^2:\overline{Z}_0'(h) \equiv\left(\frac{\delta}{i\delta h(x)}\right)^2\overline{Z}_0'(h)-\left[\left(\frac{\delta}{i\delta h(x)}\right)^2\overline{Z}_0'(h)\right]_{h=0}\overline{Z}_0'(h)\,. \qquad (23)$$

Insertion of expression (21) into the properly interpreted differential equation leads (with $W_0'[h] \equiv dW_0[h]/dh$, etc.) to

$$h(x)W_0'[h(x)]+ibm^2\{W_0''[h(x)]-W_0''[0]\}=0\,. \qquad (24)$$

Here at last is an equation devoid of any formal character. The linearity of (24) just reflects the fact that only truncated terms survive the renormalization.

A solution to (24) is sought such that $W_0[-h] = W_0[h]$ and $W_0[0]\equiv 0$, and it is straightforward to see that the desired solution is given by

$$W_0[h(x)]=W_0''[0]\int_0^{h(x)}dv\int_0^v du\,\exp[i\tfrac{1}{2}(v^2-u^2)/bm^2]\,. \qquad (25)$$

The coefficient $W_0''[0]$ is left arbitrary but it is convenient (and consistent with previous choices[3]) to choose $W_0''[0]=2/m^2$, in which case it follows that

$$W_0[h(x)]=2b\int_0^{h(x)/(bm^2)^{1/2}}ds\int_0^s dt\,\exp[i\tfrac{1}{2}(s^2-t^2)]\,, \qquad (26)$$

an expression that coincides with solutions previously given after translating between Minkowski and Euclidean space-time.[3,4] We remark that in the Euclidean space form the relevant expressions define a positive-definite characteristic functional and thus the model possesses an underlying stochastic interpretation.

While our derivation of (26) has been formal, one could and should imagine regularizing an equation such as (22) by a high-momentum cutoff which is removed at the end of the computation. This regularization is implicit in our analysis.

Finally, we remark that the solution determined for the pseudofree independent-value model does indeed implicitly contain the essentials needed to define local products for this model. Moreover, with this prescription an approach such as symbolized by (19) can actually be employed to introduce a quartic interaction. These matters are adequately discussed in Ref. 3 and are not treated here.

Coupled differential equations

It is interesting, and at first sight somewhat surprising, that the basic differential equations (9) and (14) are of different *order*. To compare the two equations, as in (14) and (15), it was in fact necessary to take an additional functional derivative of (9) at the *same* point x. Generally speaking there is "less information" in such a higher-order equation. For example, to arrive at (15) the right-hand side of (9) could have been anything that did not depend on the function h at the particular point x. In studying the independent-value model this potential ambiguity was absent by virtue of the special symmetry of the model as exemplified by (21). For the relativistic case this possible "loss of information" may be serious, and thus we now derive coupled equations where no "loss of information" occurs.

For that purpose consider the augmented Green's functional

$$Z'(h,w)\equiv \mathfrak{N}'\int \exp\Big(i\int (h\phi+w\chi)dx + i\int \{\tfrac{1}{2}[(\partial_\mu\phi)^2-(m^2+\chi^2)\phi^2] - \lambda\phi^4\}dx\Big)\,\mathfrak{D}\phi\,\mathfrak{D}\chi\,, \qquad (27)$$

where in addition to h we have introduced a "source" field w for the field variable χ, where $\mathfrak{N}'$ is chosen so that $Z'(0,0)=1$, and where $\mathfrak{D}\phi$ and $\mathfrak{D}\chi$ are as defined in (2) and have the formal property that

$$\mathfrak{D}(\phi+\Lambda)=\mathfrak{D}\phi,\quad \mathfrak{D}(\chi+M)=\mathfrak{D}\chi \qquad (28)$$

for arbitrary $\Lambda(x)$ and $M(x)$. It follows that an expression of the form $\int F'\{\phi+\Lambda,\chi+M\}\mathfrak{D}\phi\,\mathfrak{D}\chi$ is independent of both Λ and M, and thus

$$\frac{\delta}{i\delta\Lambda(x)}\int F'\{\phi+\Lambda,\chi\}\mathfrak{D}\phi\,\mathfrak{D}\chi\Big|_{\Lambda=0}\equiv 0 \qquad (29)$$

and

$$\frac{\delta}{i\delta M(x)}\int F'\{\phi,\chi+M\}\mathfrak{D}\phi\,\mathfrak{D}\chi\Big|_{M=0}\equiv 0\,. \qquad (30)$$

Applied to (27) these identities lead to the formal equations

$$\Big\{h(x)-\Big[K_x+\Big(\frac{\delta}{i\delta w(x)}\Big)^2\Big]\Big(\frac{\delta}{i\delta h(x)}\Big) - 4\lambda\Big(\frac{\delta}{i\delta h(x)}\Big)^3\Big\}Z'(h,w)=0 \qquad (31)$$

and

$$\Big[w(x)-\Big(\frac{\delta}{i\delta h(x)}\Big)^2\Big(\frac{\delta}{i\delta w(x)}\Big)\Big]Z'(h,w)=0\,. \qquad (32)$$

As may be expected, a careful formulation of such equations entails regularization and renormalization. Given a solution $Z'(h,w)$, the functional of interest is given by

$$Z'(h)\equiv Z'(h,0)\,, \qquad (33)$$

which according to (27) represents the Green's functional for the augmented $(\phi^4)_n$ model.

The coupled equations (31) and (32) characterize the functional $Z'(h,w)$ at the same "information level" as (9) characterizes $Z(h)$. To see this more clearly we may derive another relation by acting with $\delta/i\delta h(x)$ on (31) and with $\delta/i\delta w(x)$ on (32), specifically at the same point x, and then subtracting. The result is given by

$$\left[h(x)\left(\frac{\delta}{i\delta h(x)}\right)-w(x)\left(\frac{\delta}{i\delta w(x)}\right)-\left(\frac{\delta}{i\delta h(x)}\right)K_x\left(\frac{\delta}{i\delta h(x)}\right)-4\lambda\left(\frac{\delta}{i\delta h(x)}\right)^4\right]Z'(h,w)=0\,. \tag{34}$$

When this expression is evaluated at $w\equiv 0$ the second term makes no contribution, and consequently the single differential equation (14) for $Z'(h)$ that was based on the scale invariance of $\mathfrak{D}'\phi$ is recovered. We emphasize again that relativistic models may require the "information level" contained in the augmented set of equations (31) and (32) in contrast to the "lesser information" in (14).

III. DERIVATION AND ANALYSIS OF GREEN'S FUNCTION EQUATIONS

In principle, it is a straightforward matter to derive coupled Green's function equations from an equation such as (14) for the Green's functional. By definition,

$$Z'(h)\equiv\sum_{m=0}^{\infty}(m!)^{-1}i^m\int\cdots\int G'_m(x_1,x_2,\ldots,x_m)h(x_1)h(x_2)\cdots h(x_m)dx_1dx_2\cdots dx_m\,, \tag{35}$$

with $G'_0\equiv 1$, and

$$\ln Z'(h)\equiv\sum_{m=1}^{\infty}(m!)^{-1}i^m\int\cdots\int G'^{T}_m(x_1,x_2,\ldots,x_m)h(x_1)h(x_2)\cdots h(x_m)dx_1dx_2\cdots dx_m\,, \tag{36}$$

where the superscript T denotes "truncated." Insertion of (35) or (36) into (14) generates an infinite set of coupled equations for G'_m or G'^{T}_m, respectively. Proceeding straightforwardly, one would determine a *linear* set of equations relating the Green's functions, and consequently a *nonlinear* set of equations relating the trunacted Green's functions. However, experience with equations such as (14) strongly suggests (as was the case for independent-value models) that multiplicative renormalization factors lead only to connected contributions from the higher-order functional derivatives. Exploiting this feature at the outset one effectively arrives at a *linear* set of equations relating the truncated Green's functions. While this expected behavior can only be postulated, it can certainly be checked subsequently for consistency given a specific solution. Adopting this consequence of the anticipated renormalization we find for (14) the coupled set of equations

$$i\left[\sum_{r=1}^{m}\delta(x-x_r)\right]G'^{T}_m(x_1,\ldots,x_m)+K_xG'^{T}_{m+2}(x,x^*,x_1,\ldots,x_m)+4\lambda G'^{T}_{m+4}(x,x,x,x,x_1,\ldots,x_m)=0\,, \tag{37}$$

which holds for all even $m\geqslant 2$ (for odd m, $G'^{T}_m\equiv 0$ by symmetry). Here x^* denotes a "second" variable that is set equal to x but only *after* the operation K_x is carried out on the "first" x variable. For simplicity expected multiplicative renormalizations in K_x and λ are omitted, but the vacuum expectation of the two latter terms in (14) have been subtracted off [cf. the discussion regarding (22)]. This equation is homogeneous, but an overall scale factor can be adopted through a suitable normalization of G'_2 [cf. the discussion regarding (25)].

For $\lambda=0$, (37) reduces to the coupled Green's functions for the pseudofree model. To focus on their importance and to simplify notation we set $g_m\equiv G'^{T}_m(\lambda=0)$ for the pseudofree model, and therefore we find the fundamental relations

$$i\left[\sum_{r=1}^{m}\delta(x-x_r)\right]g_m(x_1,\ldots,x_m)+K_xg_{m+2}(x,x^*,x_1,\ldots,x_m)=0 \tag{38}$$

for all even $m\geqslant 2$. Again, expected multiplicative renormalizations are not included. Assume, somehow, that a solution to (38) were found. Then it is conceivable that starting with that solution some iterative procedure could be devised on the basis of (37) to formally solve the quartic interacting case. This approach has every reason to be considered seriously.

But the question naturally arises whether (38) actually determines the pseudofree truncated Green's functions g_m, or whether (37) actually determines them in the interacting case. Concern on this point was expressed in the last section and it arises here because in (37) and (38) x^* is not independent of x.

Coupled Green's function equations

Equations that fully determine the Green's functions follow from the augmented set of functional equations (31) and (32). The structure of these equations lends itself to an initial description linear in the Green's functions (rather than the truncated functions). Thus let us introduce the definition

$$Z'(h,w)\equiv\sum_{m,n=0}^{\infty}(m!n!)^{-1}i^{m+n}\int\cdots\int G'_{m,n}(x_1,\ldots,x_m;y_1,\ldots,y_n)\times h(x_1)\cdots h(x_m)w(y_1)\cdots w(y_n)dx_1\cdots dx_m\,dy_1\cdots dy_n\,, \tag{39}$$

with $G'_{0,0}\equiv 1$, whereupon (31) and (32) become

$$\Big[i\sum_{r=1}^{m}\delta(x-x_r)G'_{m-1,n}(x_1,\ldots,\hat{x}_r,\ldots,x_m;y_1,\ldots,y_n)+K_xG'_{m+1,n}(x,x_1,\ldots,x_m;y_1,\ldots,y_n)+G'_{m+1,n+2}(x,x_1,\ldots,x_m;x,x,y_1,\ldots,y_n)+4\lambda G'_{m+3,n}(x,x,x,x_1,\ldots,x_m;y_1,\ldots,y_n)\Big]=0, \tag{40}$$

for odd $m\geqslant 1$ and even $n\geqslant 0$, and

$$\Big[i\sum_{s=1}^{n}\delta(x-y_s)G'_{m,n-1}(x_1,\ldots,x_m;y_1,\ldots,\hat{y}_s,\ldots,y_n)+G'_{m+2,n+1}(x,x,x_1,\ldots,x_m;x,y_1,\ldots,y_n)\Big]=0\,, \tag{41}$$

for even $m\geqslant 0$ and odd $n\geqslant 1$, and where $\hat{x}_r$ signifies that the symbol x_r is omitted, etc. Of course, renormalizations are undoubtedly required for these equations. And it is our expectation that certain multiplicative renormalizations enter in a highly nontrivial fashion. For, as we have seen, the coupled set of functional differential equations (31) and (32) imply the "higher-order" equation (14); and thus (40) and (41) must conspire to imply the validity of (37). A study of the coupled equations (37) and (40) and (41), and their interrelationship, is of prime importance in understanding augmented quantum field theory.

Conventional equations

The relation of (40) and (41) to the conventional coupled Green's function equations for the $(\phi^4)_n$ model is readily determined. On comparing (9) and (31), and referring to (40), it is easy to see that the Green's functions defined by

$$Z(h)\equiv\sum_{m=0}^{\infty}(m!)^{-1}i^m\int\cdots\int G_m(x_1,\ldots,x_m)\times h(x_1)\cdots h(x_m)\,dx_1\cdots dx_m\,, \tag{42}$$

with $G_0\equiv 1$, satisfy, by virtue of (9), the coupled set of equations

$$\Big[i\sum_{r=1}^{m}\delta(x-x_r)G_{m-1}(x_1,\ldots,\hat{x}_r,\ldots,x_m)+K_xG_{m+1}(x,x_1,\ldots,x_m)+4\lambda G_{m+3}(x,x,x,x_1,\ldots,x_m)\Big]=0 \tag{43}$$

for all odd $m\geqslant 1$. These are the conventional coupled Green's function equations for this model, and equations of this general type have been extensively studied.[2,5]

IV. SUMMARY

In this paper we have derived formal functional differential equations for Green's functionals of augmented quantum field theory, a theory that has been proposed as an alternative approach notably when conventional quantization techniques fail. One advantage of the differential equation formulation is that the strange regularization of the field measure (5), which is unavoidably prominent in the lattice-space formulation, need never be introduced. Instead any regularization takes place within the functional differential equations, and as we have emphasized is of a fairly common variety. The differential equation for the augmented Green's functional, which was presented first, followed from the formal scale invariance of the basic field differential $\mathfrak{D}'\phi$ and was given in (14). This relation is to be compared with (9), which is the conventional differential equation for such models. A more equal comparison arises between (14) and (15), a derivative of (9); thus (15) and hence (14) may involve "less information" than appears in (9). Consequently, we derived the coupled differential equations (31) and (32), which determine the augmented Green's functional without any possible "loss of information." Equations (31) and (32) imply the earlier relation (14) for the augmented models. While we have not launched into a discussion of the solution to these equations for covariant models, the derivation of the solution for independent-value models suggests the kind of approach that will be necessary.

The means to pass from the differential equation for the Green's functional to the coupled equations for the Green's functions is basically straightforward and well known.[2] In particular, the derivation of (43) from (9) for the conventional approach, and of (40) and (41) for the augmented approach follows long-standing procedures. On the other hand, while the same procedures could in principle be applied to (14), the very different character of this equation along with the anticipated multiplicative renormalization suggests a radically different approach to the coupled equations for the Green's functions. In this case only connected components survive the anticipated infinite multiplicative renormalization and we arrive at the coupled equations represented by (37).

A further comparison of the two types of coupled Green's function equations we have discussed is useful. On the one hand, (43) as well as (40) and (41) represent *linear* and *inhomogeneous* equations for the Green's functions, and their scale is formally fixed by the inhomogeneity, namely the condition that $G_0 \equiv G'_{0,0} \equiv 1$. On the other hand, (37) which incorporates some of the anticipated consequences of multiplicative renormalization represents a *linear* and *homogeneous* equation for the truncated Green's functions, and the overall scale may be chosen as desired. This second type of coupled equations is certainly not devoid of application for it is exactly the relevant approach to independent-value models.[4] Moreover, it is not inconceivable that starting with the independent-value model solution the space-time gradients could be restored by some suitable form of iteration scheme or perturbation theory.[6]

In conclusion, it seems clear that the new coupled equations proposed, as embodied in (37) and (40) and (41), deserve careful study for potential application to highly singular covariant $(\phi^4)_n$ models such as the nonrenormalizable ($n \geq 5$) models—and perhaps even to define new, noncanonical solutions for renormalizable ($n=4$) and superrenormalizable ($n=2,3$) models. Should such a study show that a consistent formulation can be constructed, then these methods should be applied to models of more direct physical interest.

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[1]J. R. Klauder, Phys. Rev. D 14, 1952 (1976).

[2]See, e.g., D. Lurié, *Particles and Fields* (Interscience, New York, 1968); J. Rzewuski, *Field Theory* (Iliffe, London, 1969), Vol. II; H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Massachusetts, 1972).

[3]J. R. Klauder, Acta Phys. Austriaca 41, 237 (1975).

[4]M. Marinaro, Nuovo Cimento 32A, 355 (1976).

[5]See, e.g., K. Symanzik, in *Lectures in Theoretical Physics, Boulder, Colorado, 1960*, edited by W. E. Brittin *et al*. (Interscience, New York, 1961); in *Lectures on High-Energy Physics*, edited by B. Jakšić (Gordon and Breach, New York, 1966); J. G. Taylor, Nuovo Cimento Suppl. 1, 857 (1963); E. R. Caianiello, *Combinatorics and Renormalization in Quantum Field Theory* (Benjamin, New York, 1973).

[6]Cf., S. Kövesi-Domokos, Nuovo Cimento 33A, 769 (1976).