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Multiquark hadrons. II. Methods*

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Techniques for estimating the masses and decay couplings of multiquark hadrons ($Q^m\overline{Q}^n$, $n+m>3$) are developed with specific reference to the $Q^2\overline{Q}^2$ sector. The dynamics is based on a quark-bag model of light, colored, and permanently confined quarks gauge-coupled to non-Abelian colored gluons. The SU(6) of "color-spin" generated by color SU(3) and the SU(2) of relativistic $j=\frac{1}{2}$ quarks dominates the spectrum. Color-spin rules analogous to Hund's rules of atomic spectroscopy dictate that the lightest multiquark hadrons are not exotic, are of low spin, and are coupled predominantly to strange Q^3 and $Q\overline{Q}$ decay channels. Multiquark hadrons are consequently elusive and may be misclassified as conventional $Q\overline{Q}$ mesons or Q^3 baryons.

I. INTRODUCTION

In a previous paper[1] we presented the phenomenology of two-quark, two-antiquark hadrons in a quark-bag model. The discussion was based on a phenomenological Hamiltonian describing light, S-wave, colored quarks weakly coupled to massless colored gluons, all confined to the interior of a bag. Two major technical questions were left unanswered in Ref. 1 (referred to hereafter as I): First, how is the phenomenological Hamiltonian diagonalized in the space of color-flavor-spin eigenstates, and second, how are the couplings to dissociation decay channels calculated? These are the subjects of the present paper.

Although we deal specifically with $Q^2\overline{Q}^2$ mesons, some effort is made to develop techniques applicable to all multiquark ($Q^m\overline{Q}^n$; $n+m>3$) S-wave hadrons. In particular, we introduce $SU(6)_{cs}$ [the SU(6) generated by color and the relativistic "spin" of S-wave quarks] in order to diagonalize the gluon interaction terms in the Hamiltonian. The qualitative effects of the gluon interactions are summarized by analogs of Hund's rules, which single out spectroscopically prominent multiquark configurations on the basis of color-spin [$SU(6)_{cs}$] quantum numbers. They enable us to make general arguments why multiquark hadrons are less prominent than might naively be expected.

The paper is organized as follows: In Sec. II we introduce the necessary symmetry groups and define a convenient notation for the remainder of the paper. In Sec. III we construct eigenstates of the bag Hamiltonian. First we diagonalize the flavor content of relevant SU(3) representations. This is the analog of the (trivial) "magic mixing" of ω and ϕ in the $Q\overline{Q}$ sector. Secondly, the gluon exchange interaction must be diagonalized in a basis of magically mixed flavor states. This leads us to introduce $SU(6)_{cs}$. Rules are given for spectroscopically important states. In Sec. IV we summarize the recoupling transformations which determine amplitudes for dissociation decays. In Sec. V we return to the $SU(6)_{cs}$ formalism and show that the rules of Sec. III reduce the spectroscopic importance of multiquark states in general. A simple expression for the quadratic Casimir operator of SU(3) or SU(6) is derived in the Appendix.

II. SYMMETRIES, SYMMETRY BREAKING, AND NOTATION

Our quarks carry three labels[2,3,4]: color [$SU(3)_c$], flavor [$SU(3)_f$], and spin [SU(2)]. Color is gauged, unbroken, and confined. Flavor is not gauged, broken by giving the strange quark a small mass and leaving the up and down quarks massless, and of course not confined. Spin is not actually spin at all but rather the SU(2) generated by the angular momentum of fully relativistic quarks in S-wave modes in a cavity. Neither $\vec{L}$ nor $\vec{S}$ is conserved in a relativistic quark model. Nevertheless, if we fix our attention on the $j=\frac{1}{2}$ (S-wave) sector of the theory, the algebra generated by the states and their currents is an SU(2).

As discussed in I, the phenomenological Hamiltonian (H) includes a kinetic energy term diagonal in eigenstates of the strange-quark number (n_s) and independent of color and spin, and a gluon interaction term which is approximately diagonal in eigenstates of n_s but mixes color and spin representations. The eigenstates of H are therefore characterized by the following quantum numbers (in the $Q^2\overline{Q}^2$ sector):

1. the flavor multiplet of the two quarks, denoted $\underline{3}$ or $\underline{\overline{6}}$, and of the two antiquarks, denoted $\underline{\overline{3}}$ or $\underline{6}$;
2. the $SU(6)_{cs}$ multiplet of the two quarks, de-

noted $[\underline{15}]$ or $[\underline{21}]$, and of the two antiquarks, denoted $[\overline{15}]$ or $[\overline{21}]$;

3. the total spin, labeled by $2J+1$, of the quarks and antiquarks together;

4. the total color, which is always a singlet, of the quarks and antiquarks together.

It is important to understand why the other obvious quantum numbers are not diagonal. The total flavor is not a good quantum number because of magic mixing. For example, many states in the $\underline{1}$, $\underline{8}$, and $\underline{27}$ (which result from $\underline{6}\otimes\underline{6}$) are mixed to diagonalize the number of s quarks. Generally only the states at the periphery of weight diagrams (e.g., the $I=2$ multiplet in $\underline{27}$) are pure $SU(3)_f$ eigenstates.

Total $SU(6)_{cs}$ multiplets are mixed by the gluon interactions. For example

$$[21]\otimes[\overline{21}]=[1]\otimes[35]\otimes[405] \quad . \tag{2.1}$$

The $SU(3)_c\times SU(2)$ decomposition of these shows that total color singlets with $J=0$ occur in both the [1] and [405] representations of $SU(6)_{cs}$. The gluon interactions mix these multiplets.

Eigenstates of total $SU(6)_{cs}$, color, and spin for $Q^2\overline{Q}^2$ are mixtures of color and spin representations of quarks and antiquarks separately. Again an example clarifies matters. The [1] in $[21]\otimes[\overline{21}]$ is a linear combination of $(6,3)(\overline{6},3)$ and $(\overline{3},1)(3,1)$. (The notation is $[d_c(Q),2j_Q+1][d_c(\overline{Q}),2j_{\overline{Q}}+1]$. d_c is the dimension of a color multiplet, $2j+1$ is the dimension of a spin multiplet.) Clearly both $(6,3)(\overline{6},3)$ and $(\overline{3},1)(3,1)$ can be coupled to total color and spin singlets. The (1, 1) in [405] is the orthogonal linear combination.

To summarize the notation introduced above:

$[d_{cs}]$ denotes $SU(6)_{cs}$ multiplets labeled by their dimension.

$(d_c, 2j+1)$ denotes $SU(3)_c\times SU(2)$ multiplets labeled by their color and spin dimensions.

$\underline{d}_f$ denotes flavor multiplets by their dimension.

It will always be apparent from context whether the notation refers to quarks, antiquarks, or both taken together. For reference, the SU(6) representations available to Q^2, $\overline{Q}^2$, and $Q^2\overline{Q}^2$ are summarized in Table I. Generally states will be labeled with quark quantum numbers, antiquark quantum numbers, and overall quantum numbers, in that order. We will often suppress labels when they are unnecessary. Specifically, antisymmetrization fixes the flavor once the $SU(6)_{cs}$ representation is chosen, so we often suppress flavor labels. Also, Q^2 (or $\overline{Q}^2$) $SU(3)_c\times SU(2)$ multiplets belong to unique $SU(6)_{cs}$ representations (see Table I) so we may suppress $SU(6)_{cs}$ labels if the $SU(3)_c\times SU(2)$ representation is given.

III. EIGENSTATES AND INTERACTION ENERGIES

Consider first the requirements of antisymmetrization. Quarks are [6] in $SU(6)_{cs}$, thus

$$[6]\otimes[6]=[15]\oplus[21] \tag{3.1}$$

are representations for Q^2. The flavor states available to two quarks are $\overline{\underline{3}}$ and $\underline{6}$. In both cases the smaller representation is antisymmetric; the larger is symmetric. Antiquarks are in conjugate representations. There are four antisymmetric combinations:

$$[21]\underline{\overline{3}}\otimes[\overline{21}]\underline{3}\ , \tag{3.2a}$$

$$[15]\underline{6}\otimes[\overline{15}]\underline{\overline{6}}\ , \tag{3.2b}$$

$$[15]\underline{6}\otimes[\overline{21}]\underline{3}\ , \tag{3.2c}$$

$$[21]\underline{\overline{3}}\otimes[\overline{15}]\underline{\overline{6}}\ . \tag{3.2d}$$

In addition to being antisymmetrized, these multiplets are not mixed by the gluon interactions.

A. Magic mixing

The flavor multiplets in Eqs. (3.2a)–(3.2d) mix to diagonalize the strange-quark content. The problem is familiar from the $Q\overline{Q}$ sector ($\eta-\eta'$, $\omega-\phi$, $f-f'$) where $I=Y=0$ octet and singlet members mix. Typically

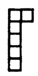
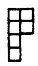
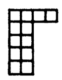
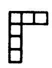
$$\begin{aligned}\eta_s&=(\tfrac{1}{3})^{1/2}\eta_{\underline{1}}-(\tfrac{2}{3})^{1/2}\eta_{\underline{8}}\ ,\\ \eta_0&=(\tfrac{2}{3})^{1/2}\eta_{\underline{1}}+(\tfrac{1}{3})^{1/2}\eta_{\underline{8}}\end{aligned} \tag{3.3}$$

where $\eta_{\underline{1}}$ and $\eta_{\underline{8}}$ are singlet and octet members, while η_0 and η_s contain zero or two strange quarks, respectively. The mixing in the $\underline{\overline{3}}\otimes\underline{3}$ of $Q^2\overline{Q}^2$ is exactly opposite to that in the $\underline{3}\otimes\underline{\overline{3}}$ of $Q\overline{Q}$ [compare Eq. (3.3) with Table II]. This has important phenomenological implications.[1]

The mixing matrices for the four $SU(3)_f$ multiplets of Eq. (3.2) are given in Table II. The magically mixed states are given as linear continuations of $SU(3)_f$ eigenstates. The $\underline{6}\otimes\underline{3}$ representation and its conjugate cause special problems. The flavor octets in $\underline{\overline{3}}\otimes\underline{\overline{6}}$ and $\underline{6}\otimes\underline{3}$ individually are not eigenstates of G parity. Appropriate linear combinations yield an f-type octet $\underline{8}_f$ and a d-type octet $\underline{8}_d$ which have different hadronic decays and may have quite different physical masses though in our (zero-width) approximation they remain degenerate. The hypercharge-zero members of $\underline{6}\otimes\underline{3}$ and $\underline{\overline{3}}\otimes\underline{\overline{6}}$ mix to diagonalize G parity[5]:

$$\begin{aligned}C_\pi^{\pm}&=\frac{1}{\sqrt{2}}[C_\pi(\underline{18})\mp C_\pi(\overline{\underline{18}})]\ ,\\ C_\pi^{s\pm}&=\frac{1}{\sqrt{2}}[C_\pi^s(\underline{18})\mp C_\pi^s(\overline{\underline{18}})]\ ,\\ C^{s\pm}&=\frac{1}{\sqrt{2}}[C^s(\underline{18})\mp C^s(\overline{\underline{18}})]\ .\end{aligned} \tag{3.4}$$

TABLE I. SU(6) and SU(3) representations for Q^2 and $\bar{Q}^2$ sectors ($\bar{Q}^2$ representations are obtained from Q^2 representations by interchanging quarks and antiquarks and drawing conjugate tableaux).

Sector	Young tableau	Dimension	SU(6) representations: $SU(3)\times SU(2)$ content	Eigenvalue of Casimir operator
$Q^{2\,a}$		21	$(6,3)$, $(\bar{3},1)$	160/3
		15	$(6,1)$, $(\bar{3},3)$	112/3
$Q^2\bar{Q}^2$		1	$(1,1)$	0
		35	$(1,3)$, $(8,1)$, $(8,3)$	48
		189	$(1,1)$, $(1,5)$, $(8,1)$, $2(8,3)$, $(8,5)$, $(10,3)$, $(\overline{10},3)$, $(27,1)$	80
		405	$(1,1)$, $(1,5)$, $(8,1)$, $2(8,3)$, $(8,5)$, $(10,3)$, $(\overline{10},3)$, $(27,1)$, $(27,3)$, $(27,5)$	112
		280	$(1,3)$, $(8,1)$, $2(8,3)$, $(8,5)$, $(10,1)$, $(10,3)$, $(10,5)$, $(\overline{10},1)$, $(27,3)$	96

SU(3) representations

Sector	Tableau	Dimension	Eigenvalue of Casimir operator
$Q^{2\,a}$		6	40/3
		3	16/3
$Q^2\bar{Q}^2$		1	0
		8	12

[a] $\bar{Q}^2$ representations are obtained by interchanging quarks and antiquarks and drawing conjugate tableaux.

The names of states are defined in Figs. 6–8 of paper I, and for the most part also in Table II. This mixing will figure in the calculation of decay couplings.

B. Diagonalizing the gluon interaction

The spectroscopically important interaction between quarks (aside from the interaction with the bag which provides confinement and sets the overall scale) is the spin-spin force mediated by one gluon exchange. The gluon interaction Hamiltonian is given by[4]

$$H_g = -\frac{\alpha_c}{R}\sum_{a=1}^{8}\sum_{i>j}\vec{\sigma}_i\cdot\vec{\sigma}_j\lambda_i^a\lambda_j^a M(m_iR,\, m_jR)\ , \qquad (3.5)$$

where $\alpha_c = g^2/4\pi$ is the color fine-structure constant ($\alpha_c = 0.55$); R is the bag radius, later to be eliminated by a boundary condition[1,4]; a labels colors and i (j) labels quarks. $\vec{\sigma}_i$ and λ_i^a are the spin and color vectors for the ith quark. To be precise, if i or j indicates an antiquark, the following replacement should be understood:

$$\sigma_i \to -\sigma_i^*\ ,\qquad \lambda_i \to -\lambda_i^*\ . \qquad (3.6)$$

M is the magnetic-interaction strength determined by an integral over bag wave functions. It is a function of quark masses. In paper I, we approximated M as follows:

$$M(m_iR,\, m_jR) \to M\left(\frac{n_s m_s}{N}\ ,\ \frac{n_s m_s}{N}\right)\ , \qquad (3.7)$$

TABLE II. Magic mixing in $Q^2\overline{Q}^2$ states.

(a) $\underline{\overline{3}}\otimes\underline{3}$

Isospin	Hypercharge	Name[1]	$\underline{1}$	$\underline{8}$	Number of $s\bar{s}$ pairs
0	0	$C^0(\underline{9})$	$(\frac{1}{3})^{1/2}$	$-(\frac{2}{3})^{1/2}$	0
0	0	$C^s(\underline{9})$	$(\frac{2}{3})^{1/2}$	$(\frac{1}{3})^{1/2}$	1

(b) $\underline{6}\otimes\underline{\overline{6}}$

Isospin	Hypercharge	Name[1]	$\underline{27}$	$\underline{8}$	$\underline{1}$	Number of $s\bar{s}$ pairs
0	0	$C^0(\underline{36})$	$(\frac{1}{10})^{1/2}$	$(\frac{2}{5})^{1/2}$	$(\frac{1}{2})^{1/2}$	0
0	0	$C^s(\underline{36})$	$(\frac{3}{5})^{1/2}$	$(\frac{1}{15})^{1/2}$	$-(\frac{1}{3})^{1/2}$	1
0	0	$C^{ss}(\underline{36})$	$(\frac{3}{10})^{1/2}$	$-(\frac{8}{15})^{1/2}$	$(\frac{1}{6})^{1/2}$	2
1	0	$C_\pi(\underline{36})$	$(\frac{1}{5})^{1/2}$	$(\frac{4}{5})^{1/2}$	0	0
1	0	$C_\pi^s(\underline{36})$	$(\frac{4}{5})^{1/2}$	$-(\frac{1}{5})^{1/2}$	0	1
$\frac{1}{2}$	± 1	$C_K(\underline{36})$	$(\frac{2}{5})^{1/2}$	$(\frac{3}{5})^{1/2}$	0	0
$\frac{1}{2}$	± 1	$C_K^s(\underline{36})$	$-(\frac{3}{5})^{1/2}$	$(\frac{2}{5})^{1/2}$	0	1

(c) $\underline{6}\otimes\underline{3}$

Isospin	Hypercharge	Name[1]	$\underline{8}$	$\underline{10}$	Number of $s\bar{s}$ pairs
1	0	$C_\pi(\underline{18})$	$(\frac{2}{3})^{1/2}$	$-\frac{1}{\sqrt{3}}$	0
1	0	$C_\pi^s(\underline{18})$	$(\frac{1}{3})^{1/2}$	$(\frac{2}{3})^{1/2}$	1
$\frac{1}{2}$	-1	$C_{\bar{K}}^s(\underline{18})$	$(\frac{2}{3})^{1/2}$	$\frac{1}{\sqrt{3}}$	1
$\frac{1}{2}$	-1	$C_{\bar{K}}(\underline{18})$	$(\frac{1}{3})^{1/2}$	$-(\frac{2}{3})^{1/2}$	0

(d) $\underline{\overline{3}}\otimes\underline{\overline{6}}$

Isospin	Hypercharge	Name[1]	$\underline{8}$	$\underline{\overline{10}}$	Number of $s\bar{s}$ pairs
1	0	$C_\pi(\underline{\overline{18}})$	$(\frac{2}{3})^{1/2}$	$-\frac{1}{\sqrt{3}}$	0
1	0	$C_\pi^s(\underline{\overline{18}})$	$(\frac{1}{3})^{1/2}$	$(\frac{2}{3})^{1/2}$	1
$\frac{1}{2}$	1	$C_K^s(\underline{\overline{18}})$	$(\frac{2}{3})^{1/2}$	$\frac{1}{\sqrt{3}}$	1
$\frac{1}{2}$	1	$C_K(\underline{\overline{18}})$	$(\frac{1}{3})^{1/2}$	$-(\frac{2}{3})^{1/2}$	0

where n_s is the number of strange quarks in the state of being considered, N is the total number of quarks, and m_s is the strange-quark mass (270 MeV).[4] $M(x,x)$ may be read off of Fig. 3 in paper I. So approximated, M may be removed from beneath the summation.

The products of $\sigma^k\lambda^a$ are among the generators of $SU(6)_{cs}$. The entire algebra is generated by these together with the 8 λ matrices and the 3 σ matrices. Specifically, define the generators of $SU(6)_{cs}$ as follows:

$$\{\alpha\} = \begin{cases} (\frac{2}{3})^{1/2}\sigma^k, & k=1,2,3 \\ \lambda^a, & a=1,\ 2,\ \ldots,\ 8 \\ \sigma^k\lambda^a\ . \end{cases} \tag{3.8}$$

The 35 α's generate an SU(6). They are normalized to $\mathrm{Tr}\alpha^2=4$ (we have chosen $\mathrm{Tr}\lambda^2=2$ and $\mathrm{Tr}\sigma^2=2$

as is conventional). The SU(6) of the antiquarks is generated by $\{-\alpha^*\}$.

It is straightforward to express Eq. (3.5) in terms of the quadratic Casimir operators of SU(2), $SU(3)_c$, and $SU(6)_{cs}$:

$$-\sum_a \sum_{i>j} \vec{\sigma}_i \cdot \vec{\sigma}_j \lambda_i^a \lambda_j^a = 8N + \tfrac{1}{2} C_6(\text{tot}) - \tfrac{4}{3} S_{\text{tot}}(S_{\text{tot}}+1) + C_3(Q) + \tfrac{8}{3} S_Q(S_Q+1) - C_6(Q) + C_3(\overline{Q}) + \tfrac{8}{3} S_{\overline{Q}}(S_{\overline{Q}}+1) - C_6(\overline{Q}) \ . \tag{3.9}$$

The Casimir operators are defined as follows:

$$C_6 = \sum_{\mu=1}^{35} \left(\sum_{i=1}^{N} \alpha_i^{\mu} \right)^2 , \tag{3.10}$$

$$C_3 = \sum_{a=1}^{8} \left(\sum_{i=1}^{N} \lambda_i^{a} \right)^2 , \tag{3.11}$$

$$4S(S+1) = \sum_{k=1}^{3} \left(\sum_{i=1}^{N} \sigma_i^{k} \right)^2 . \tag{3.12}$$

The labels Q, $\overline{Q}$, and tot refer to the representations of the quarks, antiquarks, and the entire system, respectively. The SU(2) Casimir operator is familiar. The SU(3) and SU(6) Casimir operators may readily be evaluated if the SU(2)× U(1) or SU(3)×SU(2) content of a given representation is known. Simple formulas for C_3 and C_6 are derived in the Appendix. Values for C_3 and C_6 for representations of interest are given in Table I.

The systematics of multiquark spectroscopy may be read off from Eq. (3.9). We may enumerate a pair of "Hund's rules"[6]: The magnetic interaction is most attractive (negative) for states in which

Rule 1. The quarks and antiquarks are separately in the largest possible representations of $SU(6)_{cs}$.

Rule 2. C_6(tot) is as small as possible.

Generally the Casimirs of $SU(6)_{cs}$ dominate Eq. (3.9) because they are larger than those of color or spin (see Table I) for the representations of interest. The spectroscopy is therefore less sensitive to S_{tot}, $C_3(Q)$, $C_3(\overline{Q})$, etc.[7] In Sec. V, we combine these two rules with the requirements of antisymmetrization to establish some general patterns among exotic and cryptoexotic masses.

Armed with Eq. (3.9), we may construct the eigenstates of H_g.

1. $[21]\otimes[\overline{21}]$. These are flavor nonets [see Eq. (3.2) and Table II] and (according to Rule 1 above) should contain the lightest $Q^2\overline{Q}^2$ states. First we must look for color singlets in

$$[21]\otimes[\overline{21}] = [1] \oplus [35] \oplus [405] \ . \tag{3.13}$$

$SU(3)_c \times SU(2)$ decomposition of these reveals the following singlets:

$$(1,1)\subset[1] \ ,$$
$$(1,3)\subset[35] \ , \tag{3.14}$$
$$(1,1) \text{ and } (1,5)\subset[405] \ .$$

To apply Eq. (3.9), we must know which $SU(3)_c \times SU(2)$ representations of quarks and antiquarks contribute which total color× spin multiplet. For total spin 1 and spin 2, this is trivial (see Table II); both must arise from $(6,3)\otimes(\overline{6},3)$ since $(\overline{3},1)\otimes(3,1)$ can yield only spin 0. The wave functions of the $J_{\text{tot}}=1$ and $J_{\text{tot}}=2$ states are fixed.

$$|2^+,\underline{9}\rangle \equiv |(6,3)\underline{\overline{3}};(\overline{6},3)\underline{3};(1,5)[405]\rangle \ , \tag{3.15}$$

$$|1^+,\underline{9}\rangle \equiv |(6,3)\underline{\overline{3}};(\overline{6},3)\underline{3};(1,3)[35]\rangle \ . \tag{3.16}$$

The corresponding magnetic energies are listed in Table III.

The two spin-0 states are linear combinations of $(\overline{3},1)\otimes(3,1)$ and $(6,3)\otimes(6,3)$. Graphical techniques for calculating the coefficients weighting these states have been developed by Mandula.[8] The application of his methods yields

$$|0^+\underline{9}[1]\rangle = (\tfrac{6}{7})^{1/2}|(6,3)\underline{\overline{3}};(\overline{6},3)\underline{3};(1,1)\rangle + (\tfrac{1}{7})^{1/2}|(\overline{3},1)\underline{\overline{3}};(3,1)\underline{3};(1,1)\rangle \ , \tag{3.17}$$

$$|0^+\underline{9}[405]\rangle = (\tfrac{1}{7})^{1/2}|(6,3)\underline{\overline{3}};(\overline{6},3)\underline{3};(1,1)\rangle - (\tfrac{6}{7})^{1/2}|(\overline{3},1)\underline{\overline{3}};(3,1)\underline{3};(1,1)\rangle \ . \tag{3.18}$$

H_g mixes these two states. The eigenstates are

$$|0^+,\underline{9}\rangle = 0.972|0^+\underline{9}[1]\rangle + 0.233|0^+\underline{9}[405]\rangle \ , \tag{3.19}$$

$$|0^+,\underline{9}^*\rangle = 0.233|0^+\underline{9}[1]\rangle - 0.972|0^+\underline{9}[405]\rangle \ . \tag{3.20}$$

The eigenvalues are collected in Table III. Notice

TABLE III. Magnetic interaction energies of $Q^2\overline{Q}^2$ eigenstates.

State	Wave function	Magnetic interaction energy $(R\,H_g/\alpha_c M)$ [a]
$\|0^+\underline{9}\rangle$	Eq. (3.19)	−43.36
$\|0^+\underline{36}\rangle$	Eq. (3.27)	−19.37
$\|0^+\underline{9}^*\rangle$	Eq. (3.20)	−1.97
$\|0^+\underline{36}^*\rangle$	Eq. (3.28)	22.03
$\|1^+9\rangle$	Eq. (3.16)	−16
$\|1^+\underline{36}\rangle$	Eq. (3.24)	0
$\|1^+\underline{\overline{18}}\rangle$	Eq. (3.33)	$-\frac{40}{3}$
$\|1^+\underline{\overline{18}}^*\rangle$	Eq. (3.34)	$\frac{32}{3}$
$\|2^+\underline{9}\rangle$	Eq. (3.15)	$\frac{32}{3}$
$\|2^+\underline{36}\rangle$	Eq. (3.23)	$\frac{32}{3}$

[a] This is the eigenvalue of the operator of Eq. (3.9).

that the lighter state ($\underline{9}$) is predominantly [1] as dictated by Rule 2. The small admixture of [405] comes about through the effect of spin and color within color-spin eigenstates.

2. $[15]\otimes[\overline{15}]$. These multiplets should be relatively heavy and contain truly exotic members. The calculation is analogous to $[21]\otimes[\overline{21}]$:

$$[15]\otimes[\overline{15}]=[1]\oplus[35]\oplus[189]\ , \tag{3.21}$$

with color singlets as follows:

$$\begin{aligned}&(1,1)\subset[1]\ ,\\&(1,3)\subset[35]\ ,\\&(1,1)\text{ and }(1,5)\subset[189]\ .\end{aligned} \tag{3.22}$$

The spin-1 and spin-2 states are trivial; they are formed uniquely from $(\overline{3},3)\otimes(3,3)$,

$$|2^+,\underline{36}\rangle\equiv|(\overline{3},3)\underline{6};(3,3)\underline{\overline{6}};(1,5)[189]\rangle\ , \tag{3.23}$$

$$|1^+,\underline{36}\rangle\equiv|(\overline{3},3)\underline{6};(3,3)\underline{\overline{6}};(1,3)[35]\rangle\ . \tag{3.24}$$

The two spin-0 states are linear combinations of $(\overline{3},3)\otimes(3,3)$ and $(6,1)\otimes(\overline{6},1)$. Using Mandula's[8] methods

$$\begin{aligned}|0^+\underline{36}[1]\rangle&=(\tfrac{3}{5})^{1/2}|(\overline{3},3)\underline{6};(3,3)\underline{\overline{6}};(1,1)\rangle\\&\quad+(\tfrac{2}{5})^{1/2}|(\underline{6},1)\underline{6};(\overline{6},1)\underline{\overline{6}};(1,1)\rangle\ ,\\|0^+36[189]\rangle&=(\tfrac{2}{5})^{1/2}|(\overline{3},3)\underline{6};(3,3)\underline{\overline{6}};(1,1)\rangle\\&\quad-(\tfrac{3}{5})^{1/2}|(6,1)\underline{6};(\overline{6},1)\underline{\overline{6}};(1,1)\rangle.\end{aligned} \tag{3.26}$$

Once again these are mixed by H_g:

$$\begin{aligned}|0^+,\underline{36}\rangle=-0.998|0^+\underline{36}[1]\rangle\\+0.063|0^+\underline{36}[189]\rangle\ ,\end{aligned} \tag{3.27}$$

$$\begin{aligned}|0^+,\underline{36}^*\rangle=0.063|0^+\underline{36}[1]\rangle\\+0.998|0^+\underline{36}[189]\rangle\ .\end{aligned} \tag{3.28}$$

The magnetic-interaction matrix elements for all these states may be found in Table III.

3. $[15]\otimes[\overline{21}]$ and $[21]\otimes[\overline{15}]$. The multiplets are related by charge conjugation. We discuss $[21]\otimes[\overline{15}]$ and obtain results for $[15]\otimes[\overline{21}]$ by inspection:

$$[21]\otimes[\overline{15}]=[35]\oplus[280]\ . \tag{3.29}$$

The overall color singlets are

$$\begin{aligned}&(1,3)\subset[35]\ ,\\&(1,3)\subset[280]\ ,\end{aligned} \tag{3.30}$$

which are linear combinations of $(6,3)\otimes(\overline{6},1)$ and $(\overline{3},1)\otimes(3,3)$

$$\begin{aligned}|1^+\underline{\overline{18}}[35]\rangle&=(\tfrac{1}{3})^{1/2}|(\overline{3},1)\underline{\overline{3}};(3,3)\underline{\overline{6}};(1,3)\rangle\\&\quad-(\tfrac{2}{3})^{1/2}|(6,3)\underline{\overline{3}};(6,1)\underline{\overline{6}};(1,3)\rangle\ ,\end{aligned} \tag{3.31}$$

$$\begin{aligned}|1^+\underline{\overline{18}}[280]\rangle&=(\tfrac{2}{3})^{1/2}|(\overline{3},1)\underline{\overline{3}};(3,3)\underline{\overline{6}};(1,3)\rangle\\&\quad+(\tfrac{1}{3})^{1/2}|(6,3)\underline{\overline{3}};(6,1)\underline{\overline{6}};(1,3)\rangle\ ,\end{aligned} \tag{3.32}$$

which in turn are mixed to produce eigenstates of H_g:

$$|\underline{1}^+,\underline{\overline{18}}\rangle=\frac{2\sqrt{2}}{3}|1^+\underline{\overline{18}}[35]\rangle-\tfrac{1}{3}|1^+\underline{\overline{18}}[280]\rangle\ , \tag{3.33}$$

$$|1^+,\underline{\overline{18}}^*\rangle=\tfrac{1}{3}|1^+\underline{\overline{18}}[35]\rangle+\frac{2\sqrt{2}}{3}|1^+\underline{\overline{18}}[280]\rangle\ . \tag{3.34}$$

The wave functions for the states in $[15]\otimes[\overline{21}]$ are obtained from Eqs. (3.29)–(3.34) by interchanging quarks and antiquarks. The reader should keep in mind that these two sets of states will be mixed by the available decay channels. The magnetic-interaction energies tabulated in Table III complete the information necessary to calculate $Q^2\overline{Q}^2$ masses. The recipe for masses is reviewed in paper I and discussed in detail in Ref. 4.

IV. RECOUPLING TO DECAY CHANNELS

The decays of $Q^2\overline{Q}^2$ S-wave mesons are expected to be dominated by S-wave $(Q\overline{Q})(Q\overline{Q})$ channels into which they simply "fall apart" or "dissociate".[1] To estimate decay amplitudes, we transform from the $Q^2\overline{Q}^2$ basis of the previous section to a $(Q\overline{Q})(Q\overline{Q})$ basis. The techniques for constructing these recoupling matrices are well known. The cases of interest to us have not been written down previously (to our knowledge) because the Q^2 and $\overline{Q}^2$ channels are separately unphysical.

The calculation is conveniently performed in two steps: First one recouples the color and the spin; then one recouples the flavor, remembering the mixing induced by diagonalizing n_s.

A. Color and spin

The crossing matrices for color and spin separately are given in Tables IV and V. The recoupling of H_g eigenstates is obtained by combining the wave functions of the last section with the recoupling coefficients in Tables IV and V. The results, presented in Tables IV, V, and VI of paper I, express $Q^2\overline{Q}^2$ eigenstates as linear combinations of $Q\overline{Q}$ mesons of definite color and spin.[9]

Notice that the lightest $Q^2\overline{Q}^2$ state of each total spin recouples most strongly to the two lightest $Q\overline{Q}$ states available (see Tables IV–VI, Ref. 1). For example, the lightest 0^+, $|0^+\underline{9}\rangle$ is predominantly two color-singlet pseudoscalars. This is to be

TABLE IV. Crossing matrix for color.

	$\|(Q\bar{Q})^1(Q\bar{Q})^1\rangle^1$	$\|(Q\bar{Q})^8(Q\bar{Q})^8\rangle^1$
$\|(Q^2)^6(\bar{Q}^2)^{\bar{6}}\rangle^1$	$(\frac{2}{3})^{1/2}$	$-(\frac{1}{3})^{1/2}$
$\|(Q^2)^{\bar{3}}(\bar{Q}^2)^3\rangle^1$	$(\frac{1}{3})^{1/2}$	$+(\frac{2}{3})^{1/2}$

expected in order that H_g be minimized. It has important phenomenological consequences.

B. Flavor

The SU(3) crossing matrix is given in Table VI. Were it not for the mixing of multiplets induced by the strange-quark mass, the flavor-recoupling coefficients could be read off Table VI, together with any standard table of SU(3) isoscalar factors.[10] We have rewritten mixed states as linear combinations of $SU(3)_f$ eigenstates in Table II. Tables II and VI, together with the isoscalar factors of de Swart[10] enable us to construct the relevant recoupling coefficients. The results were given in Figs. 6–8 of Ref. 1. These provide a check on the magic mixing described in Table II; the number of s and $\bar{s}$ quarks is conserved in fall-apart decays.

V. GENERAL FEATURES OF MULTIQUARK SPECTRA

Hadron masses are roughly linear in the number of quarks plus antiquarks. Without dynamical input, we would expect often to find a multiquark state ($Q^m\bar{Q}^n$) less massive than the ordinary ($Q\bar{Q}$ or Q^3) hadrons into which it might decay. No narrow exotics are known. This was the problem posed at the outset of Ref. 1 which originally led us into this subject. Studying $Q^2\bar{Q}^2$ mesons, we found no narrow exotics. Instead we found broad heavy exotics and a low-spin cryptoexotic nonet. Many states in the nonet contain "hidden" $s\bar{s}$ pairs which make them heavy and coupled to strange particles.

This is a general phenomenon. The systematics of the color-spin interaction is such that for any $Q^n\bar{Q}^m$ sector of the quark-bag model (1) the lightest multiplets are generally not exotic; (2) they are low-spin cryptoexotic states with many s or $\bar{s}$ quarks, making them heavier and coupled predominantly to obscure channels (involving hyperons, K's, η's, etc.). The exceptions occur when n or m is a multiple of 3, where the situation is more complicated.

The argument goes as follows: According to Eq. (3.9) (and Rule 1), maximizing $C_6(Q)$ and $C_6(\bar{Q})$ minimizes the mass. This generally selects $SU(6)_{cs}$ representations in which the maximum number of quarks (or antiquarks) are symmetrized.[11] "Horizontal" $SU(6)_{cs}$ Young tableaux are favored. Antisymmetry requires the $SU(3)_f$ Young tableau to be conjugate—as antisymmetric as possible. In fact, the largest $SU(6)_{cs}$ Casimir operator is associated with a $\underline{3}$ or $\underline{\bar{3}}$ of flavor (with the noted exception).[11] Exotics are associated with less symmetric $SU(6)_{cs}$ representations and are consequently heavier.

Furthermore, a nonet made from many quarks and antiquarks ($n+m>4$) must contain triplets of Q's and $\bar{Q}$'s coupled to flavor singlets [$\mathcal{A}(uds)$ or $\mathcal{A}(\bar{u}\bar{d}\bar{s})$]. Consequently the states generally contain hidden $s\bar{s}$ pairs. This elevates them to higher mass (not only are the s quarks heavier but also their magnetic interactions are weaker) and dictates that they couple (fall apart) predominantly into strange particles.

Rule 2 selects small representations of total color-spin. These generally contain only low-spin states. The lightest nonet will have low spin.

TABLE V. Crossing matrices for spin.

	$\|(Q\bar{Q})^3(Q\bar{Q})^3\rangle^1$	$\|(Q\bar{Q})^1(Q\bar{Q})^1\rangle^1$
$\|(Q^2)^3(\bar{Q}^2)^3\rangle^1$	$(\frac{1}{4})^{1/2}$	$(\frac{3}{4})^{1/2}$
$\|(Q^2)^1(\bar{Q}^2)^1\rangle^1$	$(\frac{3}{4})^{1/2}$	$-(\frac{1}{4})^{1/2}$

	$\|(Q\bar{Q})^3(Q\bar{Q})^3\rangle^3$	$\|(Q\bar{Q})^3(Q\bar{Q})^1\rangle^3$	$\|(Q\bar{Q})^1(Q\bar{Q})^3\rangle^3$
$\|(Q^2)^3(\bar{Q}^2)^3\rangle^3$	0	$(\frac{1}{2})^{1/2}$	$-(\frac{1}{2})^{1/2}$
$\|(Q^2)^3(\bar{Q}^2)^1\rangle^3$	$(\frac{1}{2})^{1/2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$\|(Q^2)^1(\bar{Q}^2)^3\rangle^3$	$(\frac{1}{2})^{1/2}$	$\frac{1}{2}$	$\frac{1}{2}$

	$\|(Q\bar{Q})^3(Q\bar{Q})^3\rangle^5$
$\|(Q^2)^3(\bar{Q}^2)^3\rangle^5$	(1)

TABLE VI. Crossing matrices for flavor.

Singlet

$$\begin{array}{l c} & \begin{array}{cc} |(Q\overline{Q})^1(Q\overline{Q})^1\rangle^1 & |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^1 \end{array} \\ \begin{array}{l} |(Q^2)^6(\overline{Q}^2)^6\rangle^1 \\ |(Q^2)^{\overline{3}}(\overline{Q}^2)^3\rangle^1 \end{array} & \begin{bmatrix} (\frac{2}{3})^{1/2} & -(\frac{1}{3})^{1/2} \\ (\frac{1}{3})^{1/2} & (\frac{2}{3})^{1/2} \end{bmatrix} \end{array}$$

Octet

$$\begin{array}{l c} & \begin{array}{cccc} |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^{8f} & |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^{8d} & |(Q\overline{Q})^8(Q\overline{Q})^1\rangle^8 & |(Q\overline{Q})^1(Q\overline{Q})^8\rangle^8 \end{array} \\ \begin{array}{l} |(Q^2)^6(\overline{Q}^2)^{\overline{6}}\rangle^8 \\ |(Q^2)^{\overline{3}}(\overline{Q}^2)^3\rangle^8 \\ |(Q^2)^6(\overline{Q}^2)^3\rangle^8 \\ |(Q^2)^{\overline{3}}(Q^2)^{\overline{6}}\rangle^8 \end{array} & \begin{bmatrix} 0 & (\frac{1}{6})^{1/2} & (\frac{5}{12})^{1/2} & (\frac{5}{12})^{1/2} \\ 0 & (\frac{5}{6})^{1/2} & (\frac{1}{12})^{1/2} & -(\frac{1}{12})^{1/2} \\ -(\frac{1}{2})^{1/2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ (\frac{1}{2})^{1/2} & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Antidecuplet

$$\begin{array}{lc} & |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^{\overline{10}} \\ |(Q^2)^{\overline{3}}(\overline{Q}^2)^{\overline{6}}\rangle^{\overline{10}} & (1) \end{array}$$

Decuplet

$$\begin{array}{lc} & |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^{10} \\ |(Q^2)^6(\overline{Q}^2)^3\rangle^{10} & (1) \end{array}$$

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$$\begin{array}{lc} & |(Q\overline{Q})^8(Q\overline{Q})^8\rangle^{27} \\ |(Q^2)^6(\overline{Q}^2)^{\overline{6}}\rangle^{27} & (1) \end{array}$$

Higher-spin nonets are heavier.

The $Q^2\overline{Q}^2$ states of Ref. 1 are a case in point. Another important example are the $Q^4\overline{Q}$ baryons.[12] The lightest multiplet is a $\frac{1}{2}^-$ nonet of the form

$$B_{ij} = \mathcal{Q}(uds)Q_i\overline{Q}_j \ . \tag{5.1}$$

Were the strange quark massless, the nonet would lie at about 1200 MeV—embarrassingly low. Unlike a Q^3 nonet, the $S=0$ isodoublet contains an $s\overline{s}$ pair making it heavier (1600–1700 MeV) and coupled predominantly to channels like $K\Sigma$, ηN, and $K\Lambda$, not πN as one would naively expect. Truly exotic $Q^4\overline{Q}$ baryons are more massive.

The moral of this section is that simple gluon exchange may provide a systematic dynamical explanation for the failure to observe relatively narrow multiquark hadrons in familiar channels. Perhaps the light 0^+ mesons are $Q^2\overline{Q}^2$ states. Further tests of the model await detailed calculations of other channels (e.g., $Q^4\overline{Q}$, $Q^3\overline{Q}^3, \ldots$) and more experimental input.

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APPENDIX

It is particularly easy to calculate the quadratic Casimir operator C_n for a given representation of SU(n) if the SU(2) content of the representation is known. [SU(n) always contains an SU(2) subgroup.] As an example, consider SU(6):

$$C_6 = \sum_{\mu=1}^{35} \alpha_\mu{}^2 \ .$$

Take the trace of C_6 in the representation $[R]$ of SU(6) (N_R-dimensional)

$$N_R C_6[R] = \sum_{\mu=1}^{35} \mathrm{Tr}\alpha_\mu{}^2 \ . \tag{A1}$$

All traces are identical:

$$C_6[R] = \frac{35}{N_R}\mathrm{Tr}\alpha_\nu{}^2 \ . \tag{A2}$$

Choose ν to be the third generator of the SU(2) subgroup

$$C_6[R] = \frac{70}{3N_R}\,\mathrm{Tr}\sigma_z{}^2 \tag{A3}$$

$$= \frac{70}{3N_R}\sum_j d_j\,\mathrm{Tr}_j\sigma_z{}^2\,. \tag{A4}$$

The sum on j covers all SU(3)×SU(2) representations contained in $[R]$. d_j is the dimension of the SU(3) representation. An analogous calculation for SU(2) itself gives

$$\mathrm{Tr}_s\sigma_z{}^2 = \frac{4(2s+1)s(s+1)}{3} \tag{A5}$$

in a representation with spin s. Finally, then,

$$C_6[R] = \frac{280}{9N_R}\sum_j d_j(2s_j+1)s_j(s_j+1)\,. \tag{A6}$$

The analogous calculation for SU(3) makes use of the isospin subgroup:

$$C_3(R) = \frac{32}{3N_R}\sum_k (2I_k+1)I_k(I_k+1)\,, \tag{A7}$$

where the sum extends over all isospin multiplets in the given SU(3) representation.

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[1]R. L. Jaffe, preceding paper, Phys. Rev. D 15, 267 (1977).

[2]A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D 9, 3471 (1974).

[3]A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, Phys. Rev. D 10, 2599 (1974).

[4]T. DeGrand, R. L. Jaffe, K. Johnson, and J. Kiskis, Phys. Rev. D 12, 2060 (1975).

[5]Hypercharge ±1 members would mix in the $SU(3)_f$ limit to diagonalize SU(3) parity. Since this is presumably violated in decays, we ignore it here. See footnote of Ref. 1 for an explicit example.

[6]F. Hund, *Linienspektren und Periodisches System der Elemente* (Springer, Berlin, 1927), p. 124.

[7]Often the limitations of the $SU(3)_c \times SU(2)$ content of an $SU(6)_{cs}$ representation prevent the color and spin Casimir operators from playing any role at all. For example, in [21]⊗[21] the $J_{tot}=2$ state occurs in [405] while one $J_{tot}=0$ state occurs in [1]. The difference between $C_6(1)$ and $C_6(405)$ overwhelms the effect of the difference in J_{tot}.

[8]J. Mandula, private communication.

[9]It is essential that the phases of Tables IV and V be defined consistently with those of the mixed states of Sec. III. We have checked this internal consistency.

[10]J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

[11]Only $SU(6)_{cs}$ tableaux of three columns or less can be combined with $SU(3)_f$ tableaux antisymmetrically. Denote a tableau (n_1, n_2, n_3) by the number of boxes in each column. For a fixed number of boxes, N, the dimension (and hence C_6) is maximized when the differences $n_i - n_j$ are minimized. The exception is the case $N=3n$; the tableau (n,n,n) has smaller dimension than $(n+1, n, n-1)$. For example, if $n=1$, the tableau (1,1,1) has $d=56$ while (2,1,0) has $d=70$. Consequently the lowest flavor configuration in $Q^3\bar{Q}^3$ is $\underline{8}\otimes\underline{8}$ rather than $\underline{1}\otimes\underline{1}$.

[12]R. L. Jaffe, invited paper presented at the Topical Conference on Baryon Resonances, 1976, Oxford [Stanford Linear Accelerator Center Report No. SLAC-PUB-1774 (unpublished)].