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Conformally flat solutions of Einstein–massless-scalar field equations *

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It is shown that there are only two distinct conformally flat solutions of the coupled Einstein scalar field equations. One of them has recently been given by Penney.

I. INTRODUCTION

When one puts certain symmetries on a space-time and assumes the Einstein field equations, the number of independent solutions is either one or two for a given source. Conformally flat space-times constitute an excellent example for this. It has been shown that there are only two independent conformally flat solutions of the Einstein-Maxwell field equations.[1,2] For perfect-fluid distribution, the Schwarzschild interior metric is the unique stationary conformally flat solution.[3] When the fluid admits an equation of state, Robertson-Walker metrics are the only conformally flat metrics.[4]

Recently, Penney[5] gave a simple solution of the conformally flat metric of the coupled massless scalar and the gravitational field equations. In this work, we prove that there are only two distinct solutions when the source is a massless scalar field. The other solutions are related to these by coordinate transformations.

II. FIELD EQUATIONS

Conformally flat space-time has a metric tensor[6]

$$g_{\mu\nu}=\phi^{-2}\eta_{\mu\nu}, \tag{1}$$

where ϕ is an arbitrary function of coordinates. The corresponding Ricci tensor is

$$R_{\mu\nu}=-2\phi^{-1}\phi_{,\mu\nu}+\eta_{\mu\nu}\phi^{-2}(3\Omega-\phi\Box\phi), \tag{2}$$

where

$$\Omega=\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu},$$
$$\Box\phi=\eta^{\mu\nu}\partial_\mu\partial_\nu\phi.$$

Conformal flatness implies both the vanishing of the conformal tensor and of

$$R_{\mu[\nu,\alpha]}=\tfrac{1}{6}g_{\mu[\nu}R_{,\alpha]}. \tag{3}$$

The field equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu}, \tag{4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. For a scalar field ψ, it reads

$$T_{\mu\nu}=\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}\Lambda g_{\mu\nu}, \tag{5}$$

with

$$\Lambda=g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}. \tag{6}$$

Equations (4) and (5) imply that

$$R_{\mu\nu}=-\kappa\psi_{,\mu}\psi_{,\nu}. \tag{7}$$

Using Eqs. (3) and (7), we get

$$\psi_{,\mu;\nu}=\gamma(4\psi_{,\mu}\psi_{,\nu}-\Lambda g_{\mu\nu}), \tag{8}$$

where

$$\gamma=\tfrac{1}{6}\Lambda^{-2}g^{\alpha\beta}\psi_{,\alpha}\Lambda_{,\beta}. \tag{9}$$

If $\Lambda=0$, instead of Eq. (8) we have

$$\psi_{,\mu;\nu}\psi_{,\alpha}-\psi_{,\mu;\alpha}\psi_{,\nu}=0. \tag{10}$$

For the solutions, we need the following identity:

$$\eta^{\alpha\beta}\phi_{,\mu\alpha}\phi_{,\nu\beta}=A\phi_{,\mu\nu}+B\eta_{\mu\nu}, \tag{11}$$

where

$$A=\phi^{-1}(2\phi\Box\phi-3\Omega),$$
$$B=\tfrac{1}{4}\phi^{-2}(3\Omega-\phi\Box\phi)(9\Omega-5\phi\Box\phi). \tag{12}$$

Equation (11) was obtained from

$$R_{\mu\alpha}R_{\nu}{}^{\alpha}=RR_{\mu\nu}, \tag{13}$$

which describes the case when the source is a scalar field. Finally, we write the Einstein-scalar field equations to be solved:

$$2\phi^{-1}\phi_{,\mu\nu}+\eta_{\mu\nu}\phi^{-2}(\phi\Box\phi-3\Omega)=\kappa\psi_{,\mu}\psi_{,\nu}, \tag{14}$$

$$(\phi^{-2}\eta^{\mu\nu}\psi_{,\mu})_{,\nu}=0. \tag{15}$$

III. SOLUTIONS

By use of Eqs. (8), (11), and (14) it is straightforward to show that

$$\psi_{,\mu}=a\phi_{,\mu} \tag{16}$$

and

$$\Omega_{,\mu}=c\phi_{,\mu}, \tag{17}$$

where a and c are functions of ϕ and its derivatives. Then Eq. (14) becomes

$$\phi_{,\mu\nu}=\kappa a^2\phi_{,\mu}\phi_{,\nu}+b\eta_{\mu\nu}, \tag{18}$$

where

$$b=\tfrac{1}{2}\phi^{-1}(3\Omega-\phi\Box\phi). \tag{19}$$

It is possible to find all solutions of Eq. (18). To this end, we separate these solutions into two classes

(a) $b=0$ case, or

$$\phi\Box\phi-3\Omega=0. \tag{20}$$

Equation (18) reduces to

$$\phi_{,\mu\nu}=\kappa a^2\phi_{,\mu}\phi_{,\nu}. \tag{21}$$

Contracting μ and ν with $\eta^{\mu\nu}$ in Eq. (21) and using Eq. (20), it follows that

$$\kappa a^2=3\phi^{-1}. \tag{22}$$

With this result, Eq. (21) becomes simpler, i.e.,

$$(\phi^{-2})_{,\mu\nu}=0, \tag{23}$$

or

$$\phi^{-2}=k_\mu x^\mu+a_0, \tag{24}$$

where k_μ is an arbitrary constant vector. The corresponding scalar field is

$$\psi=\left(\frac{3}{2\kappa}\right)^{1/2}\ln(a_0+k_\mu x^\mu), \tag{25}$$

where a_0 is an arbitrary constant which may be taken to be unity by a coordinate transformation. The solutions ϕ^{-2} and ψ are those obtained by Penney.

(b) $b\neq 0$ case. When we take one more derivative of Eq. (18) with respect to x^α, the left-hand side becomes completely symmetric with respect to the indices $\mu\nu\alpha$. Therefore, the right-hand side must also be symmetric. Hence we obtain

$$2\kappa a\phi_{,\mu}(a_{,\alpha}\phi_{,\nu}-a_{,\nu}\phi_{,\alpha})+\kappa a^2 b(\eta_{\mu\alpha}\phi_{,\nu}-\eta_{\mu\nu}\phi_{,\alpha})+b_{,\alpha}\eta_{\mu\nu}-b_{,\nu}\eta_{\mu\alpha}=0. \tag{26}$$

It is trivial to show that Eq. (26) reduces to the following set of equations

$$a_{,\mu}=\Omega^{-1}(\eta^{\alpha\beta}\phi_{,\alpha}a_{,\beta})\phi_{,\mu}, \tag{27}$$

$$b_{,\mu}=\kappa a^2 b\phi_{,\mu}. \tag{28}$$

With the help of Eq. (28), the partial-differential equations in (18) reduce to

$$(b^{-1}\phi_{,\mu})_{,\nu}=\eta_{\mu\nu} \tag{29}$$

or

$$\phi_{,\mu}=b(x_\mu+l_\mu), \tag{30}$$

where l_μ is a constant vector which may be removed by a translation. Equation (30) implies that ϕ is a function of s, where

$$s=(\eta_{\mu\nu}x^\mu x^\nu)^{1/2}. \tag{31}$$

Combining Eqs. (19) and (30), we obtain

$$\phi\ddot\phi-3\dot\phi^2+5s^{-1}\phi\dot\phi=0, \tag{32}$$

where dots on ϕ denote differentiation with respect to s. The general solution of (32) is

$$\phi^{-2}=b_0-\frac{a_0}{s^4}, \tag{33}$$

where a_0 and b_0 are arbitrary positive constants. The corresponding scalar field ψ is

$$\psi=\begin{cases}-\left(\frac{6}{\kappa}\right)^{1/2}\tanh^{-1}(\lambda^{-1}s^2), & \lambda^{-1}s^2<1\\ -\left(\frac{6}{\kappa}\right)^{1/2}\coth^{-1}(\lambda^{-1}s^2), & \lambda^{-1}s^2>1,\end{cases} \tag{34}$$

where

$$\lambda=\left(\frac{a_0}{b_0}\right)^{1/2}.$$

The space-time becomes flat when either a_0 or b_0 vanishes.

IV. DISCUSSION AND CONCLUSION

We have shown that there exist only two solutions of the conformally flat coupled Einstein–scalar-field equations. One of the solutions had been found previously by Penney. He also showed that $\rho_\mu=\phi^{-2}k_\mu$ is a Killing vector when k_μ, which appears in the metric, is a null vector. It is trivial to prove that any vector $\xi_\mu=\phi^{-2}n_\mu$ is a Killing vector, where n_μ is an arbitrary constant vector orthogonal to k_μ. Hence, under the infinitesimal transformation $x_\mu\to x'_\mu=x_\mu+\epsilon\xi_\mu$ the metric is form invariant.

There is yet another method to find the two metrics we have obtained in the preceding section. Bekenstein[7] has shown that if $g_{\mu\nu}$ and ψ form a solution of Einstein equations for a space-time containing an ordinary massless scalar field, then $\bar g_{\mu\nu}=\omega^{-2}g_{\mu\nu}$ and $\Phi=\zeta^{-1}\tanh\zeta\psi$ is the corresponding solution for a conformal scalar field Φ, where $\omega^{-1}=\cosh\zeta\psi$ and $\zeta=(\kappa/6)^{1/2}$. It is interesting to note that the vanishing of the energy-momentum tensor does not necessarily imply that the conformal scalar field Φ vanishes. Instead Φ satisfies[6]

$$(\Phi^{-1})_{|\mu\nu}=\gamma_0\bar g_{\mu\nu},$$

where γ_0 is an arbitrary constant. This equation is valid for types N, 0, or flat space-times. In our case, the type 0 and flat space are the same, because there is no source. Hence, if we assume that $\bar g_{\mu\nu}$ is a flat metric, there exist only two distinct solutions of Φ. Then using Bekenstein's theorem we obtain two distinct solutions for $g_{\mu\nu}$ and ψ. These solutions are nothing but those we obtained in the preceding section.

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[6]Comma and semicolon denote partial and covariant derivatives, respectively. In the conclusion a stroke denotes covariant derivative with respect to $\bar{g}_{\mu\nu}$.

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