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Small-scale structure of spacetime as the origin of the gravitational constant*

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We suggest a means of incorporating the Planck length as a fundamental constant determined by the structure of spacetime. In this scheme the spacetime symmetry group is taken as the de Sitter group with radius of curvature proportional to the Planck length, but we argue that the effects of this curvature are not apparent at elementary-particle length scales. To make the connection with gravity we present a formulation of the gravitational interaction as the gauge theory of the de Sitter group. We obtain an action containing the Einstein-Cartan action, but in which the dimensional gravitational constant G appears naturally as the consequence of the commutation relation of the de Sitter group. We also find a cosmological term, higher-derivative couplings of the gravitational field, and a propagating torsion field, which we discuss.

I. INTRODUCTION

The speed of light, Planck's constant, and the gravitational constant, G, provide a set of dimensional numbers in terms of which all physical quantities with dimensions of mass, length, and time can be expressed as dimensionless numbers. But, unlike the other two, the gravitational constant is tied to a particular dynamical theory. This situation is unsatisfactory in several respects, not the least of which is the problem it poses for quantum gravity. The length scale determined by G is the Planck length, approximately 10^{-33} cm. It is on such a length scale that the quantum corrections to gravity become important, but unfortunately these corrections are infinite and cannot be removed by the customary procedure of renormalization.[1] In fact, the nonrenormalizability of Einstein's theory is directly traceable to the presence of a dimensional gravitational coupling constant, and it does not seem as though the plight of quantum gravity can be remedied until the origin of the gravitational constant is understood.

Ideally, we should hope to link the existence of a fundamental length scale, the Planck length, with the structure of spacetime. There is actually an obvious way to achieve this, which has gone unnoticed because it appears at first sight to be absurd. Consider the commutation relations of the generators of the Poincaré group,

$$\begin{aligned} &[J_{ab}, J_{cd}] = i(\Sigma^e_{abc} J_{ed} + \Sigma^e_{abd} J_{ce}) , \\ &[J_{ab}, P_c] = i\Sigma^e_{abc} P_e , \\ &[P_a, P_b] = 0 , \end{aligned} \tag{1.1}$$

with

$$\Sigma^e_{abc} = \eta_{ac}\eta^e_b - \eta_{bc}\eta^e_a , \tag{1.2}$$

where η_{ab} is the Minkowski metric with signature $(+,-,-,-)$.

The P_a are the translation generators and the J_{ab} are the Lorentz rotation generators. In the absence of a fundamental length scale the commutator of two translations must vanish because P_a has dimensions of inverse length. Conversely, if this commutator does not vanish it must determine a length scale. For example,

$$[P_a, P_b] = \frac{-i}{\kappa^2} J_{ab} , \tag{1.3}$$

where κ has dimensions of length. The commutation relations can now be written as

$$[J_{\alpha\beta}, J_{\gamma\delta}] = i(\Sigma^\epsilon_{\alpha\beta\gamma} J_{\epsilon\delta} + \Sigma^\epsilon_{\alpha\beta\delta} J_{\gamma\epsilon}) , \tag{1.4}$$

where the indices $\alpha, \beta, \gamma, \delta, \epsilon$ run from 1 through 5 and

$$\Sigma^\epsilon_{\alpha\beta\delta} = \eta_{\alpha\gamma}\eta^\epsilon_\beta - \eta_{\beta\gamma}\eta^\epsilon_\alpha , \tag{1.5}$$

in which the diagonal unit metric $\eta_{\alpha\beta}$ has the signature $(+,-,-,-,-)$ and is a generalization of the Minkowski metric. The commutation relations (1.4) include that of (1.3) if we make the identification

$$P_a = \frac{1}{\kappa} J_{5a} , \quad J_{ab} = J_{\alpha=a,\beta=b} . \tag{1.6}$$

The group defined by (1.4) is a well-known alternative to the Poincaré group, and is known as the de Sitter group.[2] It is the natural extension of the Poincaré group in that the algebra is isomorphic to that of the Dirac matrices and their commutator, $-\frac{1}{4}i[\gamma_a, \gamma_b] = \sigma_{ab}$, if κ is taken to be unity. It is clear that the de Sitter algebra of (1.4) and (1.5) is just that of the group O(4,1). [If the sign of κ^2 is changed we have the group O(3,2).] This group is the spacetime symmetry group of a homogenous, isotropic, expanding universe. Such universes have a constant, uniform spacetime curvature with radius of curvature proportional to κ. Observationally, therefore, κ would appear to be on the order of the radius of the universe, so that the Poincaré group is a good approximation for any but cosmological purposes.

However, we now wish to suggest an alternative interpretation of the constant κ appearing in (1.3), namely *that κ be proportional to the Planck length.* Not only do translations no longer commute, but their commutator is exceedingly large. In fact, if a Lorentz 4-vector is transported around a closed spacetime loop with the dimensions of the Planck length, it will be rotated about an axis perpendicular to the loop by an angle on the order of one radian. We will have more to say on the desirability of this suggestion later, but for the moment let us consider how it can be reconciled with our customary notions of how 4-vectors behave under translations.

Suppose we transport a Lorentz vector around a closed spacetime loop with the dimensions of, say, an elementary particle. The elementary-particle diameter is approximately 10^{-13} cm, so that the ratio of length scales involved is 10^{20}. Hence the Lorentz vector will be rotated in a complete circle approximately 10^{20} times. The precise direction of the 4-vector after these rotations depends critically on the *exact* spacetime path chosen. If this path is changed so that it differs from the original by deviations on the order of 10^{-33} cm, the final direction of the 4-vector may be quite different. But we cannot *physically distinguish* two spacetime paths which differ by such a small amount. For measurements which probe to 10^{-13} cm there will be approximately 10^{20} indistinguishable but significantly different paths. If physical measurements consist of averages over regions of 10^{-13} cm then we may easily imagine that the net rotation of the measured 4-vector will be zero. In other words, while spacetime is very strongly curved at very short distances, this curvature is such that its effects are averaged to zero at larger distances.

The above argument provides an intuitive way of understanding how the translation group can be non-Abelian at the Planck length and yet appear Abelian at larger length scales. But there is a more convincing argument which runs as follows: Because translations no longer commute, the corresponding quantum numbers, the components of 4-momentum, are not continuous but must occur in multiples of a basic unit. This unit is proportional to the Planck mass and is very large. If, for example, we probe the structure of an elementary particle down to the Planck length we would find that its 4-momentum is always a multiple of the basic unit. But the direction of this 4-momentum is ill defined because each time the particle moves a distance proportional to the Planck length, which it will do in a time interval on the order of the Planck time, the 4-momentum will be rotated to a new direction. Thus, although the particle's actual 4-momentum is very large, its direction oscillates with a frequency on the order of the inverse Planck time, so that the average 4-momentum is zero. Or rather, we should say that the average 4-momentum depends on additional external macroscopic forces whose effect is superimposed on a microscopic random walk. In this respect the situation is analogous to the motion of molecules within some volume of a fluid. The observed motion of the fluid depends on the external macroscopic forces.

The fluctuations envisioned above are not those of quantum theory, although it is probable that quantum fluctuations of a particle's 4-momentum are necessary for the randomness which is inherent in the above description. In particular, the Heisenberg uncertainty principle will ensure that physical measurements *are* averages over distances large compared to the Planck length until elementary particles can be accelerated to energies comparable to the Planck mass. At this point the quantum fluctuations of the gravitational field and hence of the spacetime geometry become important. Ultimately it is these fluctuations which ensure the indistinguishability of spacetime paths differing by 10^{-33} cm, and the necessity of considering physical quantities as averages over regions of spacetime larger than this. The quantum fluctuations of spacetime geometry have been the source of most of the previous speculations on the structure of spacetime at the Planck length. In particular, Wheeler[3] has speculated that the topology of spacetime is subject to fluctuations at such small distances. Presumably any such quantum structure would be an addition to that described here.

But even if we accept the notion that a physically measured 4-momentum is actually an average, we may still wonder how it is that the direction of this average momentum can remain fixed, given that the actual 4-momentum is rapidly oscillating. The answer to this query is as follows: Let us imagine that the fluctuations described above cause the traversal of an actual 4-momentum around closed spacetime loops. Whenever this happens, the direction of 4-momentum will change arbitrarily. But the rotation of a 4-vector is opposite for opposite-sense traversals of closed loops. Hence, the average 4-momentum will have the same direction as that of the actual 4-momentum (but not the same magnitude of course). This is subject to the provision that fluctuations are random and cause traversals of spacetime loops equally in both senses. This will be satisfied *statistically*.

Heuristically we may express the degree to which an average vector may change direction with time as the ratio of the number of traversals of loops in one sense to the number of traversals in the opposite sense. This ratio will differ from unity by an amount proportional to the inverse square root of the number of traversals. For example, at the elementary-particle length scale the ratio differs from unity by 10^{-10} and oscillations of the average 4-momentum are negligible. At smaller length scales these oscillations may be appreciable until at the Planck length they are isotropic. Below the Planck length the average 4-momentum is indistinguishable from the actual 4-momentum which is discrete.

We may summarize this argument by an uncertainty principle. Knowledge of the direction of a 4-vector precludes knowledge of its magnitude, since only the *average* magnitude is observable in this case. Conversely, knowledge of the magnitude precludes knowledge of the direction because the 4-vector will appear to be oscillating randomly in all directions. The gravitational constant is a measure of this uncertainty. Although this uncertainty is very large rather than very small, we hope to have shown that it is only apparent at the Planck length and that the observed Poincaré invariance could be the large-scale manifestation of an entirely different small-scale spacetime symmetry.

In order to clarify the means by which we have chosen to introduce the constant κ into physics we wish to point out that the transition from the Poincaré to the de Sitter group is analogous to the transition from the Galilean to the Lorentz group. The Galilean group is characterized by the 3-rotation generators J_{ij} and the boost generators K_i. The commutator of J_{ij} with itself and K_i are those appropriate to a 3-tensor and 3-vector, respectively. The commutator of K_i with itself is

$$[K_i, K_j] = 0\,. \tag{1.7}$$

The transition to the Lorentz group is achieved by the replacement

$$[K_i, K_j] = \frac{i}{c^2} J_{ij}\,, \tag{1.8}$$

which introduces into physics a new fundamental constant c, the speed of light. Of course, one cannot immediately deduce from (1.8) that c *is* the speed of light. This remains to be shown. In the same way, while we have introduced the constant κ into physics by means of the commutator (1.3), and have declared this constant to be the Planck constant, we have yet to show that κ, so defined, has anything to do with gravity. Until this connection is made, our choice of κ is arbitrary.

This connection between the structure of spacetime as envisioned above and the theory of gravity is the problem that concerns us for the remainder of this paper. We approach this problem by attempting to interpret gravity as the gauge theory of the de Sitter group.

The concept of the gravitational interaction as the gauge theory of the Poincaré group was first put forward by Kibble,[4] who extended the idea, due to Utiyama,[5] of gravity as the gauge theory of the Lorentz group. Various treatments of gravity as a gauge theory have since appeared.[6] We choose to present a slightly different formulation of gauge theories of spacetime symmetries and apply it to the Poincaré group in Sec. II. The key feature of our approach is the separation of the differential and matrix representations of the group. This separation is an unsatisfactory feature of our approach in some respects, but it does have the virtue of simplicity. In Sec. III we attempt to extend these ideas to the de Sitter group. In this way we are able to connect the constant κ, introduced into the commutation relations of the generators of spacetime symmetries, with the gravitational constant. Conversely, knowing the gravitational constant, the assumption that the gravitational interaction is the gauge theory of a de Sitter group allows us to fix the constant κ appearing in the commutation relations of this group. This allows the reinterpretation of the gravitational constant as a natural fundamental dimensional constant intimately connected with the structure of spacetime.

II. GAUGE FIELDS FOR SPACETIME SYMMETRIES

The spacetime symmetry group connects inertial coordinate systems. Thus for the Poincaré group the transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu + \lambda^\mu{}_\nu x^\nu \tag{2.1}$$

connects infinitesimally separated inertial frames for constant parameters $\xi^\mu, \lambda^\mu{}_\nu$, where $\lambda_{\mu\nu}$ is antisymmetric and the Minkowski metric $\eta_{\mu\nu}$ is used to raise and lower indices. If $\psi(x)$ is a scalar field, a function of spacetime, then (2.1) will induce the following transformation on $\psi(x)$:

$$\psi(x) \to \psi'(x) = [1 - (\xi_\mu + \lambda_\mu{}^\nu x_\nu)\partial^\mu]\psi(x)\,. \tag{2.2}$$

We may write this as

$$\psi'(x) = [1 + i(\xi_\mu P^\mu + \tfrac{1}{2}\lambda_{\mu\nu} J^{\mu\nu})]\psi(x)\,, \tag{2.3}$$

where P^μ and $J^{\mu\nu}$ act as linear differential operators on ψ such that they form a representation of the symmetry group. For the Poincaré group we have

$$P^\mu \equiv i\partial^\mu ,$$
$$J^{\mu\nu} \equiv i(x^\nu\partial^\mu - x^\mu\partial^\nu) . \tag{2.4}$$

In this way we consider spacetime symmetries as transformations on the fields rather than on the coordinates. If we now take the parameters ξ_μ and $\lambda_{\mu\nu}$ to be spacetime dependent, then, since P^μ and $J^{\mu\nu}$ are represented only as linear differential operators, Eq. (2.3) becomes

$$\psi'(x) = [1 - \epsilon_\nu(x)\partial^\nu]\psi(x) , \tag{2.5}$$

where, in this case,

$$\epsilon_\nu(x) = \xi_\nu(x) + \lambda_\nu{}^\rho(x)x_\rho . \tag{2.6}$$

Under the transformation (2.5), $\partial_\mu\psi$ transforms as

$$(\partial_\mu\psi)' = [1 - \epsilon_\nu(x)\partial^\nu](\partial_\mu\psi) - \epsilon^\nu{}_{,\mu}(\partial_\nu\psi) , \tag{2.7}$$

which is the transformation of a covariant vector field under the coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) . \tag{2.8}$$

(2.8) is just the generalization of (2.1) to spacetime-dependent ξ^μ and $\lambda^\mu{}_\nu$, and is the most general infinitesimal coordinate transformation. We are thus led to replace the Minkowski metric $\eta_{\mu\nu}$ by the spacetime-dependent metric $g_{\mu\nu}(x)$ defined as a tensor under general coordinate transformations. Since ψ, $\partial_\mu\psi$, and $g_{\mu\nu}$ are covariant tensors, we may construct a scalar Lagrangian for ψ from these quantities alone.

Notice that, for the scalar field, the independence of ξ^μ and $\lambda^\mu{}_\nu$ is lost when we take these to be spacetime dependent because both become absorbed into $\epsilon^\mu(x)$. There is no longer any reference to the original spacetime symmetry group which has been replaced by the 4-parameter group of general coordinate transformations. The relevance of the original symmetry group becomes apparent when we consider fields $\psi(x)$ which are not scalars but are in some nontrivial representation of the symmetry group. In this case P^μ and $J^{\mu\nu}$ will be represented by *matrix* operators on $\psi(x)$ in addition to their representation as linear differential operators. We will denote the matrix representation by $\hat{P}^\mu$ and $\hat{J}^{\mu\nu}$. But $\hat{P}^\mu$ and $\hat{J}^{\mu\nu}$ are spacetime-dependent matrices because their commutation relations involve $g_{\mu\nu}(x)$ rather than $\eta_{\mu\nu}$. To circumvent this we introduce the tetrad field $e^a{}_\mu(x)$ and its inverse $e_a{}^\mu(x)$ satisfying

$$g_{\mu\nu} = e^a{}_\mu e_{a\nu} , \quad \eta_{ab} = e_{a\mu}e_b{}^\nu . \tag{2.9}$$

We can now form the spacetime-independent matrices $\hat{P}^a$ and $\hat{J}^{ab}$. For example,

$$\hat{P}^a = e^a{}_\mu \hat{P}^\mu . \tag{2.10}$$

Now, for the Poincaré group $\hat{P}^a$ is always zero, which is a consequence of the Abelian nature of the translation subgroup. In the following we will therefore set $\hat{P}^a$ to zero and return to the more general case in the next section. If we now take $\psi(x)$ to be in some nontrivial representation of the Poincaré group, its transformation under this group for spacetime-dependent group parameters is

$$\psi \to \psi' = [1 - \epsilon_\nu(x)\partial^\nu + \rho(x)]\psi , \tag{2.11}$$

where the matrix gauge parameter $\rho(x)$ is (recall that $\hat{P}^a = 0$)

$$\rho(x) = \tfrac{1}{2}\lambda_{ab}(x)\hat{J}^{ab} , \tag{2.12}$$

in which λ_{ab} is $e_{a\mu}e_{b\nu}\lambda^{\mu\nu}$. Evidently the independence of $\lambda_{\mu\nu}(x)$ from $\epsilon_\nu(x)$ has been maintained in this case. The transformation of $(\partial_\mu\psi)$ corresponding to (2.11) is

$$(\partial_\mu\psi)' = [1 - \epsilon_\nu(x)\partial^\nu + i\rho(x)](\partial_\mu\psi) - \epsilon^\nu{}_{,\mu}(\partial_\nu\psi) + i\rho_{,\mu}\psi . \tag{2.13}$$

Because of the last term in (2.13) this is no longer the transformation of a vector field. We must introduce a gauge potential $\Gamma_\mu(x)$ which acts as a matrix operator on $\psi(x)$ such that

$$(\partial_\mu + i\Gamma_\mu)\psi \tag{2.14}$$

is a vector field. This condition determines the transformation of Γ_μ to be

$$\Gamma'_\mu - \Gamma_\mu = \delta\Gamma_\mu = -\Gamma_{\mu,\nu}\epsilon^\nu - \epsilon^\nu{}_{,\mu}\Gamma_\nu - \rho_{,\mu} - i[\rho, \Gamma_\mu] . \tag{2.15}$$

Γ_μ is clearly not a vector field because of the last two terms of (2.15), which we will call the *gauge* transformation of Γ_μ. Thus

$$\delta_G\Gamma_\mu = -\rho_{,\mu} - i[\rho, \Gamma_\mu] . \tag{2.16}$$

We can now construct a scalar Lagrangian from ψ, $(\partial_\mu + i\Gamma_\mu)\psi$, and $g_{\mu\nu}$ but not yet from Γ_μ and its derivatives alone.

By analogy with the gauge transformations of internal symmetries, we take (2.16) to define a covariant derivative of the gauge parameter,

$$D_\mu\rho = \rho_{,\mu} + i[\rho, \Gamma_\mu] . \tag{2.17}$$

By taking the commutator

$$[D_\nu, D_\mu]\rho = -i[\rho, \Gamma_{\mu\nu}] \tag{2.18}$$

we obtain the *gauge-covariant* field-strength tensor $\Gamma_{\mu\nu}$

$$\Gamma_{\mu\nu} = \Gamma_{\nu,\mu} - \Gamma_{\mu,\nu} - i[\Gamma_\mu, \Gamma_\nu] . \tag{2.19}$$

Let us now expand Γ_μ in terms of the group generators $\hat{P}^a$ and $\hat{J}^{ab}$. However, we have noted already that $\hat{P}^a$ is zero for the Poincaré group, so that we may write

$$\Gamma_\mu = \tfrac{1}{2}\omega_{\mu ab}\hat{J}^{ab}. \tag{2.20}$$

Substituting (2.20) into (2.16) and (2.19) we find that

$$\delta_G\omega_{\mu ab} = -\lambda_{ab,\mu} + \omega_{\mu a}{}^c\lambda_{cb} + \omega_{\mu b}{}^c\lambda_{ac} = -D_\mu\lambda_{ab}\,, \tag{2.21}$$

which is the usual covariant derivative of λ_{ab}, and

$$\Gamma_{\mu\nu} = \tfrac{1}{2}R_{\mu\nu ab}\hat{J}^{ab}\,, \tag{2.22}$$

which defines the curvature tensor

$$R_{\mu\nu ab} = \omega_{\nu ab,\mu} - \omega_{\mu ab,\nu} - \omega_{\mu a}{}^c\omega_{\nu cb} + \omega_{\nu a}{}^c\omega_{\mu cb}\,. \tag{2.23}$$

By contracting on all indices we can form the *gauge-invariant* curvature scalar

$$R = e^{a\mu}e^{b\nu}R_{\mu\nu ab}\,, \tag{2.24}$$

and hence we can construct the Einstein-Cartan action

$$I = -\frac{1}{16\pi G}\int d^4x\, eR\,, \tag{2.25}$$

where e is the determinant of $e_{a\mu}$. Although this is certainly the simplest action, it requires the introduction of the dimensional constant G and there is no suggestion of its origin. We shall see how this defect can be remedied in the next section.

III. GAUGE THEORY OF THE DE SITTER GROUP

We will now suppose that infinitesimally separated inertial frames are connected by the transformation

$$x_\mu \to x'_\mu = x_\mu + \frac{1}{\kappa}(x^2)^{1/2}\xi_\mu + \lambda_\mu{}^\nu x_\nu\,. \tag{3.1}$$

For a *scalar* field $\psi(x)$ this induces the transformation

$$\psi \to \psi' = [1 + i(\xi_\mu P^\mu + \tfrac{1}{2}\lambda_{\mu\nu}J^{\mu\nu})]\psi\,, \tag{3.2}$$

where P^μ and $J^{\mu\nu}$ are represented by the linear differential operators

$$P^\mu \equiv \frac{i}{\kappa}(x^2)^{1/2}\partial^\mu\,,\qquad J^{\mu\nu} \equiv i(x^\nu\partial^\mu - x^\mu\partial^\nu)\,. \tag{3.3}$$

These operators satisfy the commutation relations of the de Sitter group, (1.4)–(1.6). We will not give a further justification of (3.1)–(3.3). We refer the reader to the discussion of Sec. I.

As in Sec. II, we now generalize ξ_μ and $\lambda_{\mu\nu}$ to arbitrary spacetime functions. (3.1) and (3.2) become

$$x'_\mu = x_\mu + \epsilon_\mu(x)\,, \tag{3.4}$$

$$\psi'(x) = [1 - \epsilon_\nu(x)\partial^\nu]\psi(x)\,, \tag{3.5}$$

respectively, where in this case

$$\epsilon_\mu(x) = \frac{1}{\kappa}(x^2)^{1/2}\xi_\mu(x) + \lambda_\mu{}^\nu(x)x_\nu\,. \tag{3.6}$$

As before, the independence of $\lambda_{\mu\nu}$ and ξ_μ is lost when these parameters become spacetime dependent. However, if $\psi(x)$ is not a scalar field but is in some nontrivial representation of the de Sitter group, then (3.5) must be replaced by

$$\psi'(x) = [1 - \epsilon_\nu(x)\partial^\nu + i\rho(x)]\psi(x)\,, \tag{3.7}$$

where the gauge parameter $\rho(x)$ is now

$$\rho(x) = \xi_a\hat{P}^a + \tfrac{1}{2}\lambda_{ab}\hat{J}^{ab}. \tag{3.8}$$

In this expression ξ_a is $e_{a\mu}\xi^\mu$ and similarly for the other quantities. $\hat{P}^a$ is *not* zero for the de Sitter group and must therefore be included in (3.8). We may rewrite (3.8) in the more compact notation

$$\rho(x) = \tfrac{1}{2}\lambda_{\alpha\beta}\hat{J}^{\alpha\beta}. \tag{3.9}$$

$\hat{J}^{\alpha\beta}$ is defined as in (1.6) and $\lambda_{\alpha\beta}$ is

$$\lambda_{5a} = -\frac{1}{\kappa}\xi_a\,,\quad \lambda_{ab} = \lambda_{\alpha=a,\beta=b}\,. \tag{3.10}$$

Again $(\partial_\mu\psi)$ does not transform as a vector field under the transformation (3.7) on ψ, and we must introduce the gauge potential Γ_μ. We can expand this potential in terms of the group generators

$$\Gamma_\mu = h_{a\mu}\hat{P}^a + \tfrac{1}{2}\omega_{\mu ab}\hat{J}^{ab} = \tfrac{1}{2}\omega_{\mu\alpha\beta}\hat{J}^{\alpha\beta}\,, \tag{3.11}$$

in which the components of $\omega_{\mu\alpha\beta}$ are

$$\omega_{\mu 5a} = -\frac{1}{\kappa}h_{a\mu}\,,\quad \omega_{\mu ab} = \omega_{\mu,\alpha=a,\beta=b}\,. \tag{3.12}$$

The significance of the new potential $h_{a\mu}$ will concern us shortly. For the moment, we proceed as in Sec. II to obtain the *gauge* transformation of $\omega_{\mu\alpha\beta}$ (reall that the gauge transformation is in addition to the tensor transformation and is the reason that $\omega_{\mu ab}$ and $h_{a\mu}$ are not true tensors). This transformation is

$$\delta_G\omega_{\mu\alpha\beta} = -\partial_\mu\lambda_{\alpha\beta} + \omega_{\mu\alpha}{}^\gamma\lambda_{\gamma\beta} + \omega_{\mu\beta}{}^\gamma\lambda_{\alpha\gamma} = -\mathfrak{D}_\mu\lambda_{\alpha\beta}\,, \tag{3.13}$$

which defines the gauge-covariant derivative $\mathfrak{D}_\mu$ by analogy with (2.21). In component form (3.13) is

$$\delta_G h_{a\mu} = -D_\mu\xi_a - \lambda_a{}^b h_{b\mu}\,,\qquad \delta_G\omega_{\mu ab} = -D_\mu\lambda_{ab} - \frac{1}{\kappa^2}(\xi_a h_{b\mu} - \xi_b h_{a\mu})\,, \tag{3.14}$$

where D_μ is the Poincaré covariant derivative of (2.21). Notice the new κ-dependent term in $\delta_G\omega_{\mu ab}$. Continuing as in Sec. II we find the field-strength tensor $\Gamma_{\mu\nu}$ from the commutator of $\mathfrak{D}_\mu$ and $\mathfrak{D}_\nu$. This is

$$\Gamma_{\mu\nu}=\tfrac{1}{2}C_{\mu\nu\alpha\beta}\hat{J}^{\alpha\beta}\ , \tag{3.15}$$

where

$$C_{\mu\nu\alpha\beta}=\omega_{\nu\alpha\beta,\mu}-\omega_{\mu\alpha\beta,\nu}+\omega_{\nu\alpha}{}^{\gamma}\omega_{\mu\gamma\beta}-\omega_{\mu\alpha}{}^{\gamma}\omega_{\nu\gamma\beta}\ . \tag{3.16}$$

In component notation this is

$$C_{\mu\nu ab}=R_{\mu\nu ab}-\frac{1}{\kappa^2}\,(h_{a\mu}h_{b\nu}-h_{a\nu}h_{b\mu})\ , \tag{3.17}$$

$$C_{\mu\nu 5a}=-\frac{2}{\kappa}\,S_{\mu\nu a}(h)=-\frac{1}{\kappa}\,(D_\mu h_{a\nu}-D_\nu h_{a\mu})\ . \tag{3.18}$$

We may form the obvious scalar action from $C_{\mu\nu\alpha\beta}$, $\eta_{\alpha\beta}$, and $g_{\mu\nu}$

$$I=\tfrac{1}{4}\int d^4x\ \sqrt{g}\ g^{\mu\lambda}g^{\nu\rho}C_{\mu\nu\alpha\beta}C_{\lambda\rho}{}^{\alpha\beta}\ , \tag{3.19}$$

where g is the determinant of $g_{\mu\nu}$. But before we can make sense of this action we must find an interpretation for the gauge potential $h_{a\mu}$. An attractive choice is

$$h_{a\mu}\overset{?}{=}e_{a\mu} \tag{3.20}$$

because this links the spacetime metric to the gauge theory of gravity as well as avoiding the introduction any new fields. In other treatments of gravity as the gauge theory of the *Poincaré* group the choice of (3.20) is customary.[4,6] That is, the tetrad field $e_{a\mu}$ is usually taken to be the gauge potential of the translation subgroup. (Recall that in the approach adopted here we were not required to introduce gauge potentials for the translation subgroup of the Poincaré group.) With this choice the field $S_{\mu\nu a}(e)$ defined in (3.18) is just the torsion of spacetime. We may therefore think of $C_{\mu\nu\alpha\beta}$ as a generalized curvature tensor explicitly incorporating both the curvature and torsion.

However, it does not appear that the choice of (3.20) can be made for the de Sitter group. The reason is that the metric $g_{\mu\nu}$ constructed from $e_{a\mu}$ would not be invariant under the gauge transformation

$$\delta_G e_{a\mu}=-D_\mu\xi_a\ , \tag{3.21}$$

so that $g_{\mu\nu}$ would not be a good tensor. It is not clear to the author what relation replaces (3.20) for the de Sitter group. It is possible that we should retain the metric $g_{\mu\nu}$ and the gauge potential $h_{a\mu}$ as independent fields. We would then have separate Euler-Lagrange equations for $g_{\mu\nu}$ and $\omega_{\mu\alpha\beta}$. We have not been able to resolve this issue, so we will sidestep it by asking a simpler question: Is there a suitable limit in which the action of (3.19) contains the usual Einstein-Cartan action of (2.25)?

We will suppose that a good approximation for gravitational phenomena on length scales much larger than 10^{-33} cm is obtained by the identification of (3.20). Of course this avoids the question of how gravitational phenomena appear at the Planck length, but it will be sufficient for our purposes. Using (3.20) in (3.19) and expanding $C_{\mu\nu\alpha\beta}$ in terms of its components we find

$$I=\int d^4x\, e\left(\frac{6}{\kappa^4}-\frac{1}{\kappa^2}R-\frac{1}{\kappa^2}S_{\mu\nu a}S^{\mu\nu a}+\frac{1}{4}R_{\mu\nu ab}R^{\mu\nu ab}\right). \tag{3.22}$$

The feature of this action that we wish to stress is that it contains the Einstein-Cartan action. In fact, we can make the identification

$$\kappa^2=16\pi G\ . \tag{3.23}$$

Thus the fundamental constant κ appears *naturally* as the dimensional constant associated with the gravitational interaction if the small-scale structure of spacetime is governed by the de Sitter group with parameter κ. [We should point out, however, that the precise identification of (3.23) depends on the factor multiplying the action in (3.19). We chose $\frac{1}{4}$ by analogy with Yang-Mills theories.]

It is not clear how seriously we should take the remaining terms in (3.22). Nevertheless we shall consider them in turn.

(1) The term $R_{\mu\nu ab}R^{\mu\nu ab}$ is a "higher derivative" coupling originally considered by Weyl. Such terms improve the high-energy behavior of quantum gravity, and they have been accorded recent attention because of this fact.[7] This term does not affect the low-energy, hence large-scale, properties of the gravitation field.

(2) The term $S_{\mu\nu a}S^{\mu\nu a}$ will provide a propagating torsion field. This is already a consequence of the $R_{\mu\nu ab}R^{\mu\nu ab}$ term because $\omega_{\mu ab}$ contains torsion implicitly.[8] The torsion of spacetime couples to the matter spin density, but on a macroscopic scale the spin density usually averages to zero.[6] For this reason quadratic torsion terms might be expected to have little effect on macroscopic gravitational phenomena. Other authors have shown that if a propagating torsion field exists there must be a fundamental length associated with it,[9] and conversely that the detection of torsion waves would provide evidence for the existence of a fundamental length in the geometry of spacetime.[10] It is clear from (3.22) that we would expect this fundamental length to be none other than the Planck length.

(3) $6/\kappa^4$ is a cosmological term and causes us the most trouble, since it implies a very large negative energy density proportional to $1/\kappa^2$. This is

physically unacceptable for the classical theory. We could simply subtract $6/\kappa^4$ from the action, but there is an alternative explanation for this term, From the point of view of quantum gravity, the absence of a cosmological term is unnatural because quantum corrections will generate such a term even if it is initially absent. In fact, the vacuum energy density generated by quantum fluctuations is expected to be proportional to $1/\kappa^2$ and *positive*. We might hope that this will cancel the large negative energy density implies by $6/\kappa^2$, leaving an almost vanishing cosomological constant.

IV. SUMMARY

Our main purpose has been to introduce a fundamental *dimensional* constant into physics in a *natural* way that does not depend, *a priori*, on any particular dynamical theory. We chose to do this by replacing the Poincaré group by the de Sitter group as the symmetry group of spacetime. Because translations do not commute for the de Sitter group, their commutator provides us with a dimensional constant, κ, which can be interpreted as the constant radius of curvature of spacetime independent of the presence of matter. For this reason κ is usually taken to be on the order of the radius of the universe. However, we allowed ourselves to be guided by the idea that the constant κ has some connection with the theory of gravity. Thus we formulated the gravitational interaction as the gauge theory of the de Sitter group by analogy with the usual formulation of gravity as the gauge theory of the Poincaré or Lorentz groups. We were able to find an action for this gauge theory in a limit for which Poincaré invariance is assumed to be valid. This action contains the Einstein-Cartan action, where the gravitational constant G is proportional to κ^2. We take this as an indication that the small-scale structure of spacetime is that of the de Sitter group, and that the scale of this structure sets the scale of the gravitational coupling constant G. This scale is that of the Planck length, 10^{-33} cm. This implies a large uniform spacetime curvature. In Sec. I we have argued that the observational effects of this curvature will not be apparent for measurements at length scales much larger than 10^{-33} cm. If this idea is tenable, it is possible that the gravitational interaction is more intimately connected with the structure of spacetime than has hitherto been appreciated. It is interesting that the Planck length also sets the scale for quantum gravitational effects and one expectes quantum fluctuations of the spacetime geometry to appear at this length scale. However, the quantitative treatment of these fluctuations is not amenable to conventional techniques of quantum field theory. Any additional small-scale spacetime strucutre such as that suggested here could be relevant to this problem.

The action obtained in Sec. III contains terms in addition to the Einstein-Cartan action. Although we have provided arguments suggesting that each of these terms can be tolerated, we wish to stress that we have not yet obtained a completely satisfactory theory.

Note added in proof. S. W. MacDowell and F. Mansouri [Phys. Ref. Lett. 38, 739 (1977)] have recently treated gravity (and supergravity) as the gauge theory of the de Sitter group. However, they do not require full invariance of the action under the gauge transformations (3.14). Consequently, they are able to avoid the introduction of a gauge potential $h_{a\mu}$ distinct from $e_{a\mu}$. Their action for gravity (in our notation) is $I=\int d^4x\, \epsilon^{\mu\nu\lambda\rho}\epsilon^{abcd} \times C_{\mu\nu ab} C_{\lambda\rho cd}$. The R^2 terms are just the Gauss-Bonnet invariant $\epsilon^{\mu\nu\lambda\rho}\epsilon^{abcd}R_{\mu\nu ab}R_{\lambda\rho cd}$ and can be ignored. Hence the de Sitter group may be contracted to the Poincaré group giving the usual Einstein-Cartan action and these authors do not consider the length scale determined by the de Sitter group as fundamental.

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