Internal-symmetry structure of statistical bootstrap models

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A new formalism describing statistical decay of a fireball is developed. The final state of fireball decay is defined in the quantum-mechanical sense. The formalism allows strict conservation of internal symmetries.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

Recently many versions of the statistical bootstrap model (SBM)[1,2] have been derived and applied to fireball decay.[3-9] A common feature of the models is that according to the principle of reciprocity the decay proceeds in subsequent steps. A single decay step consists of the decay of a parent fireball into daughter fireballs (or ground-state particles). The probability of one particular type of decay step is proportional to the total phase space of the daughter fireballs. The various models differ from each other in the choice of quantum numbers of the fireballs (e.g., exotics included or excluded), in the number of daughter fireballs allowed in one decay step (e.g., full bootstrap or linear bootstrap), and also in the coupling of the parent fireball to the daughter fireballs.

When introducing internal symmetries definite transformation properties are attributed to the fireballs; e.g., isoscalar or isovector fireballs are introduced. On the other hand, because of the nature of the decay mechanism (i.e., subsequent decay steps with the probabilities and *not the amplitudes* specified) it is not possible to attribute definite transformation properties to the final state of the decay. Thus, strictly speaking, though isospin or SU(3) is conserved at each decay step, internal quantum numbers are not conserved in the whole decay process.

To check whether the symmetry is or is not violated in the whole decay process the branching ratios predicted by the model should be compared with predictions of the symmetry. It is clear that final states with only a few particles are most dangerous. For many particles there are many ways of coupling into a state of definite transformation properties. According to the Wigner-Eckart theorem different reduced matrix elements may correspond to each way of coupling, thus the number of parameters is large and there are only a few constraints (or none at all) due to the symmetry. Thus it may seem that the above symmetry violation is not a really important shortcoming of the models. However, it is by no means clear that the branching ratios of a really consistent model are close to those of a (usual) symmetry-violating model. In fact, the opposite is true in the specific example of a consistent model given below (see end of Sec. III).

We want to emphasize that symmetry violation is only a possibility and is not necessarily present in all the above models. E.g., in Ref. 5, where the coupling of the parent and daughter fireballs is according to the statistical isospin weights, SU(2) is not violated in the whole decay process.

In this note a new formalism describing the statistical decay of a fireball is introduced for the no-internal-quantum-number (NIQN) case (Sec. II). The final state of the decay is defined in the quantum-mechanical sense, hence the extension to the SU(2) or SU(3) symmetric case (Sec. III) strictly conserves internal symmetries. In the NIQN case the formalism is general enough to describe a rather large class of statistical models. Section IV investigates the behavior of the models in the large-fireball-mass limit. In particular, the criterion for the existence of a maximal temperature is derived. Section V is a summary of the results. Some technical details of the SU(3) symmetric models are given in Appendix A. Appendix B contains a brief discussion of the finite-quantization-volume case.

II. MODELS WITHOUT INTERNAL QUANTUM NUMBERS

In this section we investigate models with a single, neutral ground-state particle with mass m. Generalization for the case with SU(2) or SU(3) multiplets of ground-state particles will be given in Sec. III.

In all versions of the SBM the generating functional of fireball decay is given by

$$V[Q;\phi]=\frac{\sigma[Q;\alpha^2\phi]}{\sigma[Q;\alpha^2]}\ , \tag{1}$$

where Q_μ is the four-momentum of the decaying fireball and $\phi(q_\mu)$ is the test function,

$$\sigma[Q;\alpha^2\phi]=\sum_{n=1}^{\infty}|g(n)|^2 \times\int\prod_{i=1}^{n}\frac{d^3p_i}{2p_{i0}}\alpha^2\phi(\vec{p}_i)\delta^4(Q-\sum_{i=1}^{n}p_i), \quad (2)$$

$p_{i0}=(m^2+\vec{p}_i^{\,2})^{1/2}$ and α^2 is a coupling constant of dimension GeV^{-2}. The numbers $g(n)$ differ in the specific versions of the SBM and contain *all* the dynamical information of the model.[10,11] It is essential that the $g(n)$'s are independent of Q^2.

According to the usual terminology the integrals in Eq. (2) are called Boltzmann invariant momentum-space (IMS) integrals (for $\phi\equiv 1$). However, it is clear that they are in fact proportional to the true Bose-Einstein IMS integrals for infinite quantization volume. Thus Eqs. (1) and (2) are not necessarily in contradiction with quantum mechanics. We shall discuss the case of a finite volume in Appendix B.

It is easy to verify that the various distributions which may be derived from Eqs. (1) and (2) are identical to those which may be derived from assuming a quantum-mechanical state vector of the final state of the form

$$|\psi(Q)\rangle=\frac{1}{N(Q^2)}\sum_{n=1}^{\infty}\alpha^n g(n)\frac{1}{(n!)^{1/2}} \times\int\prod_{i=1}^{n}\frac{d^3p_i}{2p_{i0}}a^\dagger(\vec{p}_i)|0\rangle\delta^4\Big(Q-\sum_{i=1}^{n}p_i\Big), \quad (3)$$

where $a^\dagger(\vec{p}_i)$ are usual creation operators with the commutators

$$[a(\vec{p}),a^\dagger(\vec{p}\,')]=2p_0\delta^3(\vec{p}-\vec{p}\,')$$

and $N(Q^2)$ is a normalization factor.

In fact the probability density of observing an n particle state with momenta $\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_n$ is given by

$$\Big|\langle\psi(Q)\Big|\prod_{i=1}^{n}a^\dagger(\vec{p}_i)\Big|0\rangle\Big|^2. \quad (4)$$

The scalar product in Eq. (4) may be easily evaluated and we immediately arrive at the generating functional of Eqs. (1) and (2). To calculate the total weight of n-particle states Eq. (4) should be integrated over the momenta and multiplied by $(n!)^{-1}$. The last factor takes into account the fact that a specific final state is reproduced $n!$ times by the momentum integration.

Introducing the unnormalized state vector

$$|\phi(Q)\rangle=N(Q^2)|\psi(Q)\rangle \quad (5)$$

we can prove that $|\phi(Q)\rangle$ satisfies the integral equation

$$|\phi(Q)\rangle=\alpha f(1)\frac{\delta(Q_0-(m^2+\vec{Q}^2)^{1/2})}{2Q_0}a^\dagger(\vec{Q})|0\rangle+\alpha\int\frac{d^3r}{2r_0}d^4r'a^\dagger(\vec{r})\frac{f(N+1)}{f(N)}|\phi(r')\rangle\delta^4(Q-r-r'), \quad (6)$$

with

$$f(n)=\frac{g(n)}{\sqrt{n!}}$$

and

$$N=\int\frac{d^3p}{2p_0}a^\dagger(\vec{p})a(\vec{p})$$

the particle-number operator. Vice versa, postulating Eq. (6) as an integral equation describing fireball decay, we arrive at Eqs. (1) and (2), i.e., the usual result. The normalization factor squared $N(Q^2)^2$ just corresponds to $\sigma[Q;\alpha^2]$.

The most natural choice of the function $f(n)$ seems to be $f(n)\equiv 1$. In this case Eq. (6) may be interpreted in the usual way recalling the reciprocity principle[4] as a linear bootstrap equation. A fireball is characterized by the state vector of its decay final state $|\phi(Q)\rangle$. The decaying fireball is either a ground-state particle [first term on the right-hand side of Eq. (6)], or it consists of a ground-state particle and another (daughter) fireball [second term on the right-hand side of Eq. (6)]. As the ground-state particle and the daughter fireball (of the second possibility) are independent of each other, the product of their wave functions should be taken [as done in Eq. (6)]. Unfortunately, the choice $f(n)\equiv 1$ corresponds to a model with rather unphysical high-mass properties (see Sec. IV). In the following we shall not confine $f(n)$ to any specific choice.

It is clear that Eq. (6) is only a formal description of fireball decay. Many other integral equations[12] may lead to the same generating functional, some of which may have better intuitive physical justification than Eq. (6). For our purposes the important feature is that Eq. (6) can be easily generalized to a consistent model with internal symmetries.

III. GENERALIZATION TO THE SU(n)-SYMMETRIC CASE

In this section we write down the analog of Eq. (6) for the case of pseudoscalar-nonet ground-state particles and SU(3) symmetry. Inclusion of vector and tensor mesons (neglecting their spins) as well as extension to higher symmetries [e.g., SU(4)] is straightforward.

To obtain the required generalization the creation operator $a^\dagger(\vec{p})$ of Eq. (6) is replaced by a matrix of creation operators [the SU(3) meson matrix]:

$$M^\dagger(\vec{p})^a_b=\begin{bmatrix} \dfrac{a^\dagger_{\pi^0}+\cos\theta a^\dagger_\eta+\sin\theta a^\dagger_{\eta'}}{\sqrt{2}} & a^\dagger_{\pi^+} & a^\dagger_{K^+} \\ a^\dagger_{\pi^-} & \dfrac{-a^\dagger_{\pi^0}+\cos\theta a^\dagger_\eta+\sin\theta a^\dagger_{\eta'}}{\sqrt{2}} & a^\dagger_{K^0} \\ a^\dagger_{K^-} & a^\dagger_{\bar{K}^0} & -a^\dagger_\eta\sin\theta+a^\dagger_{\eta'}\cos\theta \end{bmatrix}, \tag{7}$$

where θ is the mixing angle of pseudoscalar mesons relative to the ideal mixing. We assume fireballs with SU(3) nonet transformation properties, thus the (unnormalized) state vector will also carry two SU(3) indices: $|\phi(Q)\rangle^a_b$. The generalization of Eq. (6) is then given by

$$|\phi(Q)\rangle^a_b=\alpha f(1)\,\frac{\delta(Q_0-(m^2+\vec{Q}^2)^{1/2})}{2Q_0}M^\dagger(\vec{Q})^a_b|0\rangle+\alpha\sum_{c=1}^{3}\int\frac{d^3r}{2r_0}d^4r'M^\dagger(\vec{r})^a_c\frac{f(N+1)}{f(N)}|\phi(r')\rangle^c_b\delta^4(Q-r-r'), \tag{8}$$

$r_0=(m^2+\vec{r}^2)^{1/2}$, where m is the common mass of the pseudoscalar nonet, and N is the particle-number operator,

$$N=\sum_{a,b}\int\frac{d^3p}{2p_0}M^\dagger(\vec{p})^a_bM(\vec{p})^b_a. \tag{9}$$

The solution of Eq. (8) is easy to obtain:

$$|\phi(Q)\rangle^a_b=\sum_{n=1}^{\infty}\frac{\alpha^n g(n)}{(n!)^{1/2}}\int\left(\prod_{i=1}^{n}\frac{d^3p_i}{2p_{i0}}\right)M^\dagger(\vec{p}_1)^a_{a_1}M^\dagger(\vec{p}_2)^{a_1}_{a_2}\cdots M^\dagger(\vec{p}_n)^{a_{n-1}}_b\delta^4\left(Q-\sum_{i=1}^{n}p_i\right)|0\rangle, \tag{10}$$

with $g(n)\equiv f(n)(n!)^{1/2}$. The probability density of a specific final state with l_1 neutral pions (momenta $\vec{p}^1_1,\ldots,\vec{p}^1_{l_1}$), l_2 positive pions (momenta $\vec{p}^2_1,\ldots,\vec{p}^2_{l_2}$), etc. is given by the square of the scalar product:

$$\left|\langle\phi(Q)\right|^a_b\frac{1}{N^a_b(Q^2)}\prod_{i=1}^{9}\left(\prod_{j=1}^{l_i}a^\dagger(\vec{p}^i_j)\right)\left|0\rangle\right|^2, \tag{11}$$

where $N^a_b(Q^2)$ is the normalization factor of $|\phi(Q)\rangle^a_b$.

Equation (10) shows that the integral equation (8) implies a specific SU(3) coupling of the final-state particles. To evaluate this coupling the powers of the $M^\dagger$ matrix should be calculated. After this the scalar product in Eq. (11) may be easily determined. A great simplification occurs as $M^\dagger(\vec{p}_1)M^\dagger(\vec{p}_2)\cdots M^\dagger(\vec{p}_n)$ is multiplied by a symmetric function of the momenta, thus antisymmetric terms cancel. $(M^\dagger)^n$ is calculated in Appendix A. Here we quote only the result for the SU(2) case and an $I=0$ fireball. In this simple case $(M^\dagger)^n$ may be calculated directly. Only even total pion numbers occur, and the probability to observe a final state with $2l$ π^0's and k $\pi^+\pi^-$ pairs is

$$\frac{1}{N(Q^2)^2}\frac{|g(2(l+k))|^2}{[2(l+k)]!}\alpha^{2(l+k)}\binom{k+l}{l}^2\times\left(\frac{1}{2}\right)^{2l}(2l)!(k!)^2\rho_{2(l+k)}(Q^2), \tag{12}$$

where

$$\rho_n(Q^2)=\int\prod_{i=1}^{n}\frac{d^3p_i}{2p_{i0}}\delta^4\left(Q-\sum_{i=1}^{n}p_i\right), \tag{13}$$

is the invariant momentum-space integral. The specific coupling scheme mentioned above is really very simple in the SU(2) case: pions are coupled in pairs into $I=0$ states. Thus for even (odd) total pion number we have $I=0$ (1). Note that the distribution (12) is very different from the (binomial for fixed total pion number) distribution of Ref. 9.

IV. THE LARGE-FIREBALL-MASS LIMIT

In this section we study the behavior of the models for large fireball mass. In particular we want to find those models which are characterized by a maximal temperature.

Let us start with the NIQN case. As the generating functional is given in terms of an expansion, Eq. (2), the most convenient method to study the high-mass behavior is the "dominant phase space" method of Ref. 13. (The same results may be also obtained by the Laplace-transformation method of Ref. 4.) Excluding the possibility $g(n)\equiv 0$ for $n>n_0$ fixed, it is clear that for $M\to\infty$ the large-multiplicity channels will be the most important ones. Our task is then to find the multiplicity which yields the largest contribution at a fixed large value of the fireball mass M. Approximating the IMS integrals by the Lurcat-Mazur-Krzywicki[14] formula, we have for large M

$$g^2(n)\alpha^{2n}\rho_n(M^2) \propto \exp[M\beta + n \ln z(\beta) + 2n \ln\alpha + 2 \ln g(n)], \quad (14)$$

where $z(\beta) = 2\pi(m/\beta)K_1(m\beta)$ and $\beta = \beta(n/M)$ is determined by the equation

$$1 = -\frac{n}{M}\frac{\partial \ln z}{\partial\beta}. \quad (15)$$

Now it is easy to determine the dominant multiplicity by looking for the maximum of the right-hand side of Eq. (14) as a function of n, which we treat as a continuous variable. The β value corresponding to the dominant multiplicity is determined by the equation

$$\alpha^2 z(\beta) \exp\left[2\frac{g'(-M/(\partial \ln z/\partial\beta))}{g(-M/(\partial \ln z/\partial\beta))}\right] = 1, \quad g'(n) = \frac{dg(n)}{dn} \quad (16)$$

for any finite M. There is a maximal temperature T_0 in the model if and only if $\lim_{M\to\infty}\beta(M) = \beta_\infty$ turns out to be finite. Thus we see that a maximal temperature exists if $\lim_{n\to\infty}[g'(n)/g(n)] = q$ is a finite number. The maximal temperature $T_0 = \beta_\infty^{-1}$ is then determined from the equation

$$\alpha^2 z\left(\frac{1}{T_0}\right)e^{2q} = 1. \quad (17)$$

If $g'/g \to -\infty$, arbitrarily high temperatures are possible. If $g'/g \to +\infty$ from Eq. (16) we obtain $T_0 = 0$, i.e. the model overemphasizes the high-multiplicity channels and in the $M \to \infty$ limit all the decay products will be at rest.

Starting from Eq. (12) the same procedure may be applied in case of the SU(2)-symmetric model. The only difference is that we have to look for the dominant phase space as a function of two variables k, l. The result is qualitatively the same as in the NIQN case. There is a maximal temperature T_0, when $\lim_{n\to\infty}[g'(n)/g(n)] = q$ is finite. The equation determining T_0 reads as

$$\frac{\alpha^2}{2} z\left(\frac{1}{T_0}\right)e^{2q} = 1. \quad (18)$$

The above qualitative results as well as Eq. (18) are valid for both the $I = 0$ and $I = 1$ fireballs.

We were not able to study the high-mass behavior of the SU(3) symmetric model in a fully satisfactory way. The main difficulty is that the weight of a particular final state is proportional to the square of a finite sum, as given in Appendix A. For specific choices of the final states (e.g., excluding η, η' and K mesons) it can be proved that the qualitative result on the existence of the maximal temperature is the same as in the NIQN and SU(2) cases. It is clear physically (and may also be obtained from our formulas) that a larger set of possible final states only lowers the value of the maximal temperature.[8] We conclude that there is a maximal temperature in the SU(3)-symmetric model if $\lim_{n\to\infty}[g'(n)/g(n)]$ is finite. By the same arguments we can also establish upper limits for the maximal temperature. The best upper limit is obtained from the solution of

$$\alpha^2 \cos^2\theta\, z\left(\frac{1}{T_0}\right)e^{2q} = 1 \quad (\cot\theta > 2^{-1/2}). \quad (19)$$

It is amusing to note that, due to the specific nature of coupling, excluding some particles from the final states does not always increase the maximal temperature. An example of this is the SU(2) model with final states containing charged pions excluded. As can be easily checked, the maximal temperature is given by Eq. (18), i.e., it does not change at all.

V. SUMMARY

In the previous sections we have given a formalism describing fireball decay which allows strict conservation of internal quantum numbers. The formalism is a straightforward generalization of the usual one in the no-internal-quantum-number case. In this case it is a formal description of a large class of statistical decays. By the word "statistical" we understand that a given multiplicity is distributed according to IMS (phase space). The characteristic of this class of models is that the dependence on the fireball mass of the ratio of weights of multiplicities m and n is given only by the ratio of IMS (phase-space) integrals, i.e., $g(m)\rho_m(M^2)/g(n)\rho_n(M^2)$ [with $g(n)$ independent of M].

The basic equation for the SU(3)-symmetric model with the pseudoscalar-meson nonet taken as ground-state particles and SU(3)-nonet fireballs is our Eq. (8). A specific SU(3) coupling of the final states is implied by this equation, thus SU(3) is treated nonstatistically in the present model.[15] The model is completely specified by the function $f(n)$ in Eq. (8). The model can be easily generalized to include vector and tensor mesons (neglecting spins). There is a natural way of introducing SU(3) breaking. The prescription is to use the observed masses of pseudoscalar mesons in Eq. (8). The explicit solution of the basic equation is easily obtained in both the SU(3)-symmetric and broken-SU(3) cases. It is given by Eq. (10) and the formulas in Appendix A.

We have established criteria for the existence of a maximal temperature in various models. Our treatment is incomplete for the SU(3) model as we did not succeed in writing down the exact equation for the maximal temperature; we have given only an upper bound for it. Nevertheless, with the help of the formulas in Appendix A all the distributions of the SU(3) model may be calculated for finite fireball mass.

The formalism may be extended to the case of a finite quantization volume. The necessary modifications are briefly discussed in Appendix B. Naturally, the structure of these models is more complicated than in the infinite-volume case, therefore we did not attempt a study of their asymptotic properties here.

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APPENDIX A

As was explained in Sec. III in order to calculate the weight of a specific final state in the SU(3)-symmetric model we have to calculate the powers of the $M^\dagger$ matrix. We suppress in the following momentum variables and denote $a^\dagger_{\pi^0}$ by π^0, $a^\dagger_{\pi^+}$ by π^+, etc. To calculate the nth power of the $M^\dagger$ matrix note that

$$1+M^\dagger+M^{\dagger 2}+\cdots=(1-M^\dagger)^{-1} \tag{A1}$$

and

$$(1-M^\dagger)^{-1}=[\det(1-M^\dagger)]^{-1}\overline{M}, \tag{A2}$$

where

$$\overline{M}=\begin{bmatrix} \left(1+\frac{\pi^0-\eta_n}{\sqrt2}\right)(1-\eta_s)-K^0\overline{K}^0 & (1-\eta_s)\pi^++\overline{K}^0K^+ & \left(1+\frac{\pi_0-\eta_n}{\sqrt2}\right)K^++\pi^+K^0 \\ (1-\eta_s)\pi^-+K^-K^0 & \left(1-\frac{\pi^0+\eta_n}{\sqrt2}\right)(1-\eta_s)-K^+K^- & \left(1-\frac{\pi^0+\eta_n}{\sqrt2}\right)K^0+\pi^-K^+ \\ \left(1+\frac{\pi^0-\eta_n}{\sqrt2}\right)K^-+\pi^-\overline{K}^0 & \left(1-\frac{\pi^0+\eta_n}{\sqrt2}\right)K^0+\pi^+K^- & \left(1-\frac{\pi^0+\eta_n}{\sqrt2}\right)\left(1+\frac{\pi^0-\eta_n}{\sqrt2}\right)-\pi^-\pi^+ \end{bmatrix}, \tag{A3}$$

with

$$\eta_n=\eta\cos\theta+\eta'\sin\theta,\quad \eta_s=-\eta\sin\theta+\eta'\cos\theta.$$

It is now straightforward to expand $\det(1-M^\dagger)^{-1}$ in terms of powers of $\pi^0,\pi^+,\pi^-,\ldots$. The result is

$$\begin{aligned}(1-M^\dagger)^{-1}{}^a_b=&\sum_{k_1,\ldots,k_{12}=0}^{\infty}(-1)^{k_6+k_{12}}\frac{1}{\sqrt2}^{2k_9+k_5+k_6+k_{11}+k_{12}}(\cos\theta)^{k_8+k_{11}}(\sin\theta)^{k_7+k_{12}}\left(\prod_{i=1}^{12}\frac{1}{k_i!}\right)\frac{(A+k_7+k_8)!\,(A+k_9+k_{10})!}{A!}\\ &\times\frac{(B+k_{11}+k_{12})!}{B!}\overline{M}^a_b(\pi^0)^{2k_9+k_5+k_6}(\pi^+)^{k_1+k_{10}}(\pi^-)^{k_2+k_{10}}\eta^{k_7+k_{11}}\eta'^{k_8+k_{12}}(K^0)^{k_1+k_3+k_6}(\overline{K}^0)^{k_2+k_3+k_6}(K^+)^{k_2+k_4+k_5}\\ &\times(K^-)^{k_1+k_4+k_5},\end{aligned} \tag{A4}$$

where

$$A=k_1+k_2+k_3+k_4+k_5+k_6,$$

$$B=2(k_1+k_2+k_5+k_6+k_9+k_{10})+k_3+k_4+1.$$

The number of particles is given by

$$n=3(k_1+k_2+k_5+k_6)+2(k_3+k_4+k_9+k_{10})+k_7+k_8+k_{11}+k_{12}+l_M,$$

where $l_M=0$, 1, or 2 depending on the number of particles in a particular term of $\overline{M}^a_b$. It is easy to find $(M^\dagger)^n$ from (A4): it is given by the terms with particle number equal to n. The above formulas are valid in the broken-SU(3) model, too [i.e., when SU(3) breaking is introduced only by the masses].

APPENDIX B

We shall discuss here the generalization of our formalism to both the NIQN and SU(3)-symmetric cases with a finite quantization volume.[16]

We take a cubic box with volume l^3 as a quantization volume. The possible values of the momentum are then $\vec{p}=(h/l)\vec{k}$, with $k_x, k_y, k_z = 0, \pm 1, \pm 2, \cdots$. The set of normalized states is given by

$$\prod_e \frac{1}{(n_e!)^{1/2}} \prod_{i=1}^{n} a^\dagger(\vec{p}_i)|0\rangle, \tag{B1}$$

where the n_e's denote the numbers of equal-momentum particles. $\sum_e n_e = n$ = total particle number. For the commutators we have

$$[a(\vec{p}), a^\dagger(\vec{p}\,')] = \delta_{\vec{p},\vec{p}\,'}.$$

Assuming equal weight for all the possible momentum configurations of an n-particle state we arrive at the state vector of the final state of fireball decay

$$|\Psi(Q)\rangle = \frac{1}{N(Q)} \sum_n g(n) \frac{(N_E)^{1/2}}{n!} \times \sum_{\vec{k}_1, \cdots, \vec{k}_n} \prod_{i=1}^{n} a^\dagger(\vec{p}_i)|0\rangle \delta_{Q, \Sigma_i \vec{p}_i}{}^4, \tag{B2}$$

where

$$\vec{p}_i = \frac{h}{l}\vec{k}_i$$

and

$$N_E = \prod_{\vec{k}} [a^\dagger(\vec{p})a(\vec{p})]!\,.$$

Sums and products are running over all possible values of $\vec{k}_i$ (compatible with four-momentum conservation). $\delta_{Q,\Sigma_i p_i}{}^4$ is a product of four Kronecker δ symbols. Equation (B2) is the analog of Eq. (3). The factor $(N_E)^{1/2}$ in (B2) is necessary as a single (normalized) state appears $n!\prod_e n_e!$ times in the sum.

The "integral equation" [analog of Eq. (6)] reads as

$$|\phi(Q)\rangle = f(1)\delta_{Q_0,(m^2+\vec{Q}^2)^{1/2}} a^\dagger(\vec{Q})|0\rangle + \sum_{\vec{r}, r'} (N_E)^{1/2} a^\dagger(\vec{r}) \frac{f(N+1)}{f(N)} \frac{1}{(N_E)^{1/2}} \times |\phi(r')\rangle \delta_{Q, r+r'}{}^4, \tag{B3}$$

where

$$\vec{r} = \frac{h}{l}\vec{k}, \quad \vec{r}\,' = \frac{h}{l}\vec{k}\,',$$

$$N = \sum_{\vec{k}} a^\dagger(\vec{p})a(\vec{p}),$$

$$f(n) = \frac{g(n)}{n!}.$$

The normalized state vector is

$$|\Psi(Q)\rangle = \frac{1}{N(Q)} |\phi(Q)\rangle.$$

The total weight of n-particle final states is easily obtained from (B2):

$$\frac{|g(n)|^2}{N(Q)^2} \frac{1}{n!} \sum_{\vec{k}_i} \left\langle 0 \left| \prod_{i=1}^{n} a(\vec{p}_i) \prod_{i=1}^{n} a^\dagger(\vec{p}_i) \right| 0 \right\rangle \delta_{Q,\Sigma p_i}{}^4 = \frac{|g(n)|^2}{N(Q)^2} \sigma_n(Q^2), \tag{B4}$$

where $\sigma_n(Q^2)$ is the Bose-Einstein invariant phase-space or momentum-space integral.[17,18]

The generalization to the SU(3) case is given by

$$|\phi(Q)\rangle^a_b = f(1)\delta_{Q_0,(m^2+\vec{Q}^2)^{1/2}} M^{\dagger a}_{\ b}(\vec{Q}) + \sum_{\vec{r}, r'} (N_E)^{1/2} M^\dagger(\vec{r})^a_{a_1} \frac{f(N+1)}{f(N)} \frac{1}{(N_E)^{1/2}} |\phi(r')\rangle^{a_1}_b \delta_{Q, r+r'}{}^4, \tag{B5}$$

where

$$N = \sum_{\vec{k}} \sum_{a,b} M^\dagger(\vec{p})^a_b M(\vec{p})^b_a, \quad N_E = \prod_{\vec{k}} \left(\sum_{a,b} M^\dagger(\vec{p})^a_b M(\vec{p})^b_a \right)!\,.$$

The solution of Eq. (B5) is

$$|\phi(Q)\rangle^a_b = \sum_n g(n) \frac{(N_E)^{1/2}}{n!} \sum_{\vec{k}_i} M^{\dagger a_1}_{\ a}(\vec{p}_1) M^{\dagger a_1}_{\ a_2}(\vec{p}_2) \cdots M^{\dagger a_{n-1}}_{\ b}(\vec{p}_n)|0\rangle \delta_{Q,\Sigma p_i}{}^4, \tag{B6}$$

where

$$g(n) = f(n)n!\,.$$

The total weight of n-particle states may be written in the following form:

$$\frac{|g(n)|^2}{(N^a_b(Q))^2} \frac{1}{n!} \sum_{\vec{k}_i} \left\langle 0 \left| \sum_{\text{perm.}\,\alpha_n} \frac{1}{n!} M^b_{a'_{n-1}}(\vec{p}_{\alpha_n}) \cdots M^{a'_1}_a(\vec{p}_{\alpha_1}) M^{\dagger a}_{\ a_1}(\vec{p}_1) \cdots M^{\dagger a_{n-1}}_{\ b}(\vec{p}_n) \right| 0 \right\rangle \delta_{Q,\Sigma_i p_i}{}^4, \tag{B7}$$

where $N^a_b(Q)$ is the normalization factor of $|\phi(Q)\rangle^a_b$. A great simplification in the calculation of the above matrix element is that terms antisymmetric in $\vec{p}_i$ cancel. To evaluate (B7) essentially powers of the $M^\dagger$ matrix should be calculated. (B7) is a sum of quantum-mechanical invariant phase-space (momentum-space) integrals.

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