Einstein A and B coefficients for a black hole

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By quantum calculations in a classical background geometry, Hawking has shown that an isolated black hole emits thermal radiation spontaneously. Starting from Hawking's expectation value for the number of quanta emitted per mode, and using methods from statistical thermodynamics, one of us calculated earlier the probability distribution for the number of quanta per mode outgoing from a black hole placed in a thermal radiation bath. By the same methods we show here that this probability is not simply the combination of that for Hawking's spontaneous emission and that for pure scattering. From this we infer the existence of stimulated emission in *all* modes, even those which do not superradiate. We derive the probability that m quanta go out in a given mode when precisely n are incident. It satisfies a symmetry condition originally given by Hartle and Hawking for a special case. For all modes the average number of outgoing quanta contains a contribution from stimulated emission which shows up as a negative contribution to the effective absorptivity Γ. The situation is analogous to that for opacity in the theory of radiative transport. Superradiance occurs for modes in which the negative contribution dominates the pure absorptivity. We identify the Einstein A and B coefficients for a black hole. The B coefficients satisfy the usual relation from atomic physics with the role of degeneracy factor played by the exponential of black-hole entropy. This agrees with the statistical interpretation of this quantity in terms of internal black-hole configurations. The relation between the B coefficients suggests time reversibility of the radiative aspect of a black hole. This supports Hawking's view that a black hole and a white hole are essentially the same thing.

I. INTRODUCTION

Hawking[1] has shown that a Schwarzschild black hole formed by collapse spontaneously emits quanta which reach infinity with a thermal spectrum. More precisely, the *mean number* of quanta emitted in a given mode [see (4) below] is just what would be emitted by a gray body whose "absorptivity" Γ for the given mode coincides with that of the black hole, and which has a well-defined temperature $T_{\rm bh}$. This radiation temperature is of the same form as the thermodynamic black-hole temperature introduced earlier by Bekenstein.[2,3] Hawking's result has been confirmed by Parker,[4] Wald,[5] Boulware,[6] Gerlach,[7] and others. Hawking[1] has given reasons for believing his result is also valid for a rotating and charged Kerr black hole. We shall assume this is so in what follows.

From Hawking's result and the principle of maximal entropy, one of us[8] derived the following expression for the *probability* of spontaneous emission of n quanta in a given mode:

$$p_{\rm sp}(n)=(1-e^{-\beta})e^{-\beta n}, \tag{1}$$

where

$$e^{-\beta}\equiv\Gamma(e^{x}-1+\Gamma)^{-1}, \tag{2}$$

$$x\equiv(\hbar\omega-\hbar m\Omega-\epsilon\Phi)T_{\rm bh}{}^{-1}. \tag{3}$$

Here ω is the frequency of the mode, m its azimuthal quantum number, ϵ its electric charge, Ω and Φ the rotational frequency and electrical potential of the hole, respectively. Probabilities for different modes are independent. When expanded in $(1-\Gamma)e^{-x}$, (1) gives a series which for the Schwarzschild case agrees[8] with a quantum result implicit in Ref. 5. In the approximation $\Gamma=1$, (1) agrees with quantum results of Parker[4] and Hawking[9] (we comment on this approximation in Sec. IV). The *mean* number of quanta corresponding to the distribution (1) is

$$\langle n\rangle_{\rm sp}=\Gamma(e^{x}-1)^{-1}, \tag{4}$$

which is the original Hawking result.[1]

For a black hole immersed in a blackbody radiation bath, of temperature T, one assumes[8] that the mean number $\langle n\rangle_o$ of quanta in a given outgoing mode is

$$\langle n\rangle_o=\langle n\rangle_{\rm sp}+(1-\Gamma)(e^{y}-1)^{-1}, \tag{5}$$

where

$$y=\hbar\omega/T. \tag{6}$$

Here the last term represents the fraction of incident blackbody quanta returned outward by the black hole: $1-\Gamma$ is the "reflectivity" of the hole, and $(e^{y}-1)^{-1}$ is the Planck mean number of quanta in a blackbody radiation mode. The principle of maximal entropy yields the following probability that n quanta are outgoing in the given mode[8]:

$$p_o(n)=(1-e^{-\gamma})e^{-\gamma n}, \tag{7}$$

where

$$(e^{\gamma}-1)^{-1}=\Gamma(e^{x}-1)^{-1}+(1-\Gamma)(e^{y}-1)^{-1}. \tag{8}$$

It is shown in Ref. 8 that (7) is required if the second law of thermodynamics is to be obeyed for all T, and if thermodynamic equilibrium is to be possible for $T = T_{bh}$ at least for a Schwarzschild hole.

It would seem desirable to derive these and further results on quantum scattering from a black hole by formal quantum field-theoretic calculations. However, even when performed in a classical background (semiclassical approximation), such calculations are beset by ambiguities and technical difficulties. The history of atomic physics contains similar examples, and suggests that we might profit here from a statistical thermodynamics approach to the problem which would be expected to circumvent the difficulties while yielding a plethora of useful results with little effort. This is our approach in the present paper. Our single starting point is formula (7), which follows directly from the well-verified Hawking result (4) by application of the principle of maximal entropy, as explained above. All our results then follow from standard statistical reasoning together with an additional assumption (see below) made in analogy with atomic physics. It is hoped that quantum field-theoretic checks of all our results will eventually become possible.

In this paper we show that (7) does not have the form expected if ordinary scattering is the only source of reflectivity of the hole. We trace the discrepancy to the existence of stimulated emission which may be described by Einstein A and B coefficients. Zel'dovich[10] and Starobinski[11] earlier inferred from the phenomenon of superradiance[12] the existence of stimulated emission in modes with $x < 0$. This view is supported by a field-theoretic calculation of Wald.[13]

But stimulated emission at late time in modes with $x > 0$ had not been expected. However, we show that stimulated emission must be present at late times, even for $x > 0$. It is required by thermodynamics. Stimulated emission contributes negatively to the absorptivity Γ. When $x < 0$ this contribution is large enough to turn Γ negative and give rise to superradiance.

In Sec. II we show that spontaneous emission and ordinary scattering regarded as independent cannot yield the probability distribution (7) required by thermodynamics. This implies the existence of stimulated emission, even in nonsuperradiant modes. In Sec. III we derive the probability distribution $p(m|n)$ for m outgoing quanta when precisely n are incident upon the hole. This $p(m|n)$ obeys a symmetry condition originally obtained in a special case by Hartle and Hawking[14] by a direct quantum treatment, a condition which makes detailed balance with a radiation bath possible. In Sec. IV we calculate the conditional *mean number* of emerging quanta when precisely n are incident. Subtracting from it the mean number scattered in terms of a one-quantum scattering coefficient $1-\Gamma_0$, and assuming that the remainder (stimulated plus spontaneous emission) is proportional to $n+1$ as in atomic physics, we find the relation between the pure absorption coefficient Γ_0 and the effective absorptivity Γ corrected for stimulated emission. The relation is identical to that in the theory of radiative transport. In Sec. V we identify the Einstein A and B coefficients for a black hole. The B coefficients satisfy the Einstein condition with the degeneracy factor of a black-hole state being given by the exponential of the black-hole entropy.[2,3] This supports the interpretation of black-hole entropy in terms of internal black-hole configuration.[2,3,8] The connection between the B coefficients implies time reversibility, and thus supports Hawking's view[15] that black holes and white holes are essentially the same thing. In the Appendix we treat the case of fermions, calculate $p(m|n)$, show that it obeys the same symmetry relation as for bosons, and demonstrate how the Pauli principle suppresses emission by the hole.

II. NECESSITY OF STIMULATED EMISSION

We consider a rotating, charged black hole immersed in a radiation bath of temperature T. We assume the cavity in which the radiation is enclosed to be nonrotating and at zero electric potential. Then the probability that there are n quanta incident upon the hole in a mode specified by ω, m, and ϵ is given by the blackbody normalized probability distribution

$$p_{bb}(n) = (1-e^{-y})e^{-yn}, \tag{9}$$

where y is defined by (6). The probability distribution p_o for the number of quanta in the corresponding outgoing mode is given by (7), whose derivation is outlined in Ref. 8.

We shall now regard $p_o(n)$ in (7) as being composed of the probability distribution for spontaneous emission p_{sp} given by (1), and some independent probability distribution p_r for the blackbody quanta returned outward by the hole:

$$p_o(n) = \sum_{k=0}^{n} p_{sp}(n-k)p_r(k). \tag{10}$$

Comparing this with (7) we get by term-by-term inspection

$$p_r(n) = \begin{cases} (1-e^{-\gamma})(1-e^{-\beta})^{-1}, & n=0 \\ (1-e^{-\gamma})(1-e^{-\beta})^{-1}(1-e^{\gamma-\beta})e^{-\gamma n}, & n \geq 1. \end{cases} \tag{11}$$

We now ask if this form of $p_r(n)$ is what would be expected if ordinary scattering (by the potential barrier surrounding the hole) were the only process responsible for returning the incident blackbody quanta outward. The probability distribution for scattering is calculated as follows: Let Γ_o be the probability that a single quantum incident on the hole is absorbed; $1-\Gamma_o$ is the probability that it will be scattered outward. Then the probability $p_s(n)$ that n quanta are scattered outward is constructed by combining the probability distribution p_{bb} for the incident quanta with the binomial probability distribution with parameter Γ_o:

$$p_s(n)=\sum_{m=n}^{\infty}(1-e^{-y})e^{-ym}\frac{m!}{n!\,(n-m)!}\Gamma_o^{m-n}(1-\Gamma_o)^n\,. \tag{12}$$

One finds that

$$\begin{aligned} p_s(n)&=(1-e^{-\delta})e^{-\delta n}\,,\\ e^{-\delta}&\equiv(1-\Gamma_o)(e^y-\Gamma_o)^{-1}. \end{aligned} \tag{13}$$

The sum can be verified by expanding out (13) in powers of $\Gamma_o e^{-y}$ (see Appendix of Ref. 8).

We see that $p_r(n)$ and $p_s(n)$ are not of the same form regardless of the choice of Γ_o. What can cause the discrepancy? There can be no doubt about the correctness of $p_o(n)$ in (7). As shown in Ref. 8 any other distribution would clash with the second law. The construction scheme (10) is the only possible one if the spontaneous emission is independent of the "scattering"; and the scattering probability distribution $p_s(n)$ is the only one possible if the quanta are indistinguishable bosons. Thus we are forced to conclude that scattering is not the only process that returns incident quanta outward. We are led to believe that stimulated emission must also be present. To be sure, the suggestion is not new. Zel'dovich[10] and Starobinski[11] interpreted the phenomenon of black-hole superradiance[12] as implying stimulated emission in modes with $\omega<m\Omega$ (uncharged black hole). This view is verified by a quantum calculation of Wald.[13] However, we are led by thermodynamics to expect stimulated emission in all modes, even in those for which there is no classical superradiance. We now proceed to extract the contribution of stimulated emission to $p_o(n)$, and to the definition of the Einstein coefficients.

III. THE CONDITIONAL PROBABILITY $p(m|n)$

We define $p(m|n)$ as the conditional probability that a rotating charged Kerr black hole emits exactly m quanta in a given mode when precisely n are incident in the corresponding mode. The $p(m|n)$ includes spontaneous and stimulated emission as well as ordinary scattering. Effectively $p(m|n)$ is the square of the matrix element $(n+m,m|n\rangle$, in the notation of Ref. 4, which has been calculated under various restrictions for the case $n=0$ by Parker,[4] Wald,[5] and Hawking.[9] It is a crucial assumption of our arguments that $p(m|n)$ does not depend on the black hole's environment, but only on its intrinsic properties.

For a black hole immersed in a blackbody bath we clearly have

$$p_o(m)=\sum_{n=0}^{\infty}p(m|n)\,p_{bb}(n)\,. \tag{14}$$

Substituting $p_o(m)$ from (7) and $p_{bb}(n)$ from (9), and writing $z=e^{-y}$ we see that

$$\sum_{n=0}^{\infty}p(m|n)z^n=(1-e^{-\gamma(z)})(1-z)^{-1}e^{-m\gamma(z)}\,, \tag{15}$$

where $\gamma(z)$ is defined by (8). Expanding the right-hand side in powers of z we find[16] that

$$p(m|n)=\frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{1-e^{-\gamma(z)}}{1-z}e^{-m\gamma(z)}\right)_{z=0}. \tag{16}$$

The expansion considered is natural, as it is about the point $T=0$. Writing $A=\Gamma(e^x-1)^{-1}+\Gamma-1$. and changing to the new independent variable $W=(1-z)(1-\Gamma)^{-1}$ we have

$$p(m|n)=\frac{1}{n!}\frac{(-1)^n}{(1-\Gamma)^{n+1}}\frac{\partial^n}{\partial W^n}\left\{\frac{(1+AW)^m}{[1+(1+A)W]^{m+1}}\right\}_{W=(1-\Gamma)^{-1}}. \tag{17}$$

Performing the differentiation and simplifying, we get

$$p(m|n)=\frac{(e^x-1)e^{xn}\Gamma^{m+n}}{(e^x-1+\Gamma)^{n+m+1}}\sum_{k=0}^{\min(n,m)}\frac{(-1)^k(m+n-k)!}{k!(n-k)!(m-k)!}\times\left[1-2\frac{1-\Gamma}{\Gamma^2}(\cosh x-1)\right]^k, \tag{18}$$

where the sum extends up to m or n, whichever is smaller.

This expression is so complicated that it is worthwhile checking it in a simple limit. For $n=0$ we get

$$\begin{aligned} p(m|0)&=(e^x-1)(e^x-1+\Gamma)^{-m-1}\Gamma^m\\ &=(1-e^{-\beta})e^{-\beta m}\,, \end{aligned} \tag{19}$$

the last step following from the definition of $e^{-\beta}$, Eq. (2). We thus find that $p(m|0)=p_{sp}(m)$, as would be expected: The conditional probability for emission when nothing is incident is just the probability of spontaneous emission.

We notice that the sum in (18) is symmetric in

m and n. It follows that in general

$$e^{-xn}p(m|n)=e^{-xm}p(n|m). \qquad (20)$$

This result generalizes that obtained by Hartle and Hawking[14] by a Feynman integral calculation for scalar quanta and a Schwarzschild hole:

$$p(1|0)=e^{-x}p(0|1). \qquad (21)$$

Condition (20) makes detailed balance possible between the black hole and a cavity at the same temperature and electric potential, and rotating with the same frequency, insofar as the nonsuperradiant modes are concerned. We can see this by noticing that the radiation in such a cavity has the blackbody distribution

$$p_{bb}(n)=(1-e^{-x})e^{-xn}, \qquad (22)$$

where x is given by (3). Thus (20) assures us that the probability that n quanta are incident and m are outgoing equals that for m to be incident and n to be outgoing. This means there is detailed balance and thermodynamic equilibrium between hole and cavity through the given mode. However, this conclusion does not apply to a superradiant mode, one with $x<0$. In such a case (22) is not the probability distribution of the incident quanta as it is not normalizable: $\sum_{n=0}^{\infty}p_{bb}(n)\to\infty$. Hence (20) does not have a simple physical interpretation in this case. It has earlier[8] been argued that a black hole cannot reach thermodynamic equilibrium with the cavity through superradiant modes.

IV. EFFECT OF STIMULATED EMISSION ON THE ABSORPTIVITY

Let us calculate the mean number of quanta outgoing in a given mode if precisely n are incident in the corresponding ingoing mode:

$$\langle m\rangle_n=\sum_{m=0}^{\infty}mp(m|n)$$
$$=e^{xn}\sum_{m=0}^{\infty}p(n|m)me^{-xm}, \qquad (23)$$

where use has been made of (20) in the last step. It follows that

$$\langle m\rangle_n=e^{xn}\left[u\frac{\partial}{\partial u}\sum_{m=0}^{\infty}p(n|m)u^m\right]_{u=e^{-x}}. \qquad (24)$$

But from (15) we see that

$$\sum_{m=0}^{\infty}p(n|m)u^m=(1-e^{-\gamma})(1-u)^{-1}e^{-\gamma n}, \qquad (25)$$

where according to (8)

$$e^{-\gamma}=\{1+[\Gamma(e^x-1)^{-1}+(1-\Gamma)(u^{-1}-1)^{-1}]^{-1}\}^{-1}. \qquad (26)$$

Substituting in (24) and performing the differentiation[17] we get

$$\langle m\rangle_n=\Gamma(e^x-1)^{-1}+n(1-\Gamma). \qquad (27)$$

This result is reasonable: It includes the contribution (4) from spontaneous emission, and shows that on the average a fraction $1-\Gamma$ of the incident n quanta is returned outward. Were we to average (27) over a blackbody distribution for n, we would recover (5), which was our starting point. But (27) is a stronger result than (5), independent as it is of the distribution for the incident quanta.

What is the contribution of pure scattering to (27)? Arguing as in Sec. II we write

$$\langle\tilde{m}\rangle_n=\sum_{m=0}^{n}m\frac{n!}{m!(n-m)!}\Gamma_o^{\,n-m}(1-\Gamma_o)^m$$
$$=n(1-\Gamma_o), \qquad (28)$$

where the tilde refers to scattering, and the factor multiplying m is the probability that precisely m of the n incident quanta are scattered. The sum is carried out by means of the binomial theorem. We know that the factor $1-\Gamma_o$ cannot equal $1-\Gamma$, for otherwise there would be no room in (27) for the stimulated emission we know must be present. Defining $\Delta\Gamma=\Gamma_o-\Gamma$ we can write (27) as

$$\langle m\rangle_n=\langle\tilde{m}\rangle_n+\Delta\Gamma\left(n+\frac{\Gamma}{\Delta\Gamma}\,\frac{1}{e^x-1}\right). \qquad (29)$$

The last term in (29) is the contribution to $\langle m\rangle_n$ from spontaneous plus stimulated emission. We shall take as our working hypothesis that this term is precisely proportional to $n+1$ as in atomic physics. Then it follows that

$$\Delta\Gamma=\Gamma(e^x-1)^{-1}, \qquad (30)$$

and thus that

$$\Gamma=\Gamma_o(1-e^{-x}). \qquad (31)$$

We recall that $1-\Gamma_o$ is the probability that an incident quantum will be scattered outward. Therefore $0<1-\Gamma_o<1$, and it follows that the pure absorption coefficient Γ_o is also between 0 and 1. What is the physical meaning of Γ? We see from (27) that $1-\Gamma$ is the effective reflectivity of the black hole for incident quanta; thus Γ is the effective absorptivity—effective in that it includes the effects of stimulated emission. It is clear that a measurement will reveal directly only Γ but not Γ_o. Likewise any classical calculation of the reflectivity of the hole would yield $1-\Gamma$ but not $1-\Gamma_o$ [large-quantum-number limit of (27)]. We observe that Γ is the sum of Γ_o and a negative contribution due to stimulated emission, $-\Gamma_o e^{-x}$, which is present for *all* modes. This had already been conjectured in Ref. 8 (footnote 14). For modes with $x>0$ the positive contribution dominates, $0<\Gamma<1$ and $1-\Gamma<1$. Therefore the black hole returns outward on the average fewer quanta than are incident. For modes with $x<0$ the stimulated emission contribution to Γ dominates, $\Gamma<0$ and $1-\Gamma>1$. Therefore the black hole returns outward on the average more quanta than are incident upon it. In the classical limit it amplifies radiation—it superradiates. In short, there is stimulated emission in *all* modes, but only for $x<0$ does it become strong enough to cause superradiance.

For superradiant modes formula (27) for the mean number of outgoing quanta by itself shows that there has to be stimulated emission (effective reflectivity $1-\Gamma$ greater than unity). The situation is different for nonsuperradiant modes when (27) by itself does not hint at anything apart from spontaneous emission and scattering. If $\langle m\rangle_n$ is all that is considered, one could conclude that there is no stimulated emission.[13] Only when considering the probability distribution (Sec. II) does one need to appeal to stimulated emission to explain the results.

The relation between stimulated emission and superradiance is analogous to that in atomic physics. There is stimulated emission for any atomic transition. But stimulated emission manifests itself as amplification only when a population inversion is present. The analogy can be made more precise. In the theory of radiative transport one deals with the pure absorption coefficient (opacity) κ_o of matter and with the effective absorption coefficient κ which takes into account the effects of stimulated emission. For a medium in thermodynamic equilibrium, the relation is[18]

$$\kappa=\kappa_o\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]. \tag{32}$$

Thus κ is positive, and radiation traversing the medium is attenuated. From (31) we see that the situation for a black-hole mode with $x>0$ is analogous; Γ is positive, and radiation is attenuated upon bouncing off the hole. When there is a population inversion in the medium, the negative exponent in (32) is replaced by a factor larger than unity.[18] Thus κ becomes negative even though κ_o is always positive. Radiation is amplified when traversing the medium. From (31) we see that this is precisely analogous to the case $x<0$ for a (rotating or charged) black hole; Γ becomes negative, and radiation is amplified when bounced off the hole (superradiance).

We see from (31) that Γ must always be smaller than unity due to stimulated emission. This makes the often-used approximation $\Gamma=1$ of doubtful validity when computing probabilities for nonvacuum initial states since it is equivalent to neglecting a relevant physical effect. This approximation could be justified only in the high-frequency limit when $\Gamma_o\to 1$ and $x\to\infty$.

For modes with x small in absolute value, (31) gives $\Gamma\approx x\Gamma_o=\Gamma_o T_{\mathrm{bh}}{}^{-1}\hbar(\omega-m\Omega)$ for the case $\Phi=0$.[19] We see that in the neighborhood of $\omega=m\Omega$, Γ is linear in $\omega-m\Omega$ since we do not expect the pure absorptivity Γ_o, a barrier tunneling factor, to be particularly sensitive to $\omega-m\Omega$. And indeed calculations of Starobinski[11] and Starobinski and Churilov[20] show just this linear behavior for Γ. One important consequence of this is that the spontaneous emission [see (4)] remains finite and positive for modes with $x=0$.

V. THE EINSTEIN COEFFICIENTS

We are now in a position to define the Einstein coefficients for a black hole. Let us substitute (28) and (30) in (29). We get

$$\langle m\rangle_n=(1-\Gamma_o)n+\Gamma(e^x-1)^{-1}(n+1). \tag{33}$$

The coefficient of 1 in the last pair of parentheses is the coefficient of spontaneous emission $A_\downarrow$. The coefficient of n in those same parentheses is the coefficient of stimulated emission $B_\downarrow$. The coefficient of absorption $B_\uparrow$ is the complement of the scattering coefficient $1-\Gamma_o$. Thus

$$A_\downarrow=B_\downarrow=\Gamma(e^x-1)^{-1}, \tag{34a}$$

$$B_\uparrow=\Gamma_o=\Gamma(1-e^{-x})^{-1}. \tag{34b}$$

These are coefficients *per mode*. Were we to desire coefficients on a *time rate* basis, we would have to multiply our coefficients by the number of outgoing modes of given ω, m, and ϵ per unit time.

The equality of $A_\downarrow$ and $B_\downarrow$ is, of course, a result of our assumption that the last term in (29) is proportional to $n+1$, and is in analogy with the results in atomic physics. It has been common practice in the literature[11] to identify $-\Gamma$ as the Einstein coefficient of stimulated emission for superradiant modes ($x<0$). We see from (34a) that this is incorrect; the coefficient is a factor $-(e^x-1)^{-1}$ larger, and this makes quite a difference when $|x|\ll 1$. As a result early estimates of the spontaneous emission by a Kerr black hole were far off the mark for modes near the transition point $\omega=m\Omega$.

We see from (34a) and (34b) that

$$B_\downarrow=B_\uparrow e^{-x}. \tag{35}$$

We shall see that this connection is in precise analogy to the Einstein relation for atomic physics,

$$B_{\downarrow} g_u = B_{\uparrow} g_l \,, \tag{36}$$

where g_u and g_l are degeneracy factors for the atomic upper and lower levels, which are connected by the emission of the quantum in question.

Let us consider a black hole of mass M, angular momentum L, and charge Q. It has a certain black-hole entropy $S_{\rm bh}(M, L, Q)$.[1-3,8,9] For a small variation in the parameters of the hole we have[8]

$$\Delta S_{\rm bh} = (\Delta M - \Omega\Delta L - \Phi\Delta Q) T_{\rm bh}{}^{-1}. \tag{37}$$

Comparing with the definition of x, (3), we see that for the mode in question

$$S_{\rm bh}(M, L, Q) - S_{\rm bh}(M - \hbar\omega, L - \hbar m, Q - \epsilon) = x \,. \tag{38}$$

Now, according to the statistical interpretation of black-hole entropy,[2,3,8] $\exp[S_{\rm bh}(M, Q, L)]$ is the number of internal black-hole configurations compatible with the external state labeled by M, L, and Q. It can thus be thought of as the degeneracy factor for this state. Substituting the x from (38) into (35) we see that it takes the form (36) with

$$g_l = \exp[S_{\rm bh}(M - \hbar\omega, L - \hbar m, Q - \epsilon)]$$

and

$$g_u = \exp[S_{\rm bh}(M, L, Q)] \,.$$

Thus the B coefficients for a black hole are related in the same way as those for an atom whose transition from the upper to the lower level gives rise to a quantum in the mode described by ω, m, and ϵ. Since a black hole can emit a number of quanta in the same mode, we can say it behaves like a thermal ensemble of atoms.

The conclusion that the Einstein relation is satisfied by the B coefficients of a black hole is important. It serves as a check on our entire approach, and in particular on our assumption about the form of the term in brackets in (29). It also brings out very vividly the reality of internal black-hole configurations, and shows that $\exp(S_{\rm bh})$ is the number of such configurations for definite M, L, and Q,[8] and not as is sometimes claimed,[15] the density of configurations per unit interval of mass, angular momentum, and charge.

In quantum theory, the Einstein relation (36) is a direct consequence of time reversibility of the system in question. We may thus argue that by analogy a black hole, or more precisely, that aspect of it which has to do with absorption and emission of radiation, is time reversible. Now the time reversal of a black hole is a white hole. We thus find support for Hawking's[15] view that insofar as radiative properties are concerned, a black hole and a white hole are the same thing.

APPENDIX

All our previous considerations referred to Bose quanta. Here we shall treat fermions starting from Hawking's[1] conclusion that a black hole spontaneously emits a mean number $\Gamma(e^x+1)^{-1}$ of quanta in each fermion mode, with x defined by (3) and Γ having a meaning of effective absorption coefficient. In analogy with (5) we assume that the mean number of outgoing quanta in the given mode is

$$\langle n\rangle_o = \Gamma(e^x+1)^{-1} + (1-\Gamma)(e^y+1)^{-1} \,, \tag{A1}$$

when the black hole finds itself in a thermal bath. The y is defined by (6) and we assume that the chemical potential of the fermions in the bath is zero. It is clear that (A1) is consistent with the Pauli exclusion principle since $\langle n\rangle_o \leq 1$.

We now define the conditional probability $p(m|n)$ in precise analogy with Sec. III, except that because of the Pauli principle only the values 0 and 1 are allowed for m and n. Clearly, we have by normalization

$$p(0|0) + p(1|0) = 1 \,, \tag{A2}$$

$$p(0|1) + p(1|1) = 1 \,. \tag{A3}$$

The *probability* that a quantum is incident in the given mode is $p_{\rm bb}(1) = (e^y+1)^{-1}$; the *probability* that no quantum is incident is $p_{\rm bb}(0) = 1 - p_{\rm bb}(1)$, so that the *mean* number incident is the well-known Fermi-Dirac result $(e^y+1)^{-1}$. The *probability* that one quantum is outgoing in the given mode is $p_o(1) = \langle n\rangle_o$, while the *probability* that no quantum is outgoing is $p_o(0) = 1 - p_o(1)$. Thus the *mean* number outgoing is precisely $\langle n\rangle_o$.

We can clearly write in analogy with (14)

$$p_o(1) = p(1|0)\,p_{\rm bb}(0) + p(1|1)\,p_{\rm bb}(1) \,, \tag{A4}$$

$$p_o(0) = p(0|0)\,p_{\rm bb}(0) + p(0|1)\,p_{\rm bb}(1) \,. \tag{A5}$$

Equations (A2)–(A5) for $p(m|n)$ are not linearly independent; they have many solutions. But the only physical solution is

$$p(0|0) = 1 - \Gamma(e^x+1)^{-1} \,, \tag{A6}$$

$$p(1|0) = \Gamma(e^x+1)^{-1} \,, \tag{A7}$$

$$p(0|1) = \Gamma(1+e^{-x})^{-1} \,, \tag{A8}$$

$$p(1|1) = \Gamma(e^x+1)^{-1} + 1 - \Gamma \,. \tag{A9}$$

Only for it are the $p(m|n)$ independent of y, that is, of the hole's environment. The $p(m|n)$ satisfy the same condition as for bosons:

$$p(m|n)\,e^{-nx} = p(n|m)\,e^{-mx} \tag{A10}$$

for $m, n = 0, 1$. In analogy with the discussion in Sec. III we conclude that this condition is what allows detailed balance of the black hole with a cavity having the same temperature, angular velocity, and electric potential. This conclusion holds also for modes with $x<0$ in this case; the normalization difficulties of the boson case have no counterpart here.

Can one understand the form of the $p(m|n)$ physically? The $p(1|0)$ just equals the mean number of quanta emitted spontaneously, which for fermions is just the probability for spontaneous emission of one quantum. This is precisely the definition of $p(1|0)$. The $p(0|0)$ is just the complement of $p(1|0)$ by normalization. The $p(1|1)$ is defined as the probability that one quantum is outgoing given that one is incident. This could be expressed as the probability for spontaneous emission of one quantum regardless of what happens to the ingoing quantum, $\Gamma(e^x+1)^{-1}$, plus the probability the ingoing quantum is returned outward regardless of whether the black hole emits or not, $1-\Gamma$, minus the probability the black hole emits *and* the incident quantum is returned outward. By the Pauli principle the last probability must vanish. Hence $p(1|1)=\Gamma(e^x+1)^{-1}+1-\Gamma$, which agrees with (A9). The $p(0|1)$ is just the complement of $p(1|1)$. We note that in the fermion case $1-\Gamma$ cannot be regarded as a scattering coefficient. In fact there is no such thing. Whether an incident quantum is scattered depends on whether the black hole emits.

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[16] By the definition of $\gamma(z)$ we see that the right-hand side of (15) is analytic at $z=0$. Thus the coefficients $p(m|n)$ of the Taylor expansion are determined uniquely by (16).
[17] In the differentiated (term-by-term) series in (24) the ratio of successive terms tends to $ue^x[1+(e^x-1)/\Gamma]^{-1}$ as $m\to\infty$ by virtue of (18). As shown by (31), $(e^x-1)/\Gamma$ is positive for all x. Thus the differentiated series converges absolutely in a neighborhood of $u=e^{-x}$, and hence converges uniformly within this neighborhood. Since the undifferentiated series converges at $u=e^{-x}$ by the same ratio test, differentiation of the series in (25) is a valid operation.
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