Stress-tensor trace anomaly in a gravitational metric: General theory, Maxwell field*

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The renormalization of the vacuum expectation value of the stress-energy tensor of a scalar field propagating in a curved space-time with an arbitrary metric was discussed in a previous paper. A new regularization scheme was introduced which employs a continuation in the dimensionality of space-time implemented with a proper-time representation of the Green's function. Here we present a more general formulation of this method which clarifies its basic features and which explicitly displays the stress tensor as the metric functional derivative of the one-loop action functional. We apply this more general formulation to both the scalar field theory and to the electrodynamic, Maxwell theory. Although the trace of the stress tensor formally vanishes both for the massless scalar field and for the Maxwell field, the trace of the renormalized vacuum expectation value of the stress tensor does not vanish for either theory. These finite-trace anomalies cannot be removed by adding a finite local counterterm into the Lagrange function. The anomalies are intimately related to the infinite scalar counterterms that are needed to render the action finite.

I. INTRODUCTION AND SUMMARY

The vacuum expectation value of the stress-energy tensor[1] and the corresponding one-loop action functional were studied in a previous paper[2] (which we shall refer to as I) for the case of a scalar field propagating in a space-time with an arbitrary metric. A renormalization scheme was introduced in that paper which employs a continuation in the space-time dimensionality accomplished with the use of a proper-time representation.[3,4] This new method is well defined and free of ambiguity.[5] It yields an explicit stress-tensor trace anomaly of the kind which Deser, Duff, and Isham[6] indicated should exist. The previous work relied heavily on DeWitt's[7] explicit "WKB" construction of Schwinger's[3] proper-time representation. Here we shall present the theory in a more general, formal manner. We need the WKB construction only to exhibit functional dependence on the space-time dimensionality and for the explicit calculation of the renormalization counterterms and the anomalous terms in the stress-tensor trace. This more formal presentation clarifies the basic elements in the dimensionally continued, proper-time renormalization scheme. Moreover, it exhibits the vacuum expectation value of the stress tensor directly as the metric functional derivative of the one-loop action functional. This necessary connection, which guarantees that the stress tensor is conserved, was not made clear in paper I. In the present work, we shall apply the renormalization scheme to the case of the electromagnetic, Maxwell field as well as to the scalar field. The trace of the stress tensor as calculated naively from the Lagrange function appears quite differently in these two cases. The Lagrange function for a massless scalar field can be chosen so as to produce, naively, a traceless stress tensor in a space-time of arbitrary dimensionality as a consequence of the dynamical field equations. On the other hand, the trace of the stress tensor of the Maxwell field naively vanishes only in four dimensions, and then it vanishes as a consequence of the algebraic structure of the Lagrange function. Nonetheless, our renormalization scheme gives similar trace anomalies for these two theories.

We discuss the scalar theory in Sec. II. Although this discussion clarifies the work of paper I, it does not altogether supersede that paper which, in addition to containing various descriptive and calculational details omitted here, carries out the dimensional continuation with the Lagrange function chosen so that the naive stress-tensor trace identity holds for arbitrary space-time dimensionality n. Here we fix the theory so that this identity,

$$g_{\mu\nu} T^{\mu\nu} = -m^2 \phi^2, \tag{2.5}$$

holds only at $n=4$. We do this not only so as to make the scalar case parallel more closely the Maxwell case, but to simplify the theory, as was already remarked upon in paper I. We show in Sec. II that the one-loop action functional has the dimensional-continuation limit in four dimensions

$$W_1^{(n=4)} = \left(\frac{1}{4-n} + L_4\right) \int (d^4x) \sqrt{-g}\, \mathcal{A}^{(0)} + W_{1\,\mathrm{ren}}^{(n=4)}, \tag{2.45}$$

where the renormalized action functional $W_{1\,\mathrm{ren}}^{(n=4)}$ possesses a well-defined proper-time representation. The infinite counterterm involves

$$\mathcal{G}^{(0)} = \frac{1}{(4\pi)^2}\left[\tfrac{1}{180}(R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} - R_{\mu\nu}R^{\mu\nu} + R_{,\mu}{}^{;\mu}) + \tfrac{1}{2}m^4\right]. \tag{I.1.24}$$

(Here and subsequently we use equations with a prefix I to label the corresponding equations in paper I.) Note that although m^2R is of the proper scale dimension, it does not appear in the infinite counterterm. In order to examine the structure of the counterterm more closely, we note that the Weyl tensor

$$C^{\lambda}{}_{\mu\kappa\nu} = R^{\lambda}{}_{\mu\kappa\nu} - \tfrac{1}{2}(\delta^{\lambda}{}_{\kappa}R_{\mu\nu} - \delta^{\lambda}{}_{\nu}R_{\mu\kappa} - g_{\mu\kappa}R^{\lambda}{}_{\nu} + g_{\mu\nu}R^{\lambda}{}_{\kappa}) + \tfrac{1}{6}R(\delta^{\lambda}{}_{\kappa}g_{\mu\nu} - \delta^{\lambda}{}_{\nu}g_{\mu\kappa}) \tag{1.1}$$

is not altered by a conformal transformation of the metric tensor,

$$g_{\mu\nu}(x) \to \lambda(x)^2 g_{\mu\nu}(x). \tag{1.2}$$

Moreover, the quantity

$$G = R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{1.3}$$

is a topological scalar in the sense that its space-time volume integral is invariant under any metric variation,

$$\delta\int (d^4x)\sqrt{-g}\,G = 0. \tag{1.4}$$

Expressing $\mathcal{G}^{(0)}$ in terms of these quantities gives

$$\mathcal{G}^{(0)} = \frac{1}{(4\pi)^2}\left[\tfrac{1}{120}(C_{\mu\nu\lambda\kappa}C^{\mu\nu\lambda\kappa} - \tfrac{1}{3}G) + \tfrac{1}{180}R_{,\mu}{}^{;\mu} + \tfrac{1}{2}m^4\right]. \tag{1.5}$$

The topological scalar G and the total derivative $R_{,\mu}{}^{;\mu}$ can be omitted from the infinite counterterm since they do not contribute to its metric variation. We see that the massless theory is renormalized by an infinite counterterm containing the square of the Weyl tensor. This counterterm is invariant under a conformal transformation (1.2) of the metric tensor. The massive theory requires a further renormalization involving m^4 which corresponds to an infinite renormalization of the cosmological constant in the Einstein Lagrange function. The renormalized action can be expressed as the space-time volume integral of an effective, one-loop Lagrangian,

$$W_{1\,\mathrm{ren}}^{(n=4)} = \int (d^4x)\sqrt{-g}\,\mathcal{L}_{1\,\mathrm{ren}}^{(n=4)}, \tag{2.46}$$

with the effective Lagrangian written in a proper-time representation,

$$\mathcal{L}_{1\,\mathrm{ren}}^{(n=4)} = \tfrac{1}{4}\mathcal{G}^{(0)} - \frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 \times[e^{-m^2 is}F(x,x;is)], \tag{2.47}$$

where $F(x,x;is)$ is a weight in the proper-time representation of the scalar field Green's function. Here κ is an arbitrary, auxiliary scale mass which must be introduced in the dimensional-continuation process so as to keep the integrand at a fixed scale dimension appropriate to $n=4$ before the limit $n\to 4$ is taken. A change in this scale mass produces the proper-time integral of a total derivative with only the lower limit of the integration contributing. Since

$$\mathcal{G}^{(0)} = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is)]\Big|_{s=0}, \tag{I.1.24}$$

such a change is accounted for by a finite change in the finite constant L_4 which appears in the infinite counterterm displayed in Eq. (2.45) quoted above.[8] [The constant L_4 accounts for the derivative with respect to n of various dimensional-dependent factors such as $(4\pi)^{-n/2}$.]

The vacuum expectation value of the stress-energy tensor is given by the variational derivative of the action functional with respect to the metric tensor. Thus, the proper-time representation of the renormalized action yields the renormalized stress-tensor representation

$$\langle T^{\mu\nu}(x)\rangle_{\mathrm{ren}}^{(n=4)} = \frac{1}{8}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2 \overline{T}^{\mu\nu}(x;is)\Big|_{s=0} - \frac{1}{4}\frac{1}{(4\pi)^2} \times\int_0^\infty ids(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3\overline{T}^{\mu\nu}(x;is), \tag{2.52}$$

which is automatically conserved. The trace of the stress-tensor weight can be obtained by computing, with an operator technique, the effect of a conformal transformation of the metric, Eq. (1.2), and we find that

$$\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x) = -2(is)^2 is\frac{\partial}{\partial is}(is)^{-2}e^{-m^2 is}F(x,x;is) - 2m^2 is\,e^{-m^2 is}F(x,x;is). \tag{2.58}$$

When the right-hand side of this identity is put into the proper-time integral for the renormalized stress tensor, Eq. (2.52), one finds a quantity which can be integrated by parts to get an integral of a total derivative. This trivial integral yields the counterterm scalar $\mathcal{G}^{(0)}$ shown in Eq. (I.1.24) plus a term that cancels the trace of the first term on the right-hand side of Eq. (2.52). There remains a proper-time integral which represents the renormalized vacuum expectation value of the square of the scalar field. Hence,

$$\langle T^{\mu\nu}(x)\rangle_{\rm ren}^{(n=4)} g_{\mu\nu}(x) = \mathcal{C}^{(0)}(x) - m^2\langle\phi^2(x)\rangle_{\rm ren}^{(n=4)} . \tag{2.63}$$

This is the anomalous trace identity which violates the naive expectation, Eq. (2.5). The anomaly[9] $\mathcal{C}^{(0)}(x)$ cannot be removed by an additional renormalization with a local counterterm. On dimensional grounds, such a counterterm would involve only $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}R^{\alpha\beta}$, and R^2, or equivalently, the square of the Weyl tensor, $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$, the topological scalar, G, and R^2. Because of its conformal invariance, an action formed by the space-time volume integral of the square of the Weyl tensor yields a traceless stress tensor. The topological scalar produces a vanishing stress tensor. Thus, only R^2 remains as a possible counterterm with which the anomaly $\mathcal{C}^{(0)}$ could be removed by an additional renormalization. However, since under an infinitesimal version of the conformal metric transformation (1.2)

$$\delta(\sqrt{-g}R^2) = -\sqrt{-g}\,12\delta\lambda_{,\mu}{}^{;\mu}R , \tag{1.6}$$

we find that

$$\frac{2}{[-g(x)]^{1/2}} g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)} \int (d^4x)\sqrt{-g}R^2 = \frac{1}{[-g(x)]^{1/2}} \frac{\delta}{\delta\lambda(x)} \int (d^4x)\sqrt{-g}R^2 = -12R_{,\mu}{}^{;\mu}, \tag{1.7}$$

which obviously does not cancel the expression (1.5) for $\mathcal{C}^{(0)}$.

The situation with regard to the Maxwell field is discussed in Sec. III. The Maxwell Lagrange function must be supplemented by a gauge-fixing term to make the Green's function well defined. We do this initially with an arbitrary "ξ-gauge" fixing term. Gauge invariance is restored with the addition of an anticommuting, scalar, massless "ghost" field. We prove that the proper-time weight of the one-loop action functional is not changed by a variation of the ξ parameter. We then work entirely in the $\xi=1$ gauge as this simplifies the development. We again have a renormalization of the action functional with the structure exhibited in Eq. (2.45), but with a different counterterm,

$$\mathcal{C}^{(1)} = \frac{1}{(4\pi)^2}\left(-\tfrac{13}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \tfrac{44}{90}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{5}{36}R^2 - \tfrac{1}{10}R_{,\alpha}{}^{;\alpha}\right). \tag{3.51}$$

This can be expressed in terms of the square of the Weyl tensor (1.1) and the topological scalar (1.3),

$$\mathcal{C}^{(1)} = \frac{1}{(4\pi)^2}\left(\tfrac{1}{10}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - \tfrac{31}{180}G - \tfrac{1}{10}R_{,\alpha}{}^{;\alpha}\right). \tag{1.8}$$

Thus, since G and $R_{,\alpha}{}^{;\alpha}$ can be discarded, the counterterm is invariant under a conformal transformation of the metric tensor. The action can be written as the space-time volume integral of an effective Lagrangian, with the Lagrangian expressed in a proper-time representation

$$\mathcal{L}_{1\,\rm ren}^{(n=4)}(x) = -\frac{1}{(4\pi)^2}f_2(x,x) - \frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 [\underline{F}^{\mu}{}_{\mu}(x,x;is) - 2F(x,x;is)] . \tag{3.53}$$

Again, a finite change in the auxiliary scale mass κ can be compensated for by a finite change in the infinite constant that multiplies the counterterm scalar $\mathcal{C}^{(1)}$. The proper-time integral now involves the weight functions in the proper-time representations of the Green's functions for both the vector potential and the scalar "ghost" field. The term involving $f_2(x,x)$ in Eq. (3.53) arises from a derivative with respect to the dimension n acting on the weight $\underline{F}^{\mu}{}_{\mu}(x,x;is)$, a derivative that arises when the residue of the dimensional pole $1/(4-n)$ is expanded in powers of $(4-n)$ and the limit $n\to 4$ is taken. The proper-time weight $\underline{F}^{\mu}{}_{\nu}(x,x;is)$ for the vector field gives an explicit n dependence because it has a term involving $\delta^{\mu}{}_{\nu}$ whose trace produces a factor of n. This is in contrast to the scalar theory where the proper-time weight contains no explicit dependence on the space-time dimensionality n. [The terms in Eqs. (2.47) and (2.52) in addition to the proper-time integrals did not come from a derivative with respect to n. They were separated from the infinite counterterm so that the trace identity (2.63) would be valid without any additional finite renormalization.] We have, explicitly,

$$f_2 = \tfrac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \tfrac{1}{180}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{72}R^2 - \tfrac{1}{30}R_{,\alpha}{}^{;\alpha}, \tag{3.49}$$

or

$$f_2 = \tfrac{1}{120}(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - \tfrac{1}{3}G) + \tfrac{1}{72}R^2 - \tfrac{1}{30}R_{,\alpha}{}^{;\alpha}. \tag{1.9}$$

Although the contribution of f_2 in Eq. (3.53) could be removed by a finite renormalization, this would spoil the conformal invariance of the counterterm.

The renormalized stress-tensor for the Maxwell field satisfies a proper-time representation of the same form as that for the scalar field given in Eq.

(2.52) above. The ghost field contributes an effective total proper-time derivative which could be deleted without altering the renormalized stress tensor. The ghost field contribution to the action functional, however, must be retained. We see that the ghost field plays the role of an integrating factor in constructing the action from its variational derivative, the stress tensor. Moreover, the ghost field contribution to the stress-tensor proper-time weight must be retained if this weight is to be conserved. The trace of the renormalized stress tensor can be computed in a manner parallel to that used in the scalar field case discussed above. We find the anomalous trace identity

$$\langle T^{\mu\nu}(x)\rangle_{\rm ren}^{(n=4)} g_{\mu\nu}(x) = \mathcal{G}^{(1)}(x) - \frac{1}{(4\pi)^2}\,\frac{2}{[-g(x)]^{1/2}} g_{\mu\nu}(x)\,\frac{\delta}{\delta g_{\mu\nu}(x)} \int (d^4x)\sqrt{-g}\, f_2 \,. \qquad (3.69)$$

As remarked before, the metric derivative contribution could be omitted if a finite counterterm were introduced which is not conformally invariant. Using Eqs. (1.9) and (1.7), we can write Eq. (3.69) as

$$\langle T^{\mu\nu}(x)\rangle_{\rm ren}^{(n=4)} g_{\mu\nu}(x) = \mathcal{G}^{(1)}(x) + \frac{1}{(4\pi)^2}\,\tfrac{1}{6} R_{,\mu}{}^{;\mu}(x)\,. \qquad (1.10)$$

Since $\mathcal{G}^{(1)}(x)$ contains terms other than $R_{,\mu}{}^{;\mu}$, it cannot be removed by the addition of a local counterterm.[10]

Appendix A describes the connection of the scalar field stress-tensor weight $\overline{T}^{\mu\nu}(x;is)$ to the Green's function weight $F(x,x;is)$ and discusses the relationship of the weight $\overline{T}^{\mu\nu}(x;is)$ to that used in paper I. Appendix B contains some technical details on the proper-time construction of the vector field Green's function. Appendix C describes the connection of the Maxwell field stress-tensor weight with the vector and "ghost" Green's function weights.

II. SCALAR FIELD

The scalar field Lagrange function

$$\mathcal{L} = -\tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - \tfrac{1}{2}\xi R\phi^2 - \tfrac{1}{2}m^2\phi^2 \qquad (2.1)$$

yields the field equation

$$-\phi_{,\mu}{}^{;\mu} + (\xi R + m^2)\phi = 0 \qquad (2.2)$$

and the stress tensor

$$T^{\mu\nu} = \phi^{,\mu}\phi^{,\nu} - \tfrac{1}{2}g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma} - \tfrac{1}{2}g^{\mu\nu}m^2\phi^2 + \xi[G^{\mu\nu}\phi^2 + g^{\mu\nu}(\phi^2)_{,\sigma}{}^{;\sigma} - (\phi^2)^{,\mu;\nu}]\,, \qquad (2.3)$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R \qquad (2.4)$$

is the Einstein tensor. As shown in paper I, the trace of the stress tensor obeys the formal identity

$$g_{\mu\nu}T^{\mu\nu} = -m^2\phi^2 \qquad (2.5)$$

for a space-time of arbitrary dimensionality n if the parameter ξ is given by

$$\xi = \frac{n-2}{4(n-1)}\,. \qquad (2.6)$$

The parameter ξ was held to the functional dependence on n given by Eq. (2.6) for the most part in paper I so as to maintain the formal trace identity (2.5) throughout the dimensional continuation involved in the renormalization process. However, as was already remarked in I, the dimensional continuation process is simplified if ξ is held fast at its limiting value and not treated as a continuous function of n. With this prescription, the formal trace identity (2.5) is violated except at the limit when n takes on its physical value. Nonetheless, as shown in I, this prescription does yield the proper renormalized stress tensor. In the present work we shall consider only the dimensional continuation to our space-time of $n=4$ and keep

$$\xi = \tfrac{1}{6} \qquad (2.7)$$

fixed appropriate to this dimensionality. In addition to being the simpler method, it is also akin to the case of the Maxwell field where the stress tensor is traceless only at $n=4$.

We shall need the Green's function

$$G(x,x') = \langle iT(\phi(x)\phi(x'))\rangle\,, \qquad (2.8)$$

which obeys

$$[-\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu + \sqrt{-g}\,(\tfrac{1}{6}R + m^2)]\,G(x,x') = \delta(x-x')\,, \qquad (2.9)$$

or, in an operator notation,

$$G^{-1}G = 1\,. \qquad (2.10)$$

It is convenient to introduce

$$\tilde{G}(x,x') = [-g(x)]^{1/4}G(x,x')[-g(x')]^{1/4} \qquad (2.11)$$

because it is a biscalar density which, under a coordinate transformation, transforms in the same way as does the scalar product

$$\langle x\,|\,x'\rangle = \delta(x-x')\,. \qquad (2.12)$$

Now, on going to the operator notation

$$\tilde{G}(x,x') = \langle x\,|\,\tilde{G}\,|\,x'\rangle\,, \qquad (2.13)$$

we have

$$H\tilde{G}=1\,, \tag{2.14}$$

where the inverse operator H has the coordinate representation

$$H=-(-g)^{-1/4}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu(-g)^{1/4}+\tfrac{1}{6}R+m^2\,. \tag{2.15}$$

The proper-time representation is obtained by writing

$$\tilde{G}=H^{-1}=\int_0^\infty ids\, e^{-isH}, \tag{2.16}$$

so that

$$\tilde{G}(x,x')=\int_0^\infty ids\,\langle x,s\,|\,x',0\rangle\,, \tag{2.17}$$

where the proper-time–dependent transformation function is given by

$$\langle x,s\,|\,x',0\rangle=\langle x\,|\,e^{-isH}|\,x'\rangle\,. \tag{2.18}$$

This representation tacitly assumes that the real mass parameter m^2 is considered as the limit of a complex mass, $m^2\to m^2(1-i\epsilon)$, $\epsilon\to 0^+$ or (as is needed in the zero-mass case) that an analytic continuation can be performed so that the integrand in Eq. (2.17) vanishes rapidly as $s\to\infty$. The representation yields a vacuum expectation value of the time-ordered product of two scalar field operators. If the metric tensor can be expanded about the Minkowski metric with space-time asymptotically flat, it yields the value in the vacuum state characterized by vanishing asymptotic energy and momentum.

The transformation function (2.18) obeys the "Schrödinger" equation

$$-\frac{\partial}{\partial is}\langle x,s\,|\,x',0\rangle=H\langle x,s\,|\,x',0\rangle\,, \tag{2.19}$$

with the boundary condition

$$s\to 0:\ \langle x,s\,|\,x',0\rangle\to\langle x\,|\,x'\rangle=\delta(x-x')\,. \tag{2.20}$$

For the construction of the stress tensor and action functional by means of the dimensional-continuation method, we will need the short-distance limit of the transformation function for arbitrary space-time dimensionality n, with its analytic character in the proper time s displayed explicitly for small s. These requirements are met by the WKB construction

$$\langle x,s\,|\,x',0\rangle=\frac{i}{(4\pi is)^{n/2}}[-g(x)]^{1/4}\Delta^{1/2}(x,x')[-g(x')]^{1/4}F(x,x';is)\exp\left[-\frac{\sigma(x,x')}{2is}-m^2is\right]. \tag{2.21}$$

All the functions which appear here are symmetrical under the interchange of x and x'. The biscalar $\sigma(x,x')$, the "world function," is equal to one-half of the square of the distance along the geodesic between x and x'. It may be defined locally by a differential equation of the Hamilton-Jacobi form

$$\sigma_{,\mu}\sigma^{,\mu}=2\sigma\,, \tag{2.22}$$

and the coincident coordinate boundary conditions

$$x=x':\ \sigma=0=\sigma_{,\mu}=\sigma_{,\mu'} \tag{2.23}$$

and

$$x=x':\ \sigma_{,\mu;\nu}=g_{\mu\nu}=-\sigma_{,\mu,\nu'}\,. \tag{2.24}$$

Here we use a suffix to denote a derivative with respect to the variable x, and a primed suffix to denote a derivative with respect to x'. Note that $\sigma(x,x')$ has no explicit dependence on the space-time dimensionality n. The biscalar $\Delta^{1/2}(x,x')$ may be defined in terms of the Van Vleck determinant

$$[-g(x)]^{1/2}\Delta(x,x')[-g(x')]^{1/2}=-\det[-\sigma_{,\mu,\mu'}(x,x')]\,, \tag{2.25}$$

which shows that it also carries no explicit dependence on the dimensionality n. Using Eq. (2.24), we see that this defintion implies the coincident coordinate limit

$$\Delta^{1/2}(x,x)=1\,. \tag{2.26}$$

To derive the differential equation obeyed by $\Delta^{1/2}$, we first differentiate the "Hamilton-Jacobi" equation for the world function with respect to x and x':

$$\sigma_{,\alpha,\beta'}=(\sigma_{,\mu}g^{\mu\nu})_{,\alpha}\sigma_{,\nu,\beta'}+\sigma_{,\mu}g^{\mu\nu}\partial_\nu\sigma_{,\alpha,\beta'}\,. \tag{2.27}$$

We then contract this equation with the inverse of the matrix $\sigma_{,\alpha,\beta'}$ and use the variational formula

$$\delta\ln\det X=\operatorname{tr}X^{-1}\delta X \tag{2.28}$$

to secure

$$n=(\sigma_{,\mu}g^{\mu\nu})_{,\nu}+\sigma_{,\mu}g^{\mu\nu}\partial_\nu\ln[-\det(-\sigma_{,\alpha,\beta'})]\,. \tag{2.29}$$

Writing the determinant scalar density in terms of the biscalar $\Delta^{1/2}$ enables the ordinary derivatives above to combine into covariant derivatives, and we get

$$n\Delta^{1/2}=\Delta^{1/2}\sigma_{,\mu}{}^{;\mu}+2\Delta^{1/2}{}_{,\mu}\sigma^{,\mu}\,. \tag{2.30}$$

The biscalar $\Delta^{1/2}$ could have been defined simply as the solution to this differential equation with the boundary condition given by Eq. (2.26). This would, however, have obscured the fact that $\Delta^{1/2}$ contains no explicit n dependence. We can now substitute the WKB structure (2.21) into the Schrödinger equation (2.19) and use the differential equations (2.22) and (2.30) obeyed by σ and $\Delta^{1/2}$ to derive the weight function equation

$$-\frac{\partial F}{\partial is}=\tfrac{1}{6}RF+\frac{1}{is}\sigma^{,\mu}F_{,\mu}-\frac{1}{\Delta^{1/2}}(\Delta^{1/2}F)_{,\mu}{}^{;\mu}. \tag{2.31}$$

The factor $i(4\pi is)^{-n/2}$ in the WKB construction (2.21) has been chosen so that it yields a representation of the δ-function boundary condition (2.20) for small s or, equivalently, by the condition that the Green's function approach the flat-space Green's function in a short-distance limit taken in a locally flat frame as done in paper I. This determines $F(x,x;0)=1$. The $s\to 0$ limit of Eq. (2.31) requires that $F(x,x';0)$ is a constant. Hence

$$F(x,x';0)=1. \tag{2.32}$$

The differential equation (2.31) and the boundary condition (2.32) for the weight F make no reference to the space-time dimensionality n, and F has no explicit dependence on n. The only explicit dependence on n in the WKB construction (2.21) appears in the overall factor of $(4\pi is)^{-n/2}$.

The one-loop action functional $W_1[g_{\alpha\beta}]$ has the formal (divergent) definition

$$W_1=\tfrac{1}{2}i\ln \mathrm{Det}\, G^{-1}, \tag{2.33}$$

or the variational equivalent

$$\delta W_1=\tfrac{1}{2}i\,\mathrm{Tr}G\delta G^{-1}, \tag{2.34}$$

with the boundary condition that W_1 vanish in flat space-time. Recalling the definitions (2.11) and (2.14) of $\tilde{G}$ and its inverse H, and using the proper-time representation (2.16) gives

$$\delta W_1=\tfrac{1}{2}i\int_0^\infty ids\;\mathrm{Tr}e^{-isH}\{\delta H+\tfrac{1}{4}[\delta\ln(-g)]H+H\tfrac{1}{4}[\delta\ln(-g)]\}. \tag{2.35}$$

Using the cyclic symmetry of the trace, the two terms involving the variation of the determinant of the metric tensor g can be expressed as the proper-time integral of a total derivative, an integral that vanishes in the dimensional-regularization scheme:

$$-\tfrac{1}{2}i\int_0^\infty ids\,\frac{\partial}{\partial is}\mathrm{Tr}e^{-isH}\tfrac{1}{2}[\delta\ln(-g)]=0. \tag{2.36}$$

[A formal evaluation of these terms would give $\sim \mathrm{Tr}\tilde{G}H\delta\ln(-g)\sim\delta^{(n)}(0)\int(d^nx)\,\delta\ln(-g)$ which would be deleted by a partial renormalization.] The variational relation (2.35) for the one-loop action functional can be integrated to give the formal expression

$$W_1=-\tfrac{1}{2}i\int_0^\infty\frac{ids}{is}\mathrm{Tr}e^{-isH}, \tag{2.37}$$

or

$$W_1=\int(d^nx)\sqrt{-g}\,\mathcal{L}_1. \tag{2.38}$$

where the one-loop effective Lagrangian $\mathcal{L}_1$ is given by

$$\mathcal{L}_1=-\frac{i}{2\sqrt{-g}}\int_0^\infty\frac{ids}{is}\langle x|e^{-isH}|x\rangle=\frac{1}{2}\frac{1}{(4\pi)^{n/2}}\int_0^\infty\frac{ids}{(is)^{1+n/2}}e^{-m^2is}F(x,x;is), \tag{2.39}$$

with the last equality following from the WKB construction (2.21).

The renormalized action is obtained by a continuation in the dimension n. This is accomplished by exhibiting the explicit n dependence in the proper-time representation,

$$\mathrm{Tr}e^{-isH}=i(4\pi is)^{-n/2}w(is), \tag{2.40}$$

and by observing that the weight $w(s)$ is finite and differentiable at $s=0$. We begin the dimensional continuation to $n=4$ from sufficiently small values of n so that three integrations by parts can be performed with no contribution from the $s=0$ end point,

$$W_1=\frac{1}{2}\frac{1}{(4\pi)^{n/2}}\frac{2}{n}\frac{1}{\frac{1}{2}n-1}\frac{1}{\frac{1}{2}n-2}\times\int_0^\infty\frac{ids}{(is)^{n/2-2}}\left(\frac{\partial}{\partial is}\right)^3 w(is). \tag{2.41}$$

We now introduce an arbitrary, auxiliary scale mass κ so that the weight $w(is)$ keeps a fixed scale dimension appropriate to $n=4$. This is effected by the replacement

$$\frac{1}{(is)^{n/2-2}}\to\frac{1}{(\kappa^2 is)^{n/2-2}}\simeq 1-(\tfrac{1}{2}n-2)\ln\kappa^2 is \tag{2.42}$$

in the integrand above. After integrating the total derivative contribution, we can pass to the $n\to 4$ limit and obtain

$n\to 4$:

$$W_1^{(n=4)}=\left(\frac{1}{4-n}+L_4+\tfrac{1}{4}\right)\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2 w(is)\Big|_{s=0}-\frac{1}{2}\frac{1}{(4\pi)^2}\frac{\partial}{\partial n}\left[\left(\frac{\partial}{\partial is}\right)^2 w(is)\Big|_{s=0}\right]\Big|_{n=4}-\frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 w(is), \tag{2.43}$$

where

$$L_4=\tfrac{1}{2}+\frac{\partial}{\partial n}\ln(4\pi)^{n/2}\Big|_{n=4}\,. \tag{2.44}$$

The term with the factor $[1/(4-n)+L_4+\frac{1}{4}]$ diverges in the $n\to 4$ limit. As we shall soon see, this term involves the space-time volume integral of quantities that depend locally upon the metric tensor. Hence this infinite term can be discarded if an infinite but local counterterm is added to the Lagrange function, and a finite, renormalized effective action is secured. Note that a change in the arbitrary, auxiliary scale mass $\kappa\to\kappa'$ is accomodated by a finite change in the infinite counterterm involving $L_4\to L_4+\ln(\kappa'/\kappa)$. The weight $w(is)$ has no explicit dependence on the dimensionality n and the dimensional derivative term in Eq. (2.43) vanishes. We have displayed this potential contribution in Eq. (2.43) so as to record a result of general validity which will be needed below.

We can write the action W_1 as the space-time volume integral (2.38) of the effective Lagrangian $\mathcal{L}_1$, and write the Lagrangian $\mathcal{L}_1$ in terms [Eq. (2.39)] of the proper-time weight $F(x,x';is)$ of the Green's function. As shown in paper I, the weight $F(x,x';is)$ can be expanded in a double power series in $(x-x')$ and s, with the coefficients determined by the differential equation (2.31) obeyed by $F(x,x';is)$, to determine

$$\begin{aligned}\mathcal{a}^{(0)}&=\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2[e^{-m^2 is}F(x,x;is)]\Big|_{s=0}\\&=\frac{1}{(4\pi)^2}[\tfrac{1}{180}(R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}-R_{\mu\nu}R^{\mu\nu}+R_{,\mu}{}^{;\mu})+\tfrac{1}{2}m^4]\,.\end{aligned} \tag{I.1.24}$$

Thus the dimensional-continuation limit (2.43) can be written as

$$W_1^{(n=4)}=\left(\frac{1}{4-n}+L_4\right)\int(d^4x)\sqrt{-g}\,\mathcal{a}^{(0)}+W_{1\,\text{ren}}^{(n=4)}, \tag{2.45}$$

where the renormalized one-loop action is given by

$$W_{1\,\text{ren}}^{(n=4)}=\int(d^4x)\sqrt{-g}\,\mathcal{L}_{1\,\text{ren}}^{(n=4)}, \tag{2.46}$$

with

$$\mathcal{L}_{1\,\text{ren}}^{(n=4)}=\tfrac{1}{4}\mathcal{a}^{(0)}-\frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3\times[e^{-m^2 is}F(x,x;is)]\,. \tag{2.47}$$

Here a term $\frac{1}{4}\mathcal{a}^{(0)}$ has been placed in the renormalized Lagrangian which could have been included in the infinite counterterm. We have made this separation so as to make the numerical factor $[1/(n-4)+L_4]$ in the counterterm correspond to that which appears in the renormalization of a field with a scale dimension of mass squared (where the counterterm involves not the second but the first proper-time derivative of a weight function). We have done this so that the trace of the counterterm is precisely $-m^2$ times the counterterm of the vacuum expectation value of ϕ^2—so that the formal trace identity (2.5) holds for the two counterterms.

We turn now to the renormalization of the stress-energy tensor. Since the stress tensor is formally defined in terms of the metric variation of the field action, which can be written as the operator scalar product

$$\int(d^nx)\sqrt{-g}\,\mathcal{L}=-\tfrac{1}{2}(\phi,G^{-1}\phi)\,, \tag{2.48}$$

we have the formal identities

$$\begin{aligned}[-g(x)]^{1/2}\langle T^{\mu\nu}(x)\rangle&=-\left\langle\left(\phi,\frac{\delta G^{-1}}{\delta g_{\mu\nu}(x)}\phi\right)\right\rangle\\&=i\,\mathrm{Tr}G\frac{\delta G^{-1}}{\delta g_{\mu\nu}(x)}\\&=2\frac{\delta}{\delta g_{\mu\nu}(x)}W_1\,.\end{aligned} \tag{2.49}$$

Accordingly, if we define

$$[-g(x)]^{1/2}\,\overline{T}^{\mu\nu}(x;is)=2\frac{\delta}{\delta g_{\mu\nu}(x)}w(is)\,, \tag{2.50}$$

the dimensional-continuation limit (2.43) yields immediately the proper-time representation of the renormalized stress-energy tensor,

$$\langle T^{\mu\nu}(x)\rangle^{(n=4)}=\left(\frac{1}{4-n}+L_4\right)\frac{1}{(4\pi)^2}\frac{1}{2}\left(\frac{\partial}{\partial is}\right)^2\overline{T}^{\mu\nu}(x;is)\Big|_{s=0}+\langle T^{\mu\nu}(x)\rangle_{\text{ren}}^{(n=4)}\,, \tag{2.51}$$

with

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle_{\text{ren}}^{(n=4)}=&\frac{1}{8}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2\overline{T}^{\mu\nu}(x;is)\Big|_{s=0}\\&-\frac{1}{4}\frac{1}{(4\pi)^2}\\&\times\int_0^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3\overline{T}^{\mu\nu}(x;is)\,.\end{aligned} \tag{2.52}$$

This exhibits the renormalized stress tensor as the metric tensor variational derivative of the renormalized one-loop action. Since the weight $w(is)$ is a functional of the metric tensor $g_{\alpha\beta}(x)$, which is invariant under general coordinate transformations, the definition (2.50) implies that the stress-tensor proper-time weight is conserved,

$$\overline{T}^{\mu\nu}(x;is)_{;\nu}=0\,. \tag{2.53}$$

Hence, the proper-time representation (2.52) yields a conserved renormalized stress tensor.

To compute the trace of the stress tensor, we note that with $n=4$, the operator H, which has the coordinate representation

$$H=-(-g)^{1/4}\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu(-g)^{-1/4}+\tfrac{1}{6}R+m^2, \tag{2.15}$$

transforms simply under a space-time-dependent conformal transformation of the metric tensor,

$$g_{\alpha\beta}(x)\rightarrow\lambda(x)^2 g_{\alpha\beta}(x). \tag{2.54}$$

The terms involving derivatives of λ arising from the gradient operators in Eq. (2.15) are canceled by terms involving derivatives of λ which are produced by the transformation of $\frac{1}{6}R$, and we have the operator transformation law

$$H[\lambda^2 g_{\alpha\beta}]=\lambda^{-1}H[g_{\alpha\beta}]\lambda^{-1}-m^2(\lambda^{-2}-1). \tag{2.55}$$

Now

$$[-g(x)]^{1/2}\,\overline{T}^{\mu\nu}(x;is)=2\frac{\delta}{\delta g_{\mu\nu}(x)}w(is), \tag{2.50}$$

with

$$\mathrm{Tr}e^{-isH}=i(4\pi is)^{-2}w(is), \tag{2.40}$$

which give

$$[-g(x)]^{1/2}\,\overline{T}^{\mu\nu}(x;is)=i(4\pi is)^2 is\,\mathrm{Tr}e^{-isH}\,2\frac{\delta H}{\delta g_{\mu\nu}(x)}. \tag{2.56}$$

Hence

$$\int(d^4x)\,[-g(x)]^{1/2}\,\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)\,\delta\lambda(x)=-i(4\pi is)^2 is\,\mathrm{Tr}e^{-isH}(\delta\lambda H+H\delta\lambda-m^2 2\delta\lambda). \tag{2.57}$$

The cyclic symmetry of the trace enables the first factor of H to be placed adjacent to e^{-isH} so that the two terms which involve a factor of H can be expressed in terms of a proper-time derivative acting on e^{-isH}. We can now make use of the WKB construction exhibited in Eqs. (2.18) and (2.21) to secure

$$\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=-2(is)^2 is\frac{\partial}{\partial is}(is)^{-2}e^{-m^2 is}F(x,x;is)-2m^2 is\,e^{-m^2 is}F(x,x;is). \tag{2.58}$$

This gives

$$\left(\frac{\partial}{\partial is}\right)^2\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)\bigg|_{s=0}=-4m^2\frac{\partial}{\partial is}e^{-m^2 is}F(x,x;is)\bigg|_{s=0} \tag{2.59}$$

and

$$\left(\frac{\partial}{\partial is}\right)^3\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=-2\frac{\partial}{\partial is}is\frac{\partial}{\partial is}\left[\left(\frac{\partial}{\partial is}\right)^2+m^2\frac{\partial}{\partial is}\right]e^{-m^2 is}F(x,x;is)-4m^2\left(\frac{\partial}{\partial is}\right)^2 e^{-m^2 is}F(x,x;is). \tag{2.60}$$

Using these results and the integral

$$\int_0^\infty ids\,(\ln\kappa^2 is)\frac{\partial}{\partial is}is\frac{\partial}{\partial is}f(is)=f(0), \tag{2.61}$$

the trace of the renormalized stress-tensor proper-time representation (2.52) is now easily calculated:

$$\langle T^{\mu\nu}(x)\rangle_{\text{ren}}^{(n=4)}g_{\mu\nu}(x)=\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2 e^{-m^2 is}F(x,x;is)\bigg|_{s=0}+m^2\frac{1}{(4\pi)^2}\int_0^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^2 e^{-m^2 is}F(x,x;is). \tag{2.62}$$

The first quantity which appears here is precisely the scalar $\mathcal{a}^{(0)}$ displayed in Eq. (I.1.24) which occurs in the infinite counterterm needed to renormalize the one-loop action, Eq. (2.45). The remaining integral is easily seen to be the dimensionally continued, proper-time representation for the renormalized vacuum expectation value of the square of the scalar field. Thus

$$\langle T^{\mu\nu}(x)\rangle_{\text{ren}}^{(n=4)}g_{\mu\nu}(x)=\mathcal{a}^{(0)}(x)-m^2\langle\phi^2(x)\rangle_{\text{ren}}^{(n=4)}. \tag{2.63}$$

This is the trace anomaly: The trace of the renormalized stress tensor does not obey the formal identity (2.5) which would delete the anomalous term $\mathcal{a}^{(0)}(x)$. The anomaly $\mathcal{a}^{(0)}(x)$ cannot be removed by putting a local counterterm into the Lagrange function. [It should be mentioned that the trace of the infinite counterterm needed to renormalize the stress tensor given in Eq. (2.51) is easily shown to be exactly the infinite counterterm which renormalizes $\langle\phi^2\rangle$.]

The stress-tensor weight $\overline{T}^{\mu\nu}(x;is)$ can be expressed in terms of the Green's function weight $F(x,x';is)$. This is done in Appendix A. The weight $\overline{T}^{\mu\nu}(x;is)$ used in the present work differs from the weight $T^{\mu\nu}(x;is)$ used in paper I by a factor of $e^{-m^2 is}$ and, more importantly, by the addition of a quantity that is effectively a total proper-time derivative. This effective total derivative does not alter $\langle T^{\mu\nu}\rangle$ but interchanges the roles of divergence and trace in the two weights. Thus, the divergence of the new weight $\overline{T}^{\mu\nu}(x;is)$ vanishes and its trace for the massless theory is an effective total derivative, while the trace of the old weight vanishes for the massless theory and its divergence is an effective total derivative.

III. MAXWELL FIELD

We turn now to apply our proper-time, dimensional-continuation method to the one-loop action functional and stress-tensor vacuum expectation value of a massless vector field, the electromagnetic Maxwell field. A term that fixes the gauge must be included in the Lagrange function so as to make the Green's function, and the one-loop action which involves the determinant of this Green's function, nonsingular. We shall initially use an arbitrary "ξ-gauge-fixing term." (Since the letter ξ has already been used for an important parameter, we shall denote the gauge-fixing parameter by ζ.) Gauge invariance is restored by adjoining a massless ghost field to the Lagrange function,

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\zeta}(A^{\mu}{}_{;\mu})^2 - \chi^{\dagger}_{,\mu}\chi^{,\mu}. \tag{3.1}$$

Here χ is the non-Hermitian, anticommuting, scalar ghost field and

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu,\nu} - A_{\nu,\mu}. \tag{3.2}$$

The Lagrange function yields the field equations

$$F^{\mu\nu}{}_{;\nu} + \frac{1}{\zeta}A^{\nu}{}_{;\nu}{}^{;\mu} = 0, \tag{3.3}$$

$$\chi_{,\mu}{}^{;\mu} = 0, \tag{3.4}$$

and the stress-energy tensor

$$\begin{aligned} T^{\mu\nu} = {} & F^{\mu}{}_{\lambda}F^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ & + \frac{1}{\zeta}\{g^{\mu\nu}[\tfrac{1}{2}(A^{\lambda}{}_{;\lambda})^2 + A^{\lambda}{}_{;\lambda,\kappa}A^{\kappa}] \\ & - A^{\lambda}{}_{;\lambda}{}^{,\mu}A^{\nu} - A^{\lambda}{}_{;\lambda}{}^{,\nu}A^{\mu}\} \\ & + (\chi^{\dagger,\mu}\chi^{,\nu} + \chi^{\dagger,\nu}\chi^{,\mu} - g^{\mu\nu}\chi^{\dagger}_{,\lambda}\chi^{,\lambda}). \end{aligned} \tag{3.5}$$

We shall need the Green's functions

$$\begin{aligned} {}^{(\zeta)}\underline{G}_{\mu\mu'}(x,x') &= \langle iT(A_{\mu}(x)A_{\mu'}(x'))\rangle \\ &= [-g(x)]^{-1/4}\,{}^{(\zeta)}\underline{\tilde{G}}_{\mu\mu'}(x,x')[-g(x')]^{-1/4} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} G(x,x') &= \langle iT(\chi(x)\chi^{\dagger}(x'))\rangle \\ &= [-g(x)]^{-1/4}\tilde{G}(x,x')[-g(x')]^{-1/4}, \end{aligned} \tag{3.7}$$

which we shall express in terms of operators

$${}^{(\zeta)}\underline{\tilde{G}}^{\mu}{}_{\mu'}(x,x') = \langle x,\mu|\,{}^{(\zeta)}\underline{\tilde{G}}\,|x',\mu'\rangle \tag{3.8}$$

and

$$\tilde{G}(x,x') = \langle x|\tilde{G}|x'\rangle. \tag{3.9}$$

Here we have used a suffix (ζ) in order to indicate explicitly that the vector-field Green's function is calculated in a gauge fixed by the ζ parameter. Note that the vector-field Green's function is written as a mixed tensor with one upper and one lower index so that contractions with these indices can be treated as ordinary matrix multiplication. The inhomogeneous Green's function equations now appear as the simple operator statements

$${}^{(\zeta)}\underline{H}\,{}^{(\zeta)}\underline{\tilde{G}} = \underline{1} \tag{3.10}$$

and

$$H\tilde{G} = 1. \tag{3.11}$$

Here of course

$$\langle x|1|x'\rangle = \langle x|x'\rangle = \delta(x-x'), \tag{3.12}$$

while with our index convention

$$\begin{aligned} \langle x,\mu|\underline{1}|x',\mu'\rangle &= \langle x,\mu|x',\mu'\rangle \\ &= \delta^{\mu}{}_{\mu'}\delta(x-x). \end{aligned} \tag{3.13}$$

The differential operator representations of the operators ${}^{(\zeta)}\underline{H}$ and H are obtained easily by writing out the Lagrange function (3.1) in terms of ordinary derivatives and then identifying the field equations:

$$\begin{aligned} {}^{(\zeta)}\underline{H}^{\mu}{}_{\nu} = {} & -(-g)^{-1/4}\partial_{\alpha}(g^{\mu\lambda}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\lambda}) \\ & \times(-g)^{1/2}\partial_{\beta}(-g)^{-1/4}g_{\lambda\nu} \\ & - \frac{1}{\zeta}(-g)^{1/4}g^{\mu\lambda}\partial_{\lambda}(-g)^{-1/2}\partial_{\nu}(-g)^{1/4}, \end{aligned} \tag{3.14}$$

$$H = -(-g)^{-1/4}\partial_{\alpha}(-g)^{1/2}g^{\alpha\beta}\partial_{\beta}(-g)^{-1/4}. \tag{3.15}$$

Since $(g^{\mu\lambda}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\lambda})$ is antisymmetrical in both the index pairs (μ,α) and (λ,β), we have the divergence conditions

$$(-g)^{-1/4}\partial_{\mu}(-g)^{1/4}\,{}^{(\zeta)}\underline{H}^{\mu}{}_{\nu} = \frac{1}{\zeta}H(-g)^{-1/4}\partial_{\nu}(-g)^{1/4} \tag{3.16a}$$

and

$$ {}^{(\zeta)}\underline{H}^{\mu}{}_{\nu} g^{\nu\lambda}(-g)^{1/4}\partial_\lambda(-g)^{-1/4} = \frac{1}{\zeta} g^{\mu\lambda}(-g)^{1/4} \times \partial_\lambda(-g)^{-1/4} H . \tag{3.17a} $$

We introduce a symbol D which is represented with the appropriate factors of $(-g)^{\pm 1/4}$ and $g^{\mu\nu}$ according to where it appears so that these divergence conditions can be written in a simple operator form

$$ D\, {}^{(\zeta)}\underline{H} = \frac{1}{\zeta} HD \tag{3.16b} $$

and

$$ {}^{(\zeta)}\underline{H} D = \frac{1}{\zeta} DH . \tag{3.17b} $$

The dependence of ${}^{(\zeta)}\underline{H}$ on the gauge-fixing parameter ζ is also simply expressed in terms of these D symbols. The ζ variation of Eq. (3.14) gives, in operator notation,

$$ \delta\, {}^{(\zeta)}\underline{H} = \frac{\delta\zeta}{\zeta^2} DD , \tag{3.18} $$

where the D which stands on the left in this equation is identical to the D operation which appears on the left-hand side of Eq. (3.17b), while the D which stands on the right in this equation is identical to the D operation which appears on the left-hand side of Eq. (3.16b). Thus, for example,

$$ {}^{(\zeta)}\underline{H}\,\delta\, {}^{(\zeta)}\underline{H} = \frac{\delta\zeta}{\zeta^3} DHD , \tag{3.19} $$

where the D symbols on the right-hand side of this equation are represented by the quantities on the right-hand sides of Eqs. (3.16a) and (3.17a). We shall need a final property of the D symbols. If we have the operator trace (including index summation) of an expression of the type illustrated in Eq. (3.19), we can move the D symbol on the right to the extreme left by using the cyclic symmetry of the trace, and the index summation contained in the trace then produces

$$ D^2 = -H . \tag{3.20} $$

With these notational developments in hand, we can now turn to discuss the one-loop action functional for the Maxwell field system which has the formal representation

$$ W_1 = \tfrac{1}{2} i \ln \mathrm{Det}\, {}^{(\zeta)}\underline{G}^{-1} - i \ln \mathrm{Det}\, G^{-1} . \tag{3.21} $$

Here the ghost contribution appears with a relative factor of 2 and the opposite sign because the non-Hermitian ghost field is equivalent to two Hermitian, anticommuting fields. A metric variation gives

$$ \delta W_1 = \tfrac{1}{2} i \,\mathrm{Tr}\, {}^{(\zeta)}\underline{G}\,\delta\, {}^{(\zeta)}\underline{G}^{-1} - i\, \mathrm{Tr} G\delta G^{-1} , \tag{3.22} $$

where the first trace includes a diagonal sum over tensor indices as well as the diagonal coordinate integration. We proceed exactly as in the development used in the scalar field case, Eqs. (2.34) to (2.37). We write the Green's functions $\underline{G}$ and G in terms of the operators $\underline{\tilde{G}}$ and $\tilde{G}$ and then represent these operators by an exponential proper-time integral. Since

$$ G^{-1} = (-g)^{1/4} H (-g)^{1/4} \tag{3.23} $$

and

$$ ({}^{(\zeta)}\underline{G}^{-1})^{\mu\mu'} = (-g)^{1/4}\, {}^{(\zeta)}\underline{H}^{\mu}{}_{\nu} g^{\nu\mu'} (-g)^{1/4} , \tag{3.24} $$

the variations $\delta\, {}^{(\zeta)}\underline{G}^{-1}$ and δG^{-1} involve variations of $(-g)^{1/4}$ and $g^{\nu\mu'}$ in addition to $\delta\, {}^{(\zeta)}\underline{H}$ and δH. However, the variations of $(-g)^{1/4}$ and $g^{\nu\mu'}$ appear as proper-time total derivatives [cf. Eq. (2.36)] and they give a vanishing contribution in the dimensional-regularization scheme. The proper-time representation of the variational formula (3.22) may now be integrated to get the formal expression

$$ W_1 = -\tfrac{1}{2} i \int_0^\infty \frac{i ds}{is} \mathrm{Tr} e^{-is\, {}^{(\zeta)}\underline{H}} + i \int_0^\infty \frac{i ds}{is} \mathrm{Tr} e^{-isH} . \tag{3.25} $$

We can now show that the action is independent of the gauge-fixing parameter ζ. Using Eq. (3.18) we have

$$ \delta W_1 = \frac{\delta\zeta}{\zeta^2} \tfrac{1}{2} i \int_0^\infty i ds\ \mathrm{Tr} e^{-is\, {}^{(\zeta)}\underline{H}} DD . \tag{3.26} $$

Expanding the exponential in a power series and using Eq. (3.17b) repeatedly gives

$$ e^{-is\, {}^{(\zeta)}\underline{H}} D = D e^{-i(s/\zeta) H} . \tag{3.27} $$

The cyclic symmetry of the trace can now be exploited to place the D symbol on the extreme right-hand side in Eq. (3.26) over to the left where, in view of Eq. (3.27), it becomes contracted with the other D symbol to produce $-H$ [Eq. (3.20)]. Thus

$$ \begin{aligned} \delta W_1 &= -\frac{\delta\zeta}{\zeta} \tfrac{1}{2} i \int_0^\infty i d(s/\zeta)\ \mathrm{Tr} H e^{-i(s/\zeta)H} \\ &= \frac{\delta\zeta}{\zeta} \tfrac{1}{2} i \int_0^\infty i ds' \frac{\partial}{\partial i s'} \mathrm{Tr} e^{-is'H} \\ &= 0 . \end{aligned} \tag{3.28} $$

Since we have now proved that the action is independent of ζ, we shall henceforth restrict our development to case $\zeta = 1$ which will simplify our work, and we shall omit the suffix (ζ). With $\zeta = 1$, two of the derivatives in the operator $\underline{H}$ displayed in Eq. (3.14) cancel one another except for the order in which they operate. Hence, with $\xi = 1$, $\underline{H}$ has the coordinate representation

$$\underline{H}^{\mu}{}_{\nu} = -(-g)^{-1/4}\partial_{\alpha} g^{\mu\lambda} g^{\alpha\beta}(-g)^{1/2}\partial_{\beta}(-g)^{-1/4} g_{\lambda\nu} + R^{\mu}{}_{\nu}\,. \tag{3.29}$$

To proceed with our renormalization method, we need WKB constructions for the vector and scalar ghost field proper-time transformation functions. The ghost field transformation function is essentially identical to that described in Eqs. (2.18) through (2.32) in the preceding section except that the mass m should be taken to vanish and the scalar curvature term $\frac{1}{6}R$ should be omitted from Eq. (2.31). The vector-field transformation function

$$\langle x, \mu, s \,|\, x', \mu', 0\rangle = \langle x, \mu \,|\, e^{-is\underline{H}} \,|\, x', \mu'\rangle \tag{3.30}$$

obeys the Schrödinger equation

$$-\frac{\partial}{\partial is}\langle x, \mu, s \,|\, x', \mu', 0\rangle = \underline{H}^{\mu}{}_{\nu}\langle x, \nu, s \,|\, x', \mu', 0\rangle \tag{3.31}$$

and has the WKB construction

$$\langle x, \mu, s \,|\, x', \mu', 0\rangle = \frac{i}{(4\pi i s)^{n/2}}[-g(x)]^{1/4}\,\Delta^{1/2}(x,x')[-g(x')]^{1/4}\underline{F}^{\mu}{}_{\mu'}(x,x';is)\exp\left[-\frac{\sigma(x,x')}{2is}\right]. \tag{3.32}$$

Inserting this construction into the Schrödinger equation (3.31) with $\underline{H}^{\mu}{}_{\nu}$ given by Eq. (3.29) yields, on taking account of Eqs. (2.22) and (2.30), the weight equation

$$-\frac{\partial}{\partial is}\underline{F}^{\mu}{}_{\mu'} = R^{\mu}{}_{\nu}\underline{F}^{\nu}{}_{\mu'} + \frac{1}{is}\sigma^{;\lambda}\underline{F}^{\mu}{}_{\mu';\lambda} - \frac{1}{\Delta^{1/2}}(\Delta^{1/2}\underline{F}^{\mu}{}_{\mu'})_{;\lambda}{}^{;\lambda}\,. \tag{3.33}$$

The weight function $\underline{F}^{\mu}{}_{\mu'}(x,x';is)$ is regular at $s=0$ with the coincident coordinate limit

$$\underline{F}^{\mu}{}_{\mu'}(x,x;0) = \delta^{\mu}{}_{\mu'} \tag{3.34}$$

required by the boundary condition

$$s\to 0:\ \ \langle x, \mu, s \,|\, x', \mu', 0\rangle \to \delta^{\mu}{}_{\mu'}\delta(x-x')\,. \tag{3.35}$$

Hence, the $s=0$ limit of the weight equation (3.33) requires that

$$\sigma^{;\lambda}(x,x')\underline{F}^{\mu}{}_{\mu'}(x,x';0)_{;\lambda} = 0\,, \tag{3.36}$$

which together with the coincident coordinate limit (3.34) identifies

$$\underline{F}^{\mu}{}_{\mu'}(x,x';0) = \delta^{\mu}{}_{\mu'}(x,x')\,, \tag{3.37}$$

where[11] $\delta^{\mu}{}_{\mu'}(x,x')$ is the parallel displacement bivector along the geodesic defined by the world function $\sigma(x,x')$. It is convenient to write

$$\underline{F}^{\mu}{}_{\mu'}(x,x';is) = \delta^{\mu}{}_{\mu'}(x,x')F(x,x';is) + \underline{\bar{F}}^{\mu}{}_{\mu'}(x,x';is)\,, \tag{3.38}$$

where $F(x,x';is)$ is the scalar field weight defined by Eq. (2.31) with the $\frac{1}{6}R$ term deleted,

$$-\frac{\partial F}{\partial is} = \frac{1}{is}\sigma^{,\lambda}F_{,\lambda} - \frac{1}{\Delta^{1/2}}(\Delta^{1/2}F)_{,\lambda}{}^{;\lambda}\,, \tag{3.39}$$

with the boundary condition

$$F(x,x';0) = 1\,. \tag{3.40}$$

Substituting the decomposition (3.38) into the weight equation (3.33) gives

$$-\frac{\partial}{\partial is}\underline{\bar{F}}^{\mu}{}_{\mu'} = R^{\mu}{}_{\nu}\underline{\bar{F}}^{\nu}{}_{\mu'} + \frac{1}{is}\sigma^{;\lambda}\underline{\bar{F}}^{\mu}{}_{\mu';\lambda} - \frac{1}{\Delta^{1/2}}(\Delta^{1/2}\underline{\bar{F}}^{\mu}{}_{\mu'})_{;\lambda}{}^{;\lambda} + R^{\mu}{}_{\nu}\delta^{\nu}{}_{\mu'}F - 2\frac{1}{\Delta^{1/2}}(\Delta^{1/2}F)^{,\lambda}\delta^{\mu}{}_{\mu';\lambda} - F\delta^{\mu}{}_{\mu';\lambda}{}^{;\lambda}\,, \tag{3.41}$$

which defines $\underline{\bar{F}}^{\mu}{}_{\mu'}$ when subject to the boundary condition

$$\underline{F}^{\mu}{}_{\mu'}(x, x'; 0) = 0 . \quad (3.42)$$

The first two terms of the power-series development in s of the coincident coordinate limit $\underline{F}^{\mu}{}_{\mu'}(x, x; is)$ will be needed for our work. They are computed in Appendix B. The decomposition (3.38) is convenient because, as shown in Appendix B, the index contraction $\underline{F}^{\mu}{}_{\mu}(x, x; is)$ is independent of the dimensionality n (at least for the first two terms in s that we will need).

The WKB construction (3.32) for the vector-field transformation function and the similar construction for the scalar, ghost field transformation function exhibited in the preceding section in Eq. (2.21) [but with the $\frac{1}{6}R$ term omitted from Eq. (2.31) which defines the weight] enable the formal expression (3.25) for the one-loop action functional to be written in terms of an effective Lagrangian,

$$W_1 = \int (d^n x)\sqrt{-g}\,\mathcal{L}_1 , \quad (3.43)$$

with

$$\mathcal{L}_1 = \frac{1}{2}\frac{1}{(4\pi)^{n/2}} \int_0^\infty \frac{ids}{(is)^{1+n/2}} [\underline{F}^{\mu}{}_{\mu}(x, x; is) - 2F(x, x; is)] . \quad (3.44)$$

The decomposition (3.38) now displays all the dependence on the dimensionality n explicitly,

$$\mathcal{L}_1 = \frac{1}{2}\frac{1}{(4\pi)^{n/2}} \int_0^\infty \frac{ids}{(is)^{1+n/2}} [\underline{F}^{\mu}{}_{\mu}(x, x; is) + (n-2)F(x, x; is)] . \quad (3.45)$$

Thus, the proper-time weight of the action,

$$w(is) = -i(4\pi is)^{n/2}(\mathrm{Tr}e^{-is\underline{H}} - 2\,\mathrm{Tr}e^{-isH}) , \quad (3.46)$$

can be expressed as

$$w(is) = \int (d^n x)[-g(x)]^{1/2} [\underline{F}^{\mu}{}_{\mu}(x, x; is) + (n-2)F(x, x; is)] . \quad (3.47)$$

The dimensional continuation discussed in the preceding section gives the limit

$$n \to 4:\quad W_1^{(n=4)} = \left(\frac{1}{4-n} + L_4 + \tfrac{1}{4}\right)\frac{1}{2}\frac{1}{(4\pi)^2}\left(\frac{\partial}{\partial is}\right)^2 w(is)\Big|_{s=0} - \frac{1}{2}\frac{1}{(4\pi)^2}\frac{\partial}{\partial n}\left[\left(\frac{\partial}{\partial is}\right)^2 w(is)\Big|_{s=0}\right]\Big|_{n=4} - \frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 w(is) , \quad (2.43)$$

where it is understood that the weight $w(is)$ in the infinite counterterm and in the integral is to be evaluated at $n=4$. The counterterm involves

$$\frac{1}{2}\left(\frac{\partial}{\partial is}\right)^2 [\underline{F}^{\mu}{}_{\mu}(x, x; is) + 2F(x, x; is)]\Big|_{s=0} = \underline{f}_2{}^{\mu}{}_{\mu} + 2f_2 . \quad (3.48)$$

In Appendix B we evaluate the coincident coordinate limit

$$\underline{f}_2{}^{\mu}{}_{\mu'} = \tfrac{1}{2}R^{\mu}{}_{\lambda}R_{\mu'}{}^{\lambda} - \tfrac{1}{6}RR^{\mu}{}_{\mu'} - \tfrac{1}{6}R^{\mu}{}_{\mu';\alpha}{}^{;\alpha} - \tfrac{1}{12}R^{\mu}{}_{\alpha\beta\gamma}R_{\mu'}{}^{\alpha\beta\gamma} , \quad (B20)$$

while f_2 is the $\xi=0$ limit of the formula (A24) given in the Appendix of paper I,

$$f_2 = \tfrac{1}{72}R^2 + \tfrac{1}{30}R_{,\alpha}{}^{;\alpha} - \tfrac{1}{180}R_{\alpha\beta}R^{\alpha\beta} + \tfrac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} . \quad (3.49)$$

Hence,

$$W_1^{(n=4)} = \left(\frac{1}{4-n} + L_4 + \tfrac{1}{4}\right)\int (d^4x)\sqrt{-g}\,\mathcal{A}^{(1)} + W_{1\,\mathrm{ren}}^{(n=4)} , \quad (3.50)$$

where

$$\mathcal{A}^{(1)} = \frac{1}{(4\pi)^2}(\underline{f}_2{}^{\mu}{}_{\mu} + 2f_2) = \frac{1}{(4\pi)^2}\left(-\tfrac{13}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \tfrac{44}{90}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{5}{36}R^2 - \tfrac{1}{10}R_{,\alpha}{}^{;\alpha}\right) . \quad (3.51)$$

The renormalized, one-loop action functional now appears as

$$W_{1\,\mathrm{ren}}^{(n=4)} = \int (d^4x)\sqrt{-g}\,\mathcal{L}_{1\,\mathrm{ren}}^{(n=4)} , \quad (3.52)$$

with

$$\mathcal{L}_{1\,\text{ren}}^{(n=4)}(x) = -\frac{1}{(4\pi)^2} f_2(x,x) - \frac{1}{(4\pi)^2}\frac{1}{4}\int_0^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 [\underline{F}^\mu{}_\mu(x,x;is) + 2F(x,x;is)]\,. \tag{3.53}$$

The construction of the renormalized vacuum expectation value of the stress-energy tensor for the Maxwell field parallels that of the scalar field discussed in the preceding section, Eqs. (2.48)–(2.52). With the definition

$$[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is) = 2\frac{\delta}{\delta g_{\mu\nu}(x)} w(is)\,, \tag{3.54}$$

the dimensional-continuation limit (2.43) gives

$$\langle T^{\mu\nu}(x)\rangle^{(n=4)} = \left(\frac{1}{4-n} + L_4 + \frac{1}{4}\right)\frac{1}{(4\pi)^2}\frac{1}{2}\left(\frac{\partial}{\partial is}\right)^2 \overline{T}^{\mu\nu}(x;is)\bigg|_{s=0} + \langle T^{\mu\nu}(x)\rangle_{\text{ren}}^{(n=4)}\,, \tag{3.55}$$

with

$$\langle T^{\mu\nu}\rangle_{\text{ren}}^{(n=4)} = -\frac{1}{2}\frac{1}{(4\pi)^2}\frac{\partial}{\partial n}\left[\left(\frac{\partial}{\partial is}\right)^2 \overline{T}^{\mu\nu}(x;is)\bigg|_{s=0}\right]\bigg|_{n=4} - \frac{1}{(4\pi)^2}\frac{1}{4}\int^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3 \overline{T}^{\mu\nu}(x;is)\,. \tag{3.56}$$

This exhibits the renormalized stress tensor as the metric variational derivative of the renormalized one-loop action. Since the weight $w(is)$ is a functional of the metric tensor $g_{\alpha\beta}(x)$, which is invariant under general coordinate transformations, the definition (3.54) implies that the stress-tensor proper-time weight is conserved with n arbitrary,

$$\overline{T}^{\mu\nu}(x;is)_{;\nu} = 0\,. \tag{3.57}$$

Hence, the proper-time representation (3.56) yields a conserved, renormalized stress tensor. Note that Eq. (3.53) can be used to express the dimensional derivative contribution to the renormalized stress tensor as

$$-\frac{1}{2}\frac{1}{(4\pi)^2}\frac{\partial}{\partial n}\left[\left(\frac{\partial}{\partial is}\right)^2 \overline{T}^{\mu\nu}(x;is)\bigg|_{s=0}\right]\bigg|_{n=4} = -\frac{1}{(4\pi)^2}\frac{2}{[-g(x)]^{1/2}}\frac{\delta}{\delta g_{\mu\nu}(x)}\int (d^4x)\sqrt{-g}\,f_2\,. \tag{3.58}$$

We compute the trace of the Maxwell stress tensor with the method used in the preceding section for the scalar case. We consider the conformal metric variation

$$\delta g_{\alpha\beta}(x) = 2\delta\lambda(x) g_{\alpha\beta}(x)\,. \tag{3.59}$$

On referring to the coordinate representations of the operators $\underline{H}$ (with $\zeta = 1$) and H given in Eqs. (3.14) and (3.15), which display the metric tensor explicitly, we see that at $n=4$

$$\delta\underline{H} = -2\delta\lambda\underline{H} - 2\delta\lambda DD + 4D\delta\lambda D - 2DD\delta\lambda \tag{3.60}$$

and

$$\delta H = -2\delta\lambda H - 2D\delta\lambda D - 2H\delta\lambda\,. \tag{3.61}$$

It follows from Eqs. (3.54) and (3.46) that

$$[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is) = i(4\pi is)^2 is\left[\operatorname{Tr} e^{-is\underline{H}} 2\frac{\delta\underline{H}}{\delta g_{\mu\nu}(x)} - 2\operatorname{Tr} e^{-isH} 2\frac{\delta H}{\delta g_{\mu\nu}(x)}\right]. \tag{3.62}$$

Hence

$$\int (d^4x)[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is) g_{\mu\nu}(x)\delta\lambda(x) = -2i(4\pi is)^2 is[\operatorname{Tr} e^{-is\underline{H}}(\delta\lambda\underline{H} + \delta\lambda DD - 2D\delta\lambda D + DD\delta\lambda)$$
$$- 2\operatorname{Tr} e^{-isH}(\delta\lambda H + D\delta\lambda D + H\delta\lambda)]\,. \tag{3.63}$$

Now Eq. (3.27) and the discussion preceding it imply that

$$e^{-is\underline{H}}\,D = De^{-isH} \tag{3.64a}$$

and

$$De^{-is\underline{H}} = e^{-isH}D\,. \tag{3.64b}$$

Hence, using the cyclic symmetry of the trace,

$$\mathrm{Tr}\, e^{-is\underline{H}}(\delta\lambda DD - 2D\delta\lambda D + DD\delta\lambda) = \mathrm{Tr}(\delta\lambda D e^{-isH} D - 2De^{-isH}\delta\lambda D + De^{-isH}D\delta\lambda) = \mathrm{Tr}(2e^{-isH}D\delta\lambda D + 2e^{-isH}\delta\lambda H)\,. \tag{3.65}$$

Here the last two traces do not include a diagonal sum over the implicit tensor indices—the vector indices of the two D symbols are contracted—and in the last equality we use

$$-D^2 = H\,. \tag{3.20}$$

The use of Eq. (3.65) results in a large cancellation of terms in Eq. (3.63), and we again use the cyclic symmetry of the trace to secure

$$\int (d^4x)[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)\,\delta\lambda(x) = -2i(4\pi is)^2 is(\mathrm{Tr}\, e^{-is\underline{H}}\underline{H}\delta\lambda - 2\,\mathrm{Tr}\, e^{-isH}H\delta\lambda) \tag{3.66}$$

and

$$\begin{aligned}\overline{T}^{\mu\nu}(x;is)g_{\mu\nu} &= -2(is)^2 is\frac{\partial}{\partial is}(is)^{-2}[\underline{F}^{\mu}{}_{\mu}(x,x;is) - 2F(x,x;is)]\\ &= -2(is)^2 is\frac{\partial}{\partial is}(is)^{-2}\,[\underline{\overline{F}}^{\mu}{}_{\mu}(x,x;is) + 2F(x,x;is)]\,.\end{aligned} \tag{3.67}$$

The trace of the stress tensor is nearly at hand. The previous result gives

$$\begin{aligned}-\frac{1}{4}\frac{1}{(4\pi)^2}\int_0^\infty ids\,(\ln\kappa^2 is)\left(\frac{\partial}{\partial is}\right)^3\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x) &= -\frac{1}{4}\frac{1}{(4\pi)^2}\int_0^\infty ids\,(\ln\kappa^2 is)\frac{\partial}{\partial is}is\frac{\partial}{\partial is}\left\{-2\left(\frac{\partial}{\partial is}\right)^2[\underline{F}^{\mu}{}_{\mu}(s,s;is) + 2F(x,x;is)]\right\}\\ &= \frac{1}{(4\pi)^2}\frac{1}{2}\left(\frac{\partial}{\partial is}\right)^2[\underline{\overline{F}}^{\mu}{}_{\mu}(x,x;is) + 2F(x,x;is)]\Big|_{s=0} = \mathfrak{A}^{(1)}(x)\,,\end{aligned} \tag{3.68}$$

which is the integrand in the action counterterm exhibited in Eqs. (3.50) and (3.51). Thus, Eqs. (3.56) and (3.58) yield

$$\langle T^{\mu\nu}(x)\rangle_{\mathrm{ren}}^{(n=4)} g_{\mu\nu}(x) = \mathfrak{A}^{(1)}(x) - \frac{1}{(4\pi)^2}\frac{2}{[-g(x)]^{1/2}}g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\int (d^4x)\sqrt{-g}f_2\,. \tag{3.69}$$

This is the trace anomaly for the stress-energy tensor of the Maxwell field. Clearly the metric variational derivative contribution which appears here can be removed by the addition of a finite counterterm to the Lagrange function. This counterterm, however, is not invariant under a conformal variation of the metric tensor, Eq. (3.59). Moreover, the anomaly $\mathfrak{A}^{(1)}(x)$ cannot be removed by the addition of a counterterm to the Lagrange function.

The stress-tensor weight $\overline{T}^{\mu\nu}(x;is)$ can be expressed in terms of the vector and scalar (ghost) Green's function proper-time weights $\underline{F}^{\mu}{}_{\mu'}(x,x';is)$ and $F(x,x';is)$. This is done in Appendix C. The ghost field contribution to the stress-tensor weight [coming from the χ-field terms in Eq. (3.5)] essentially cancels the contribution of the longitudinal part of the vector potential to the stress-tensor weight [coming from the terms in Eq. (3.5) with the coefficient $1/\zeta$]. These contributions combine to form a total proper-time derivative which can be deleted in our renormalization scheme. Thus, the Maxwell stress-tensor vacuum expectation value can be expressed entirely in terms of the Green's function of the gauge-invariant field strength tensor.[12] We see that the ghost field plays the role of an "integrating factor": It enables the stress-tensor vacuum expectation value to be given as the metric variational derivative of an action functional. As a corollary, we note that this total proper-time derivative is needed to make the stress-tensor weight $\overline{T}^{\mu\nu}(x;is)$ conserved, although, of course, it is not needed for the conservation of the stress-tensor expectation value.

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APPENDIX A

Here we shall compute the stress-tensor weight

$$[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is)=i(4\pi is)^{n/2}is\ \mathrm{Tr}e^{-isH}2\frac{\delta H}{\delta g_{\mu\nu}(x)} \tag{2.56}$$

for the scalar field in terms of the Green's function weight $F(x,x;is)$ and compare this weight with that used in paper I. The definitions (2.11) and (2.14) imply that

$$H=(-g)^{-1/4}G^{-1}(-g)^{-1/4}. \tag{A1}$$

Hence, using the cyclic symmetry of the trace and the explicit WKB construction (2.21) for the terms produced when the factors of $(-g)^{-1/4}$ are varied, we get

$$[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is)=i(4\pi is)^{n/2}is\ \mathrm{Tr}(-g)^{-1/4}e^{-isH}(-g)^{-1/4}2\frac{\delta G^{-1}}{\delta g_{\mu\nu}(x)}$$
$$-g^{\mu\nu}(x)[-g(x)]^{1/2}(is)^{n/2}is\frac{\partial}{\partial is}(is)^{-n/2}e^{-m^2is}F(x,x;is). \tag{A2}$$

In view of the WKB construction (2.21) and the relation [Eq. (2.49)]

$$[-g(x)]^{1/2}\langle T^{\mu\nu}(x)\rangle=-\left\langle\left(\phi,\frac{\delta G^{-1}}{\delta g_{\mu\nu}(x)}\phi\right)\right\rangle, \tag{A3}$$

we see that the first line of Eq. (A2) can be evaluated by using the correspondence

$$\phi(x)\phi(x')\leftrightarrow 2is\Delta^{1/2}(x,x')F(x,x';is)\exp\left[-\frac{\sigma(x,x')}{2is}-m^2is\right] \tag{A4}$$

in

$$T^{\mu\nu}=\phi^{,\mu}\phi^{,\nu}-\tfrac{1}{2}g^{\mu\nu}\phi_{,\sigma}\phi^{,\sigma}-\tfrac{1}{2}g^{\mu\nu}m^2\phi^2+\tfrac{1}{6}[G^{\mu\nu}\phi^2+g^{\mu\nu}(\phi^2)_{,\sigma}{}^{;\sigma}-(\phi^2)^{,\mu;\nu}]. \tag{2.3}$$

This evaluation entails the coincident coordinate limits

$$x=x':\quad \sigma=0=\sigma_{,\mu}=\sigma_{,\mu'}, \tag{2.23}$$

$$\sigma_{,\mu;\nu'}=-g_{\mu\nu}, \tag{2.24}$$

$$\Delta^{1/2}=1, \tag{2.26}$$

$$\Delta^{1/2}{}_{,\mu}=\Delta^{1/2}{}_{,\mu'}=0, \tag{I.A11}$$

and

$$\Delta^{1/2}{}_{,\mu;\nu'}=-\tfrac{1}{6}R_{\mu\nu}. \tag{I.A13}$$

We compute in this way

$$\overline{T}^{\mu\nu}(x;is)=g^{\mu\nu}(x)e^{-m^2is}\left(1-is\frac{\partial}{\partial is}\right)F(x,x;is)+2is(g^{\mu\lambda}g^{\nu\kappa}-\tfrac{1}{2}g^{\mu\nu}g^{\lambda\kappa})e^{-m^2is}F_{,\lambda,\kappa'}(x,x';is)\Big|_{x=x'}$$
$$+2is\tfrac{1}{6}(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\lambda}g^{\nu\kappa})e^{-m^2is}[F(x,x;is)]_{,\lambda;\kappa}. \tag{A5}$$

The trace of this expression gives, at $n=4$,

$$\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=4e^{-m^2is}\left(1-is\frac{\partial}{\partial is}\right)F(x,x;is)+2ise^{-m^2is}F_{,\mu}{}^{;\mu}(x,x';is)\Big|_{x=x'}. \tag{A6}$$

The coincident coordinate limit of Eq. (2.31) gives

$$\frac{\partial}{\partial is}F(x,x;is)=F_{,\mu}{}^{;\mu}(x,x'is)\Big|_{x=x'}. \tag{A7}$$

Hence

$$\overline{T}^{\mu\nu}(x,is)g_{\mu\nu}(x)=-2(is)^2is\frac{\partial}{\partial is}(is)^{-2}e^{-m^2is}F(x,x;is)-2m^2ise^{-m^2is}F(x,x;is), \tag{A8}$$

and we recover the result (2.58) derived by formal operator manipulations. The conservation of $\overline{T}^{\mu\nu}(x;is)$ could also be established directly with the use of Eq. (2.31), but there is no need to do this somewhat lengthy calculation (a similar calculation was presented in paper I).

The stress-tensor weight displayed in Eq. (A5) differs from the weight $T^{\mu\nu}(x;is)$ used in paper I, Eq. (I.3.21). Fixing $\xi=\frac{1}{6}$ in $T^{\mu\nu}(x;is)$ as is done in $\overline{T}^{\mu\nu}(x;is)$, we have the connection

$$\overline{T}^{\mu\nu}(x;is)=2ise^{-m^2is}T^{\mu\nu}(x;is)-\frac{2}{n}g^{\mu\nu}(x)(is)^{n/2}is\frac{\partial}{\partial is}(is)^{-n/2}e^{-m^2is}F(x,x;is)\,. \tag{A9}$$

The old weight $T^{\mu\nu}(x;is)$ is regular at $s=0$. Hence, Eq. (A9) can be put into the stress-tensor proper-time representation (2.52), and an integration by parts can be done which reduces the triple proper-time derivative to a double derivative, yielding the representation used in paper I, Eq. (I.1.38) (with the $G^{\mu\nu}$ term omitted as is appropriate when $\xi=\frac{1}{6}$ is held fixed). The essential difference between the two weights is that one emphasizes the trace, the other the divergence. For simplicity let us consider $n=4$ and $m=0$. Then

$$T^{\mu\nu}(x;is)g_{\mu\nu}(x)=0\,, \tag{A10}$$

while

$$\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=-2(is)^2is\frac{\partial}{\partial is}(is)^{-2}F(x,x;is)\,. \tag{A11}$$

On the other hand,

$$\overline{T}^{\mu\nu}(x;is)_{;\nu}=0\,,$$

while

$$T^{\mu\nu}(x;is)_{;\nu}=\tfrac{1}{4}(is)^2\frac{\partial}{\partial is}(is)^{-2}F(x,x;is)^{;\mu}\,. \tag{A12}$$

APPENDIX B

The coincident coordinate limits of various derivatives of the parallel displacement bivector[11] $\delta^\mu{}_{\mu'}(x,x')$ are needed in our work. These are obtained from the defining differential equation

$$\sigma^{,\lambda}(x,x')\delta^\mu{}_{\mu'}(x,x')_{;\lambda}=0\,, \tag{B1}$$

with the coincident coordinate boundary condition

$$\delta^\mu{}_{\mu'}(x,x)=\delta^\mu{}_{\mu'}\,, \tag{B2}$$

and from the coincident coordinate limit of derivatives of the world function which have been reviewed in paper I,

$$x=x':\ \sigma_{,\alpha}=0\,, \tag{I.2.24}$$

$$\sigma_{,\alpha;\beta}=g_{\alpha\beta}\,, \tag{I.2.25}$$

$$\sigma_{,\alpha;\beta;\gamma}=0\,, \tag{I.A2}$$

and

$$\sigma_{,\alpha;\beta;\gamma;\delta}=\tfrac{1}{3}(R_{\delta\alpha\beta\gamma}+R_{\gamma\alpha\beta\delta})\,. \tag{I.A4}$$

Thus, the coincident coordinate limit of the first derivative of Eq. (B1) yields

$$x=x':\ \delta^\mu{}_{\mu';\alpha}=0\,. \tag{B3}$$

The second derivative of Eq. (B1) gives the coincident limit

$$x=x':\ \delta^\mu{}_{\mu';\alpha;\beta}+\delta^\mu{}_{\mu';\beta;\alpha}=0\,, \tag{B4}$$

which, with

$$\delta^\mu{}_{\mu';\alpha;\beta}-\delta^\mu{}_{\mu';\beta;\alpha}=-\delta^\lambda{}_{\mu'}R^\mu{}_{\lambda\alpha\beta}\,, \tag{B5}$$

yields

$$x=x':\ \delta^\mu{}_{\mu';\alpha;\beta}=\tfrac{1}{2}R^\mu{}_{\mu'\alpha\beta}\,. \tag{B6}$$

Finally, we take four derivatives of Eq. (B1) with pairs of derivative indices identified to get the coincident limit

$$x=x':\ 2\sigma^{,\lambda}{}_{;\alpha}{}^{;\alpha}{}_{;\beta}\delta^\mu{}_{\mu';\lambda}{}^{;\beta}+2\sigma^{,\lambda}{}_{;\alpha;\beta}{}^{;\beta}\delta^\mu{}_{\mu';\lambda}{}^{;\alpha}+2\delta^\mu{}_{\mu';\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}+2\delta^\mu{}_{\mu';\beta;\alpha}{}^{;\alpha;\beta}=0\,. \tag{B7}$$

It follows from Eq. (I.A4) that the coincident limits $\sigma^{,\lambda}{}_{;\alpha}{}^{;\alpha}{}_{;\beta}$ and $\sigma^{,\lambda}{}_{;\alpha;\beta}{}^{;\beta}$ are symmetrical in (λ,β) and (λ,α), respectively, while, according to Eq. (B6), they are contracted with quantities that are antisymmetrical in these index pairs and thus they give vanishing contributions. There remains

$$\delta^\mu{}_{\mu';\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}+\delta^\mu{}_{\mu';\beta;\alpha}{}^{;\alpha;\beta}=0\,. \tag{B8}$$

The order of the derivative indices in the last term here can be altered with the aid of the curvature tensor. Using the coincident limits (B2) and (B3), we get

$$x=x':\ \delta^\mu{}_{\mu';\beta;\alpha}{}^{;\alpha;\beta}=\delta^\mu{}_{\mu';\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}+2\delta^\lambda{}_{\mu';\alpha;\beta}R^\mu{}_\lambda{}^{\alpha\beta}+\delta^\mu{}_{\mu';\alpha;\beta}R^{\alpha\beta}+R^\mu{}_{\mu'\alpha\beta}{}^{;\alpha;\beta}\,. \tag{B9}$$

The last term which appears here vanishes by virtue of the contracted Bianchi identity. The next-to-last term also vanishes because Eq. (B6) implies that the coincident limit $\delta^\mu{}_{\mu'\alpha\beta}$ is antisymmetrical in $\alpha\beta$. Thus, on using Eq. (B6), we get

$$x=x': \quad \delta^{\mu}{}_{\mu';\beta;\alpha}{}^{;\alpha;\beta}=\delta^{\mu}{}_{\mu'\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}+R^{\mu}{}_{\lambda\alpha\beta}R_{\mu'}{}^{\lambda\alpha\beta}, \tag{B10}$$

and Eq. (B8) yields

$$x=x': \quad \delta^{\mu}{}_{\mu';\alpha}{}^{;\alpha}{}_{;\beta}{}^{;\beta}=-\tfrac{1}{2}R^{\mu}{}_{\alpha\beta\gamma}R_{\mu'}{}^{\alpha\beta\gamma}. \tag{B11}$$

We need coincident limits of the first two terms of the expansion

$$\underline{\overline{F}}^{\mu}{}_{\mu'}=isf_1{}^{\mu}{}_{\mu'}+(is)^2 f_2{}^{\mu}{}_{\mu'}+\cdots \tag{B12}$$

for our work. Inserting this expansion into Eq. (3.41) gives

$$\begin{aligned}-f_1{}^{\mu}{}_{\mu'}=&\ \sigma^{,\lambda}f_1{}^{\mu}{}_{\mu';\lambda}+R^{\mu}{}_{\nu}\delta^{\nu}{}_{\mu'}\\ &-2\frac{1}{\Delta^{1/2}}\Delta^{1/2,\lambda}\delta^{\mu}{}_{\mu';\lambda}-\delta^{\mu}{}_{\mu';\lambda}{}^{;\lambda}\end{aligned} \tag{B13}$$

and

$$\begin{aligned}-2f_2{}^{\mu}{}_{\mu'}=&\ R^{\mu}{}_{\nu}f_1{}^{\nu}{}_{\mu'}+\sigma^{,\lambda}f_2{}^{\mu}{}_{\mu';\lambda}\\ &-\frac{1}{\Delta^{1/2}}(\Delta^{1/2}f_1{}^{\mu}{}_{\mu'})_{;\lambda}{}^{;\lambda}\\ &+R^{\mu}{}_{\nu}\delta^{\nu}{}_{\mu'}f_1-2\frac{1}{\Delta^{1/2}}(\Delta^{1/2}f_1)^{,\lambda}\delta^{\mu}{}_{\mu';\lambda}\\ &-f_1\delta^{\mu}{}_{\mu';\lambda}{}^{;\lambda},\end{aligned} \tag{B14}$$

where f_1 is the first term in the expansion of the scalar field weight $F(x,x';is)$ defined by Eq. (3.39). The coincident limits listed above and

$$x=x': \quad \Delta^{1/2}=1, \tag{2.26}$$

$$\Delta^{1/2,\lambda}=0 \tag{I.A11}$$

yield the coincident limit of Eq. (B13),

$$x=x': \quad f_1{}^{\mu}{}_{\mu'}=-R^{\mu}{}_{\mu'}. \tag{B15}$$

Taking two derivatives of Eq. (B13), and again using the coincident limits listed above and the fact that

$$x=x': \quad \Delta^{1/2}{}_{,\mu;\nu}=\tfrac{1}{6}R_{\mu\nu} \tag{I.A13}$$

is symmetrical in $\mu\nu$, we get

$$\begin{aligned}x=x': \quad -f_1{}^{\mu}{}_{\mu';\alpha}{}^{;\alpha}=&\ 2f_1{}^{\mu}{}_{\mu';\alpha}{}^{;\alpha}-R^{\mu}{}_{\mu';\alpha}{}^{;\alpha}\\ &-\delta^{\mu}{}_{\mu';\lambda}{}^{;\lambda}{}_{;\alpha}{}^{;\alpha}\end{aligned} \tag{B16}$$

and

$$x=x': \quad f_1{}^{\mu}{}_{\mu';\alpha}{}^{;\alpha}=-\tfrac{1}{3}R^{\mu}{}_{\mu';\alpha}{}^{;\alpha}-\tfrac{1}{6}R^{\mu}{}_{\alpha\beta\gamma}R_{\mu'}{}^{\alpha\beta\gamma}. \tag{B17}$$

We use these results in the coincident limit of Eq. (B14),

$$\begin{aligned}x=x': \quad -2f_2{}^{\mu}{}_{\mu'}=&\ R^{\mu}{}_{\nu}f_1{}^{\nu}{}_{\mu'}-\Delta^{1/2}{}_{,\lambda}{}^{;\lambda}f_1{}^{\mu}{}_{\mu'}\\ &-f_1{}^{\mu}{}_{\mu';\lambda}{}^{;\lambda}+R^{\mu}{}_{\mu'}f_1,\end{aligned} \tag{B18}$$

along with a result quoted in paper I [Eq. (I.A20) with $\xi=0$]

$$x=x': \quad f_1=\tfrac{1}{6}R \tag{B19}$$

to secure the coincident limit

$$\begin{aligned}x=x': \quad f_2{}^{\mu}{}_{\mu'}=&\ \tfrac{1}{2}R^{\mu}{}_{\lambda}R_{\mu'}{}^{\lambda}-\tfrac{1}{6}R^{\mu}{}_{\mu'}R\\ &-\tfrac{1}{6}R^{\mu}{}_{\mu';\alpha}{}^{;\alpha}-\tfrac{1}{12}R^{\mu}{}_{\alpha\beta\gamma}R_{\mu'}{}^{\alpha\beta\gamma}.\end{aligned} \tag{B20}$$

APPENDIX C

The stress-tensor weight

$$[-g(x)]^{1/2}\overline{T}^{\mu\nu}(x;is)=i(4\pi is)^{n/2}is\left[\mathrm{Tr}e^{-is\underline{H}}2\frac{\delta\underline{H}}{\delta g_{\mu\nu}(x)}-2\,\mathrm{Tr}e^{-isH}2\frac{\delta H}{\delta g_{\mu\nu}(x)}\right] \tag{3.62}$$

for the Maxwell field can be expressed in terms of Green's function weights using the techniques described for the scalar field case in Appendix A. Since

$$\underline{H}^{\mu}{}_{\nu}=(-g)^{-1/4}\underline{G}^{-1\mu\lambda}g_{\lambda\nu}(-g)^{-1/4} \tag{C1}$$

and

$$H=(-g)^{-1/4}G^{-1}(-g)^{-1/4}, \tag{C2}$$

we get

$$\begin{aligned}[-g(x)]^{1/2}T^{\mu\nu}(x;is)=&\ i(4\pi is)^{n/2}is\left[\mathrm{Tr}(-g)^{-1/4}e^{-is\underline{H}}(-g)^{-1/4}2\frac{\delta\underline{G}^{-1}}{\delta g_{\mu\nu}(x)}-\mathrm{Tr}(-g)^{-1/4}e^{-isH}(-g)^{-1/4}4\frac{\delta G^{-1}}{\delta g_{\mu\nu}(x)}\right]\\ &-g^{\mu\nu}(x)[-g(x)]^{1/2}(is)^{n/2}is\frac{\partial}{\partial is}(is)^{-n/2}[\underline{F}^{\lambda}{}_{\lambda}(x,x;is)-2F(x,x;is)]\\ &+2[-g(x)]^{1/2}(is)^{n/2}is\frac{\partial}{\partial is}(is)^{-n/2}\underline{F}^{\mu\nu}(x,x;is).\end{aligned} \tag{C3}$$

The metric variation of the inverse Green's function operators produces the stress tensor

$$T^{\mu\nu}=(F^{\mu}{}_{\lambda}F^{\nu\lambda}-\tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})+\{g^{\mu\nu}[\tfrac{1}{2}(A^{\lambda}{}_{;\lambda})^2+A^{\lambda}{}_{;\lambda,\kappa}A^{\kappa}]-A^{\lambda}{}_{;\lambda}{}^{,\mu}A^{\nu}-A^{\lambda}{}_{;\lambda}{}^{,\nu}A^{\mu}\}+(\chi^{\dagger,\mu}\chi^{\nu}+\chi^{\dagger,\nu}\chi^{,\mu}-g^{\mu\nu}\chi^{\dagger}{}_{,\lambda}\chi^{,\lambda}) \tag{3.5}$$

with the fields represented by

$$A^{\mu}(x)A^{\nu}(x')\leftrightarrow 2is\Delta^{1/2}(x,x')\underline{F}^{\mu\nu}(x,x';is)\exp\left[-\frac{\sigma(x,x')}{2is}\right] \tag{C4}$$

and

$$\chi^{\dagger}(x)\chi(x')\leftrightarrow -2is\Delta^{1/2}(x,x')F(x,x';is)\exp\left[-\frac{\sigma(x,x')}{2is}\right], \tag{C5}$$

where the minus sign in the last formula arises from the anticommutativity of the ghost field.

We evaluate first the contribution arising from the "classical" piece of the stress tensor,

$$^{(F)}T^{\mu\nu}=F^{\mu}{}_{\lambda}F^{\nu\lambda}-\tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{C6}$$

a contribution which we denote by $^{(F)}\overline{T}^{\mu\nu}(x;is)$. This evaluation again entails the coincident coordinate limits

$$x=x':\quad \sigma=0=\sigma_{,\mu}=\sigma_{,\mu'}, \tag{2.23}$$

$$\sigma_{,\mu;\nu'}=-g_{\mu\nu}, \tag{2.24}$$

$$\Delta^{1/2}=1, \tag{2.26}$$

$$\Delta^{1/2}{}_{,\mu}=\Delta^{1/2}{}_{,\mu'}=0, \tag{I.A11}$$

and

$$\Delta^{1/2}{}_{,\mu;\nu'}=-\tfrac{1}{6}R_{\mu\nu} \tag{I.A13}$$

After a little calculation, we find

$$\begin{aligned}^{(F)}\overline{T}^{\mu\nu}(x;is)=&\left[g^{\mu\nu}\delta^{\lambda}{}_{\kappa}\left(\frac{3-n}{2}\right)+\delta^{\mu}{}_{\kappa}g^{\nu\lambda}(n-2)\right]F^{\kappa}{}_{\lambda}(x,x;is)\\&-\tfrac{1}{6}[R^{\mu\nu}\delta^{\lambda}{}_{\kappa}-R^{\mu}{}_{\kappa}g^{\nu\lambda}-R^{\nu}{}_{\kappa}g^{\mu\lambda}+R\delta^{\mu}{}_{\kappa}g^{\nu\lambda}-\tfrac{1}{2}g^{\mu\nu}(R\delta^{\lambda}{}_{\kappa}-R^{\lambda}{}_{\kappa})]2isF^{\kappa}{}_{\lambda}(x,x;is)\\&+[g^{\mu\alpha}g^{\nu\beta}\delta^{\lambda}{}_{\kappa}-g^{\mu\alpha}g^{\nu\lambda}\delta^{\beta}{}_{\kappa}-\delta^{\mu}{}_{\kappa}g^{\nu\beta}g^{\lambda\alpha}+\delta^{\mu}{}_{\kappa}g^{\nu\lambda}g^{\alpha\beta}\\&-\tfrac{1}{4}g^{\mu\nu}(2\delta^{\lambda}{}_{\kappa}g^{\alpha\beta}-\delta^{\beta}{}_{\kappa}g^{\lambda\alpha}-\delta^{\beta}{}_{\kappa}g^{\lambda\alpha})]2isF^{\kappa}{}_{\lambda;\alpha;\beta'}(x,x';is)\big|_{x=x'}.\end{aligned} \tag{C7}$$

Note that at $n=4$

$$^{(F)}\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=0. \tag{C8}$$

The contribution of the terms that remain in the stress tensor (3.5) can be computed in the same fashion. All these terms essentially cancel, however, and this cancellation is much more simply demonstrated with a more abstract approach. Hence, we consider these remaining quantities in terms of diagonal, coordinate-space matrix elements of operators. In so doing, we must bear in mind that the operator corresponding to a derivative,

$$\partial_{\mu}\langle x|=\langle x|D_{\mu}, \tag{C9}$$

obeys

$$D_{\mu}|x\rangle=-\partial_{\mu}|x\rangle. \tag{C10}$$

Thus, the contribution of the longitudinal vector potential piece of the stress tensor (3.5),

$$\begin{aligned}^{(L)}T^{\mu\nu}=&g^{\mu\nu}[\tfrac{1}{2}(A^{\lambda}{}_{;\lambda})^2+A^{\lambda}{}_{;\lambda,\kappa}A^{\kappa}]\\&-A^{\lambda}{}_{;\lambda}{}^{,\mu}A^{\nu}-A^{\lambda}{}_{;\lambda}{}^{,\nu}A^{\mu},\end{aligned} \tag{C11}$$

can be written in terms of the diagonal, coordinate-space matrix element of the operator

$$\begin{aligned}L^{\mu\nu}(is)=&g^{\mu\nu}[-\tfrac{1}{2}(De^{is\underline{H}}D)+D_{\lambda}(De^{-is\underline{H}})^{\lambda}]\\&-D^{\mu}(De^{-is\underline{H}})^{\nu}-D^{\nu}(De^{-is\underline{H}})^{\mu}.\end{aligned} \tag{C12}$$

We now recall that

$$De^{-is\underline{H}}=e^{isH}D \tag{3.64b}$$

and

$$D^2=-H. \tag{3.20}$$

Hence,

$$L^{\mu\nu}(is)=g^{\mu\nu}[\tfrac{1}{2}e^{-isH}H+(De^{-isH}D)] -D^{\mu}e^{-isH}D^{\nu}-D^{\mu}e^{-isH}D^{\nu}\,. \tag{C13}$$

The ghost piece

$$^{(G)}T^{\mu\nu}=\chi^{\dagger,\mu}\chi^{,\nu}+\chi^{\dagger,\nu}\chi^{,\mu}-g^{\mu\nu}\chi^{\dagger}{}_{,\lambda}\chi^{,\lambda} \tag{C14}$$

gives the operator

$$\mathcal{G}^{\mu\nu}(is)=D^{\mu}e^{-isH}D^{\nu}+D^{\nu}e^{-isH}D^{\mu} -g^{\mu\nu}(De^{-isH}D)\,. \tag{C15}$$

Here one must be careful to get the overall sign right. There is a minus sign arising from the anticommutativity of the ghost field as illustrated in Eq. (C5), and there is another minus sign in the operator correspondence (C10). We have now found that the longitudinal and ghost contributions essentially cancel, giving

$$L^{\mu\nu}(is)+\mathcal{G}^{\mu\nu}(is)=\tfrac{1}{2}g^{\mu\nu}e^{-isH}H \tag{C16}$$

and

$$^{(L+G)}\overline{T}^{\mu\nu}(x,is)=-g^{\mu\nu}(x)(is)^{n/2}is\frac{\partial}{\partial is} \times(is)^{-n/2}F(x,x;is)\,. \tag{C17}$$

The complete stress-tensor proper-time weight is the sum of $^{(F)}\overline{T}^{\mu\nu}(x;is)$, $^{(L+G)}\overline{T}^{\mu\nu}(x;is)$, and the proper-time derivative terms remaining in Eq. (C3),

$$\overline{T}^{\mu\nu}(x;is)={}^{(F)}\overline{T}^{\mu\nu}(x;is)+(is)^{n/2}is\frac{\partial}{\partial is}(is)^{-n/2}\{2\underline{F}^{\mu\nu}(x,x;is)-g^{\mu\nu}(x)[\underline{F}^{\lambda}{}_{\lambda}(x,x;is)-F(x,x;is)]\}\,. \tag{C18}$$

The proper-time derivative terms that appear here could be deleted, for they give a vanishing contribution in our dimensionally continued renormalization scheme. Thus, the stress tensor could be written entirely in terms of the weight $^{(F)}\overline{T}^{\mu\nu}(x;is)$, which is related to the proper-time weight of the gauge-invariant field strength Green's function. In particular, if this were done, the ghost field would not contribute to the stress tensor. This deletion, however, would upset the conservation of the weight $\overline{T}^{\mu\nu}(x,is)$, and so we keep these proper-time derivative terms. Finally, we note that at $n=4$, the trace of the stress-tensor weight Eq. (C18) is given by

$$\overline{T}^{\mu\nu}(x;is)g_{\mu\nu}(x)=-(is)^2is\frac{\partial}{\partial is}(is)^{-2}[2\underline{F}^{\lambda}{}_{\lambda}(x,x;is)-4F(x,x;is)]\,, \tag{C19}$$

in agreement with the result computed by operator techniques, Eq. (3.67).

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[1]We use the phrase "vacuum expectation value" to generally denote the matrix element of an operator between vacuum states in the remote past and future, divided by the transformation function of these two vacuum states. This definition is unambiguous when the space-time is asymptotically flat. In more general circumstances, the boundary conditions on the Green's functions are more subtle and, correspondingly, so is the definition of the vacuum expectation value. In some cases, it is the matrix element of the stress tensor with both states representing an initial vacuum that is of interest. However, these ambiguities do not affect our work significantly. The trace anomalies which we compute are unambiguous since they depend only upon the short-distance behavior of the theory and not upon its global properties. Alterations of our (rather implicit) boundary conditions can be accommodated by adding appropriate homogeneous solutions to the Green's functions or to their proper-time representation.

[2]L. S. Brown, Phys. Rev. D 15, 1469 (1977).

[3]J. Schwinger, Phys. Rev. 82, 664 (1951).

[4]Some use of dimensionally continued, parametric integral representations in quantum field theory has been made by previous authors. M. R. Brown and M. J. Duff [Phys. Rev. D 11, 2124 (1975)] used such a technique to calculate the renormalized one-loop effective Lagrangian for the special case of a self-interacting scalar field in a slowly varying external scalar background field. They also indicated an extension of this calculation to non-Abelian gauge theories. The parametric integral representation employed by these authors is not the proper-time representation used by us. P. Candelas and D. J. Raine [Phys. Rev. D 12, 965 (1975)] have used a parametric integral representation (which is not a proper-time representation) to compute Green's functions in de Sitter space with n dimensions, and have used these Green's functions to compute renormalized, one-loop effective Lagrangians for de Sitter space by dimensional regularization. J. S. Dowker and R. Critchley [Phys. Rev. D 13, 3224 (1976)] have introduced a "ζ-function" renormalization scheme which they implement by a proper-time representation, and which they applied to compute the stress tensor and effective action for de Sitter space. This ζ-function renormalization scheme is not the dimensional-continuation method which we use.

[5]Stress tensors for quantum fields propagating in curved space-time have recently been computed by several authors using an ambiguous, point-separation tech-

nique. P. C. W. Davies, S. A. Fulling, and W. G. Unruh [Phys. Rev. D 13, 2720 (1976)] and S. M. Christensen [*ibid.* 14, 2490 (1976)] use this technique and simply discard various ill-defined terms. S. L. Adler, J. Lieberman, and Y. J. Ng [Ann. Phys. (N.Y.) (to be published)] use this technique and perform a complicated averaging process to make ill-defined terms meaningful. However, their results do not agree with those of lowest-order perturbation theory for nearly flat space-times.

[6]S. Deser, M. J. Duff, and C. J. Isham, Nucl. Phys. B111, 45 (1976).

[7]B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965); B. S. DeWitt, Phys. Rep. 19C, 295 (1975).

[8]It is perhaps worth emphasizing that any renormalization procedure necessarily entails the introduction of a dimensional parameter of some sort. In our method, the variation of the renormalized action with respect to the scale mass yields a finite constant times the renormalization counterterms, a constant that is essentially the lowest-order β function of the renormalization-group language.

[9]The expression (I.1.24) for our trace anomaly $\mathcal{Q}^{(0)}(x)$ agrees with that presented in Eq. (7.3) of S. M. Christensen [Phys. Rev. D 14, 2490 (1976)], who remarks, incorrectly, that "something is wrong." It also agrees with the expressions found for the special case of the Robertson-Walker universe by P. C. W. Davies, S. F. Fulling, S. M. Christensen and T. S. Bunch, King's College London report (unpublished) and T. S. Bunch and P. C. W. Davies, King's College London report (unpublished). In lowest order of perturbation about flat space-time, only the term involving $R_{,\mu}{}^{;\mu}$ survives. Our coefficient for this term agrees with that found in the lowest-order perturbation calculation performed by D. M. Capper and M. J. Duff [Nuovo Cimento 23A, 173 (1974)].

[10]Our total coefficient for the $R_{,\mu}{}^{;\mu}$ term in the anomaly, $[1/(4\pi)^2](-\frac{1}{10}+\frac{1}{6})$, agrees with the lowest-order perturbation calculation of Capper and Duff, Ref. 9.

[11]The properties of the parallel displacement bivector (and also the world function σ) are discussed in J. L. Synge, *Relativity: The General Theory* (Interscience, New York, 1960) and in the book of DeWitt in Ref. 7.

[12]The cancellation of the longitudinal and "ghost" terms in the stress tensor was shown by Adler, Lieberman, and Ng (Ref. 5) within the context of their point-separation method.