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Scaling in the mean at asymptotic energies*

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We investigate, in detail, the consequences of the assumption that "scaling in the mean" and Koba-Nielsen-Olesen (KNO) scaling remain valid at asymptotic energies for which $p_L \gg p_T$, m and $\langle n \rangle \gg 1$. We argue that the scaling function $\phi(t)$ can be fit by the simple function e^{-t} with no free parameters. We show that, asymptotically, the semi-inclusive distributions satisfy Feynman scaling and vanish at $x=0$, and that the inclusive distributions satisfy scaling in the mean, vanish at $x=0$, and break Feynman scaling through an implicit s dependence through the variable $\langle n \rangle$. For Slattery's fit to the KNO function, we obtain the inclusive distributions for various values of $\langle n \rangle$. We assume that scaling in the mean holds for two-particle semi-inclusive distributions and obtain the normalization conditions for the two-particle scaling function. We obtain an expression for the two-particle inclusive distribution and define correlation functions in terms of the scaling functions. We show explicitly that even if the semi-inclusive distributions factorize, the inclusive distributions do not.

I. INTRODUCTION

Inspired by the Koba-Nielsen-Olesen (KNO) scaling law for the multiplicity distribution,[1] Dao *et al.* have proposed[2] that the single-particle semi-inclusive cross sections could scale in the same way as functions of both the longitudinal and transverse momentum. This assumption is commonly referred to as the "scaling-in-the-mean-hypothesis." Both Dao *et al.*[2] and Ezell *et al.*[3] have shown that the scaling-in-the-mean hypothesis is in good agreement with the data. The implications of the scaling-in-the-mean hypothesis have been considered by several authors,[4,5,6] including the present author.[5] We shall refer to our previous comment on the subject as (I) below.

The question of whether the data require scaling in the mean is by no means settled. Svensson and Sollin have shown[7] that the data of Dao *et al.* can be fit with a simple ansatz that satisfies Feynman scaling, which scaling in the mean does not.[5] The violation of Feynman scaling arises from an implicit dependence on s through the variable $\langle n \rangle$, the average multiplicity. However, since $\langle n \rangle$ is a very slowly varying function of s, it can be argued[6] that it is not possible to tell, on the basis of presently available data,[8] whether this energy dependence is actually present or not. The question of scaling in the mean for transverse momentum will be even more difficult to resolve than for longitudinal momentum. The data seem to indicate that the average value of transverse momentum varies slowly, if at all, with both s and n. If, in fact, asymptotically, the average value of transverse momentum turns out to be a constant, independent of both s and n, then the assumption of scaling in the mean in transverse momentum would convey no additional information at asymptotic energies.

We shall address the same question that we considered in (I), namely, if scaling in the mean and KNO scaling remain valid at asymptotic energies, what consequences would follow? We hope that some of these consequences may be amenable to direct experimental confirmation or contradiction.

Scaling in the mean could hold only for the pionization component and not for the diffractive component. Thus, in reactions of the type $p+p \to p+p+n\pi$, Dao *et al.* and Ezell *et al.* consider only the pion distribution and ignore the leading baryons. However, even in pionization events, the leading particles will carry off, on the average, about one half of the total c.m. energy. In such events, the multiplicity is usually fairly large, so that the energy remaining to the pions must be shared by a large number of particles so that the average value of the longitudinal momentum $\langle p_L \rangle$ will not be large. In diffractive events, the multiplicity will be small, but the leading particles will carry off almost all of the available energy. Thus, as long as the product $\langle n \rangle \langle p_T \rangle$ rises slower than $\sqrt{s}$, as the data seem to indicate, eventually, as s continues to increase, the asymptotic condition $\langle p_L \rangle \gg \langle p_T \rangle$ must eventually hold. Unfortunately, this is not yet the case with the 300-GeV/c data of Dao *et al.* and is certainly not true of the lower-energy data of Ezell *et al.* It is for this reason that it is not possible to distinguish clearly between scaling in the mean and the Svensson-Sollin ansatz[7] on the basis of the data at these energies. Thus, the asymptotic results that we shall obtain below cannot be checked against data at these energies either. However, we hope that it may be possible to check at least some of these results with data from the CERN ISR, from cosmic rays, and, in the future, from ISABELLE.

In Sec. II we consider the form of the scaling function ϕ for the semi-inclusive distributions. In Sec. III we consider the consequences of scaling in the mean for the single-particle inclusive distribution. In Sec IV we consider the consequences of the assumption that scaling in the mean is also valid for the higher-order semi-inclusive distributions, in particular, the two-particle semi-inclusive distributions. For simplicity, we shall consider a model where all produced particles are identical and spinless. Application of the energy-conservation sum rules then becomes straightforward. Since it is reasonable to assume that, if scaling in the mean is valid when n is the charged multiplicity, then it would also be true when n is the total multiplicity (although this would be difficult to verify since to determine the total multiplicity experimentally one must be sure that all neutrals are detected), and since there is experimental evidence to indicate that the average number of neutral particles in an event is directly proportional to the number of charged prongs,[9] no generality is really lost. The additional assumptions that must be made in order to apply these results to the real world where one must differentiate between different particle species and, in particular, between charged particles and neutrals, have been discussed in detail elsewhere.[6,10]

II. THE SCALING FUNCTION FOR THE SEMI-INCLUSIVE DISTRIBUTIONS

Scaling in the mean, for the noninvariant semi-inclusive cross section, integrated over transverse momentum, can be written[2]

$$\frac{\langle p_L\rangle_n}{n\sigma_n}\frac{d\sigma_n}{dp_L}=\phi\left(\frac{p_L}{\langle p_L\rangle_n}\right). \tag{1}$$

The factor $\langle p_L\rangle_n/n\sigma_n$ multiplying the semi-inclusive cross section has been chosen so that the normalization conditions[4] can be satisfied:

$$\int\frac{d\sigma_n}{dp_L}dp_L=n\sigma_n, \tag{2}$$

$$\frac{\int(d\sigma_n/dp_L)p_L dp_L}{\int(d\sigma_n/dp_L)dp_L}=\langle p_L\rangle_n. \tag{3}$$

Since we assume that the target and projectile are identical, we will have symmetry between hemispheres. We thus lose no generality if we take the variable to be not the longitudinal momentum itself, but rather the magnitude of the longitudinal momentum which is always positive. The range of integration in (2) and (3) is then 0 to ∞ instead of $-\infty$ to ∞. Then[4] (2) and (3) are satisfied if ϕ satisfies[11]

$$\int_0^\infty dt\,\phi(t)=\int_0^\infty dt\,t\phi(t)=1. \tag{4}$$

We shall call the scaling variable $\bar{x}_n$,

$$\bar{x}_n=p_L/\langle p_L\rangle_n, \tag{5}$$

which should not be confused with the usual Feynman scaling variable x,

$$x=2p_L/\sqrt{s}. \tag{6}$$

We must use the *noninvariant* inclusive cross section in (1) because the right-hand side is a function only of the dimensionless variable $\bar{x}_n$, and so the left-hand side of (1) must also be dimensionless. If we multiplied by an additional factor of E to get the invariant inclusive cross section, the left-hand side would have the dimensions of energy.

We assume that the product $\langle n\rangle\langle p_T\rangle$ increases significantly slower than $\sqrt{s}$, while the energy available to the reaction in the c.m. frame is $\sqrt{s}$. Hence, as $s\to\infty$, essentially all of the available energy must go into the longitudinal kinetic energy of the produced particles. Thus, for sufficiently large s, for any particle with $x\neq 0$, we can write

$$E=(p_L{}^2+p_T{}^2+m^2)^{1/2}\cong p_L=\tfrac{1}{2}x\sqrt{s}. \tag{7}$$

We can then use the energy-conservation sum rule[12] to show that

$$\langle p_L\rangle_n=\sqrt{s}/n, \tag{8}$$

$$\bar{x}_n=\tfrac{1}{2}nx, \tag{9}$$

so that we can immediately obtain from (1) for the invariant semi-inclusive distribution.

$$F_n(p_L,s)=\frac{1}{\sigma_n}E\frac{d\sigma_n}{dp_L}=n\bar{x}_n\phi(\bar{x}_n)=\frac{n^2x}{2}\phi\left(\frac{nx}{2}\right). \tag{10}$$

Two things are apparent from (10). First, since there is no explicit or implicit s dependence, each of the semi-inclusive cross sections satisfies Feynman scaling. Second, as $x\to 0$, each of the semi-inclusive cross sections goes to zero linearly with x, so that there is a "hole" in the center of each distribution. These two facts are not unrelated. We must have

$$\int\frac{dp_L}{E}F_n(p_L,s)=n, \tag{11}$$

which, of course, is a fixed constant. On the other hand, there is a theorem due to Bali, Brown, Peccei, and Pignotti[13] that states that, if Feynman scaling is satisfied, $F_n(p_L,s)=f(x)$, then for large s, the integral on the left-hand side of (11) must behave like $f(0)\ln s$. Hence, the only way to avoid a contradiction is to require that $f(0)=0$. In short, the fact that, asymptotically, each of the semi-inclusive cross sections satisfies Feynman scaling, with a scaling function that vanishes at the origin, is a consequence of the scaling-in-the-mean hypothesis, irrespective of the particular choice of the scaling function.

The function ϕ is determined by the data. Both Dao *et al.* and Ezell *et al.* fit their data with the form

$$\phi(t) = ae^{-(bt+ct^2)}. \tag{12}$$

Dao *et al.* obtain $a = 0.91 \pm 0.15$, $b = 0.83 \pm 0.04$, $c = 0.03 \pm 0.01$, while Ezell *et al.* obtain $a = 1.0 \pm 0.02$, $b = 1.08 \pm 0.01$, $c = -0.012 \pm 0.03$. Both Dao *et al.* and Ezell *et al.* obtain the best fit without imposing the constraints (4), but the constraints are satisfied by their fits to a very good approximation.

Since the parameter c is small in both cases, one might try a fit of the form ae^{-bt}, setting $c = 0$. The constraints (4) immediately give us $a = b = 1$, so our function, with no free parameters, is simply

$$\phi(t) = e^{-t}. \tag{13}$$

In Fig. 1 we plot this function (13) and the functions fitted to the data by Dao *et al.* and by Ezell *et al.* The three curves are quite close over the range of their arguments for which data exist and, in addition, the two fits lie on opposite sides of (13). Hence, not only do the data satisfy the scaling law (1), but the scaling function can be fit remarkably well with a simple exponential. A glance at Fig. 2 of Ref. 1 convinces us that it is impossible to distinguish between these three curves on the basis of the data of Dao *et al.* The error bars are smaller on the data of Ezell *et al.*, but these data are all at lower energy and thus further away from asymptopia. Hence, for simplicity we choose the form (13) for the scaling function.

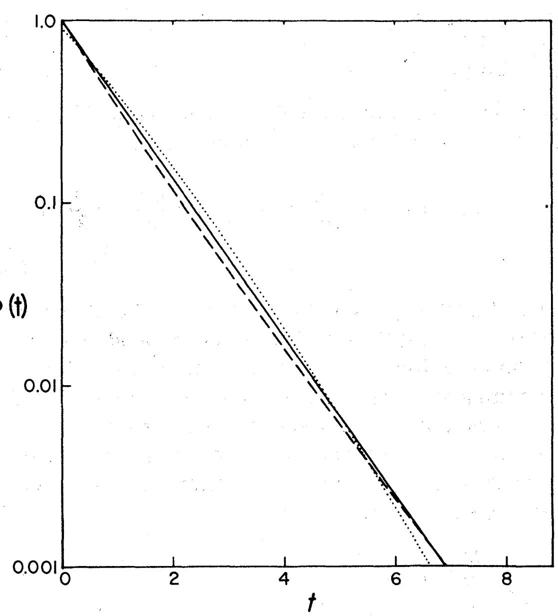

FIG. 1. The scaling functions $\phi(t)$. $\phi(t) = e^{-t}$ (solid line), the Dao *et al.* fit, $\phi(t) = 0.91e^{-(0.83t + 0.03t^2)}$ (dotted line), and the Ezell *et al.* fit, $\phi(t) = e^{-(1.08t - 0.012t^2)}$ (dashed line), where t is the scaling variable, $t = p_L/\langle p_L\rangle_n$. Since these curves are almost indistinguishable over the range, $0 \le t \le 5$, for which data exist, we argue for the first choice, which has no free parameters, on the basis of simplicity.

An additional point is that scaling in the mean cannot be valid for all n, even at asymptotic energies. The limit on the magnitude of the longitudinal momentum is $0 \le p_L \le \frac{1}{2}\sqrt{s}$. Therefore, given (8), for a given n, the function $\phi(\bar{x}_n)$ can be nonzero only for $\bar{x}_n \le \frac{1}{2}n$, and the limits of the normalization integrals (4) should also by 0 to $\frac{1}{2}n$. Hence, ϕ should depend on n, which would contradict the scaling in the mean hypothesis. However, because of the exponential falloff of ϕ with its argument, for sufficiently large n, say, $n \ge 7$ or 8, such boundary effects can be neglected, and no contradiction results. In the experimental data, which is measured in n_c, the *charged* multiplicity, which is always *less* than the true multiplicity. This could explain why scaling in the mean seems to hold[2,3] for n_c as low as 4.

III. THE INCLUSIVE DISTRIBUTION

The invariant inclusive distribution is given by a sum of semi-inclusive distributions:

$$\frac{E}{\sigma}\frac{d\sigma}{dp_L} = F(p_L, s) = \sum_n \alpha_n F_n(p_L, s), \tag{14}$$

where

$$\alpha_n = \sigma_n/\sigma, \quad \sum_n \alpha_n = 1.$$

In order to say anything about the inclusive distribution, we must make an assumption about α_n. The data seem to indicate that α_n satisfies Koba-Nielsen-Olesen (KNO) scaling[1]

$$\alpha_n = \frac{1}{\langle n\rangle}\psi\left(\frac{n}{\langle n\rangle}\right). \tag{15}$$

Approximating the sum over n by an integral over the variable $z = n/\langle n\rangle$, we obtain

$$F(p_L, s) = \frac{\langle n\rangle^2 x}{2}\int dz\, z^2\psi(z)\phi\left(\frac{\langle n\rangle xz}{2}\right). \tag{16}$$

Using the energy sum rule for the inclusive cross section, we can use the same argument that we used to obtain (8) for the semi-inclusive distribution to obtain the average value of p_L in the inclusive reaction as

$$\langle p_L\rangle = \sqrt{s}/\langle n\rangle\,, \tag{17}$$

and defining

$$\bar{x} = p_L/\langle p_L\rangle = \tfrac{1}{2}\langle n\rangle x\,, \tag{18}$$

(17) can be written as

$$F(p_L, s) = \frac{\langle n\rangle^2 x}{2}\,\Phi\left(\frac{\langle n\rangle x}{2}\right) = \langle n\rangle \bar{x}\Phi(\bar{x})\,, \tag{19}$$

with

$$\Phi(\bar{x}) = \int_0^\infty dz\, z^2\psi(z)\phi(\bar{x}z)\,. \tag{20}$$

[We should note that this is *not* the same as the function Φ in (I) as we have removed a factor of $\bar{x}$.]

We recall that the KNO function must satisfy the normalization conditions,[1,14]

$$\int_0^\infty dz\,\psi(z) = \int_0^\infty dz\, z\psi(z) = 1\,,$$
$$\int_0^\infty dz\, z^2\psi(z) = \langle n^2\rangle/\langle n\rangle^2\,, \tag{21}$$

so that we have, defining $v = \bar{x}z$,

$$\int d\bar{x}\,\Phi(\bar{x}) = \int dz\, z\psi(z)\int dv\,\phi(v) = 1\,,$$
$$\int \bar{x}\,d\bar{x}\,\Phi(\bar{x}) = \int dz\,\psi(z)\int v\,dv\,\phi(v) = 1\,. \tag{22}$$

If we assume that ϕ is given by (13), (20) becomes

$$\Phi(\bar{x}) = \int_0^\infty z^2\psi(z)e^{-\bar{x}z}dz \tag{23}$$

and the function Φ is just the Laplace transform of the function $z^2\psi(z)$. From (7), (17), (18), and (19), it is clear that the inclusive cross section also satisfies scaling in the mean

$$\frac{\langle p_L\rangle}{\langle n\rangle\sigma}\frac{d\sigma}{dp_L} = \Phi\left(\frac{p_L}{\langle p_L\rangle}\right)\,. \tag{24}$$

However, since $\langle n\rangle$ depends on s, the invariant inclusive distribution does *not* satisfy Feynman scaling, as we noted in (I).

The form of the scaling function Φ will depend on the form of the KNO function ψ. Two simple forms for the KNO function that satisfy the normalization conditions (22) are

$$\psi_1(z) = \delta(z-1)\,,$$
$$\psi_2(z) = \tfrac{1}{2},\quad 0 \le z \le 2;\quad \psi_2(z) = 0,\quad z > 2\,. \tag{25}$$

A more physically meaningful choice, however, would be Slattery's fit[15] to the multiplicity data with normalization (21)

$$\psi_3(z) = (Az + Bz^3 + Cz^5 + Dz^7)e^{-Ez}\,, \tag{26}$$

$$A = 1.90,\ B = 16.9,\ C = -3.32,\ D = 0.166,\ E = 3.04.$$

The corresponding Φ functions are

$$\Phi_1(\bar{x}) = e^{-\bar{x}}\,, \tag{27}$$

$$\Phi_2(\bar{x}) = (\bar{x})^{-3}\{1 - e^{-2\bar{x}}[(\bar{x}+1)^2 + \bar{x}^2]\}\,, \tag{28}$$

$$\Phi_3(\bar{x}) = \frac{3!A}{(E+\bar{x})^4} + \frac{5!B}{(E+\bar{x})^6} + \frac{7!C}{(E+\bar{x})^8} + \frac{9!D}{(E+\bar{x})^{10}}\,. \tag{29}$$

In Fig. 2 we plot the function $\bar{x}\Phi(\bar{x})$ that determines the shape of the inclusive distribution. The solid line corresponds to Slattery's fit. For comparison we have also plotted $\bar{x}\Phi(\bar{x})$ for the other two choices for the KNO function. It is clear that the shape of the inclusive distribution is not strongly dependent on the details of the shape of the KNO function.

In order to see how the inclusive distribution will vary with $\langle n\rangle$ and through $\langle n\rangle$ with s, we plot in Fig. 3 the inclusive distribution

$$F(p_L, s) = \frac{\langle n\rangle^2 x}{2}\Phi_3\left(\frac{\langle n\rangle x}{2}\right) \tag{30}$$

as a function of the Feynman scaling variable, x, for various values of $\langle n\rangle$. We use Φ_3 obtained from Slattery's fit to the KNO function as the most physically meaningful choice.

The curves in Fig. 3 were obtained under the assumption that the approximation (7) is valid. This is not the case at any fixed, finite energy at $x = 0$, and there will thus be a dip rather than a "hole" at this point. We can estimate the error in using the approximation (7) by explicitly calculating the in-

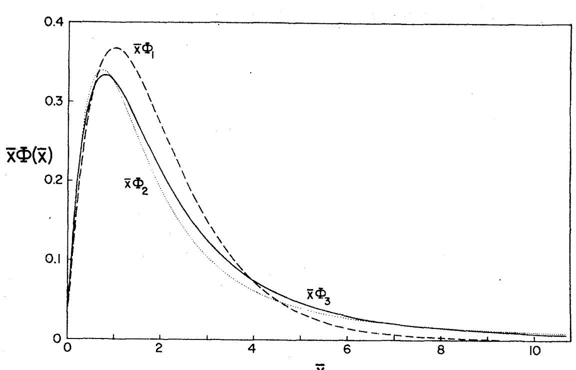

FIG. 2. The function $\bar{x}\,\Phi(\bar{x})$ that determines the shape of the inclusive distribution, for various choices of the KNO scaling function, $\psi(z)$. Φ_1 (dashed line) corresponds to $\psi(z) = \delta(z-1)$. Φ_2 (dotted line) corresponds to $\psi(z) = \frac{1}{2}$, $0 \le z \le 2$; $\psi(z) = 0$, $z > 2$. Φ_3 (solid line) corresponds to Slattery's fit to the data for the KNO function and therefore corresponds most closely to the real world. It should be clear that the shape of the inclusive distribution is not very dependent on the specific choice of KNO function.

clusive distribution at $x=0$. Since we wish to integrate over transverse momentum, we make the approximation of replacing p_T everywhere with $\langle p_T\rangle$ and we define

$$m_T=(\langle p_T\rangle^2+m^2)^{1/2}. \tag{31}$$

Using (1) without the approximation (7) we obtain for the semi-inclusive distribution at $x=0$

$$F_n(0,s)=n^2(m_T/\sqrt{s})\phi(0). \tag{32}$$

And, from (32) and (14) we obtain the inclusive distribution at $x=0$

$$F(0,s)=\langle n^2\rangle(m_T/\sqrt{s})\phi(0), \tag{33}$$

and since

$$\Phi(0)=\int dz\, z^2\psi(z)\phi(0)=(\langle n^2\rangle/\langle n\rangle^2)\phi(0), \tag{34}$$

(33) could also be written

$$F(0,s)=\langle n\rangle^2(m_T/\sqrt{s})\Phi(0). \tag{35}$$

As discussed by Ernest and Schmitt,[6] as long as $\langle n\rangle$ increases with s slower than $s^{1/4}$, $F(0,s)$ will tend asymptotically to zero. In fact, the multiplicity data for pp interactions at higher energies can be fit with the form[16]

$$\langle n_c\rangle=2.04\ln(s)-4.33, \tag{36}$$

with s in units of GeV^2; $\langle n_c\rangle$ is the average *charged* multiplicity, whereas what appears in (19), is the total multiplicity $\langle n\rangle$. Making the reasonable assumption that, on the average, the same number of positive, neutral, and negative particles are produced, we would have $\langle n\rangle=\frac{3}{2}\langle n_c\rangle$. Thus $\langle n\rangle=60$ corresponds to $\langle n_c\rangle=40$, which from (36) would

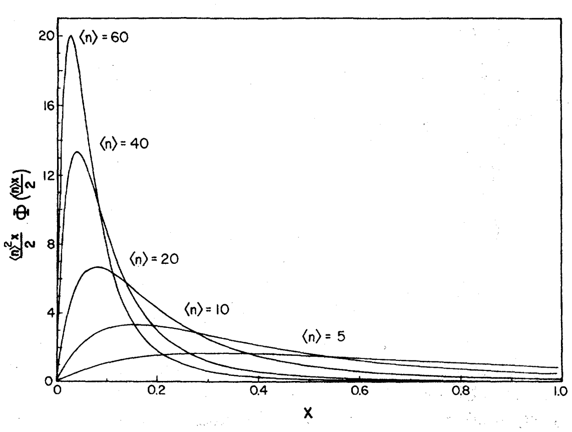

FIG. 3. The inclusive cross section $(\langle n\rangle^2 x/2)\,\Phi_3(\langle n\rangle x/2)$, corresponding to Slattery's fit to the KNO function, as a function of the Feynman scaling variable x for various values of the average multiplicity $\langle n\rangle$. $\langle n\rangle$ is a slowly varying function of s, and $\langle n\rangle=60$ would correspond to a c.m. energy, $\sqrt{s}=5\times10^4$ GeV.

occur at a c.m. energy, $\sqrt{s}=5\times10^4$ GeV.

If a dip is to be observable,[17] the value of the inclusive distribution at $x=0$ should be significantly smaller than the maximum value of approximately $\langle n\rangle/e$ which occurs near the point $x=2/\langle n\rangle$. Taking $m_T\cong0.4$ GeV, and for Slattery's function $\langle n^2\rangle/\langle n\rangle^2=1.3$, we can estimate the value of $F(0,s)$ and thus, the error in using the approximation (7) in this region. Thus, at $\sqrt{s}=25$ GeV, in the Fermilab energy range, we would have $F(0,s)=3.5$ compared to the maximum value of F at this energy of 4.8, whereas, at the top of the CERN ISR energy range $\sqrt{s}=60$ GeV, we would have $F(0,s)=2.8$ compared to a maximum value of 6.8.

The results that we have obtained were for a model where all produced particles are identical, whereas in pp interactions we must have two leading baryons. Dao *et al.* have removed the leading particles by considering only negative secondary tracks. In nondiffractive events the leading particles carry off, on the average, approximately half of the available energy, whereas in diffractive events the leading particles will have most of the energy. Thus the energy available to the nonleading particles in p-p collisions will be considerably less than $\sqrt{s}$ and it is not true $\langle p_L\rangle\gg\langle p_T\rangle$, m for the 300-GeV/c data of Dao *et al.* Thus even though scaling in the mean is satisfied by the data at Fermilab energies, a violation of Feynman scaling is not apparent for p-p interactions in this energy range. Svensson and Sollin[7] obtained a good fit to the data of Dao *et al.* with the ansatz

$$\begin{aligned}F_n(p_L,s)&=ae^{-bx(n+1)/\langle n\rangle},\\ F(p_L,s)&=\int dz\,\psi(z)e^{-bxz}=f(x).\end{aligned} \tag{37}$$

The semi-inclusive distribution has no hole at $x=0$ and an s dependence through the variable $\langle n\rangle$. The inclusive cross section has no hole or dip at $x=0$, no s dependence, and thus satisfies Feynman scaling.

It is probably the case that if scaling in the mean holds up to the highest CERN ISR energies, a violation of Feynman scaling should be apparent. Unfortunately there has been no investigation of whether scaling in the mean does, in fact hold for the CERN ISR data. However, a violation of Feynman scaling at the CERN ISR has been observed by the British-Scandanavian-M.I.T. collaboration,[18] who found a rise of approximately 12% in the inclusive $\pi^{\pm}$ distribution over the ISR energy range in the region of $x=0$. They have not observed the type of structure that we have obtained in Fig. 3. Enrst and Schmitt[7,19] have speculated that the observed behavior could be explained by scaling in the mean with the additional assumption that the average multiplicity behaves like $s^{1/3}$. The question is by no means settled and further analysis of the data in the ISR energy range would prove fruitful.

In $p\bar{p}$ annihilation, there are no leading particles and, as a first approximation, one can take all of the produced particles to be pions. At a given energy, the average multiplicity will be larger than in pp interactions. Our simple model should thus provide a better approximation to annihilation reactions than to p-p interactions. If scaling in the mean holds for $p\bar{p}$ interactions over the Fermilab energy range, a violation of Feynman scaling and a dip in the region of $x=0$ should be observable. It would be interesting if experimentalists with access to the raw data could tell us whether, in fact, scaling in the mean does hold in $p\bar{p}$ annihilation reactions.

IV. THE TWO-PARTICLE INCLUSIVE DISTRIBUTIONS

If scaling the mean is valid for the single-particle semi-inclusive cross sections, it is reasonable to assume that a similar scaling law should hold for higher-order multiparticle semi-inclusive cross sections. In the case of the two-particle semi-inclusive distribution, we assume that the scaling law takes the form

$$\frac{\langle p_L\rangle_n^2}{n(n-1)\sigma_n}\frac{d^2\sigma_n}{dp_L dp_L'}=\phi_2\left(\frac{p_L}{\langle p_L\rangle_n},\frac{p_L'}{\langle p_L\rangle_n}\right). \quad (38)$$

Now, of course, we cannot use the magnitude of the longitudinal momentum instead of the longitudinal momentum itself as the appropriate variable since we must allow for the case where p_L and p_L' have opposite signs. Hence, the limits of all integrals below should be understood to be from $-\infty$ to ∞ and the functions should be normalized accordingly. Since there is no data on scaling in the mean for two-particle inclusive cross sections (a situation which we hope that experimentalists will soon rectify), we shall discuss no specific functional forms. It is clear that the normalization condition,

$$\int\frac{d^2\sigma_n}{dp_L dp_L'}dp_L dp_L'=n(n-1)\sigma_n\,, \quad (39)$$

is satisfied if ϕ_2 satisfies

$$\int dt\,dt'\,\phi_2(t,t')=1\,. \quad (40)$$

Now, let us consider the energy sum rule, assuming $E\cong|p_L|$:

$$\int\frac{d^2\sigma_n}{dp_L dp_L'}|p_L|dp_L=(\sqrt{s}-|p_L'|)\frac{d\sigma_n}{dp_L'} \quad (41)$$

or, integrating once again and using the energy sum rule and the multiplicity sum rule for the single-particle distribution,

$$\int\frac{d^2\sigma_n}{dp_L dp_L'}|p_L|dp_L dp_L'=(n-1)\sqrt{s}\sigma_n\,. \quad (42)$$

Using (38) and recalling (4), (41) and (42) give, respectively,

$$\int dt\,|t|\,\phi_2(t,t')=\frac{n-|t'|}{n-1}\phi_1(t')\,, \quad (43)$$

$$\int dt\,dt'\,|t|\,\phi_2(t,t')=1\,. \quad (44)$$

Equation (43) obviously cannot be satisfied if ϕ_1 and ϕ_2 do not depend on n, so scaling in the mean cannot be valid for all n for the two-particle inclusive cross section. However, because of the exponential falloff of ϕ_1 with its argument, data will only exist for relatively small values of t', say $t'<4$ or 5. Hence, for small t and sufficiently large n, we can write (43) as

$$\int dt\,|t|\,\phi_2(t,t')\cong\phi_1(t')\,, \quad (45)$$

and scaling in the mean *can* be a good approximation for large n. We shall assume that (40), (44), and (45) determine the normalization of ϕ_2. We should also note that, as a direct consequence of (39) and (42), we get the same average value $\langle p_L\rangle_n$ as before,[20]

$$\langle p_L\rangle_n=\frac{\int(d^2\sigma/dp_L dp_L')\,|p_L|\,dp_L dp_L'}{\int(d^2\sigma/dp_L dp_L')dp_L dp_1'}=\frac{\sqrt{s}}{n}\,, \quad (46)$$

which is the same as (8). We can also use (43) and (4) to write

$$\int|t||t'|\phi_2(t,t')dt\,dt'=\frac{n}{n-1}-\frac{1}{n-1}\int|t'|^2\phi_1(t')dt'\,. \quad (47)$$

If we assume that $\phi_1(t)$ has the exponential form (13), then the last integral is just

$$\int_0^\infty t^2e^{-t}dt=2\,, \quad (48)$$

so we have

$$\int|t||t'|\phi_2(t,t')dt\,dt'=\frac{n-2}{n-1}\,, \quad (49)$$

which, for sufficiently large n, we can write

$$\int|t||t'|\phi_2(t,t')dt\,dt'\cong1\,, \quad (50)$$

and below we shall assume that all n's are large enough so that (50) is valid. Thus, while we have assumed no specific form for the function ϕ_2, we have obtained a number of constraints that it must satisfy.

Let us now define a new function of the scaling variables C_2 by the equation

$$\phi_2(t,t')=\phi_1(t)\phi_1(t')+C_2(t,t'). \tag{51}$$

Then we must have

$$\int dt\,dt'C_2(t,t')=\int dt\,dt'\,|t|\,C_2(t,t')=\int dt\,dt'\,|t|\,|t'|\,C_2(t,t')=0. \tag{52}$$

C_2 thus does not correspond to the usual two-particle semi-inclusive correlation function[21] which would have to satisfy

$$\int dp\,dp'\rho_{2n}(p,p')=-n, \tag{53}$$

but is a semi-inclusive correlation function in the sense of Koba and Olesen.[22]

In analogy with (10), we can use (38) to write for the invariant semi-inclusive distribution,

$$\begin{aligned}F_n(p_L,p_L',s)&=\frac{1}{\sigma_n}EE'\frac{d^2\sigma_n}{dp_Ldp_L'}\\&=n(n-1)\bar{x}_n\bar{x}_n'\phi_2(\bar{x}_n,\bar{x}_n')\\&=\frac{n^3(n-1)xx'}{4}\phi_2\left(\frac{nx}{2},\frac{nx'}{2}\right);\end{aligned} \tag{54}$$

then the inclusive distribution has the form

$$F(p_L,p_L',s)=\sum_n\alpha_nF_n(p_L,p_L',s) \tag{55}$$

and, in place of (17), we have

$$F(p_L,p_L',s)=\frac{\langle n\rangle^3xx'}{4}\int dz\,z^3(\langle n\rangle z-1)\psi(z)\times\phi_2\left(\frac{\langle n\rangle xz}{2},\frac{\langle n\rangle x'z}{2}\right), \tag{56}$$

which, for sufficiently large $\langle n\rangle$, can be approximated by

$$\begin{aligned}F(p_L,p_L',s)&=\frac{\langle n\rangle^4xx'}{4}\int dz\,z^4\psi(z)\times\phi_2\left(\frac{\langle n\rangle xz}{2},\frac{\langle n\rangle x'z}{2}\right)\\&=\langle n\rangle^2\bar{x}\bar{x}'\Phi_2(\bar{x},\bar{x}'),\end{aligned} \tag{57}$$

with

$$\Phi_2(\bar{x},\bar{x}')=\int_0^\infty dz\,z^4\psi(z)\phi_2(\bar{x}z,\bar{x}'z). \tag{58}$$

And we could write

$$\begin{aligned}\int|\bar{x}|\,d\bar{x}\,d\bar{x}'\,\Phi(\bar{x},\bar{x}')&=\int dz\,z\psi(z)\int|v|\,dv\,dv'\,\phi_2(v,v')\\&=1,\\\int|\bar{x}|\,|\bar{x}'|\,d\bar{x}\,d\bar{x}'\,\Phi(\bar{x},\bar{x}')&=\int dz\,\psi(z)\int|v|\,|v'|\,dv\,dv'\,\phi_2(v,v')\\&=1,\end{aligned} \tag{59}$$

but

$$\int d\bar{x}\,d\bar{x}'\,\Phi(\bar{x},\bar{x}')=\int dz\,z^2\psi(z)\int dv\,dv'\,\phi_2(v,v')=\int dz\,z^2\psi(z)=\langle n^2\rangle/\langle n\rangle^2, \tag{60}$$

which depends on the choice of KNO function. If we assume that $C_2=0$ so that the two-particle semi-inclusive distribution completely factorizes, we have

$$\begin{aligned}\Phi_2(\bar{x},\bar{x}')&=\int_0^\infty dz\,z^4\psi(z)\phi_1(\bar{x}z)\phi_1(\bar{x}'z)\\&=\int_0^\infty dz\,z^4\psi(z)e^{-(x+x')z}\\&=\tilde{\Phi}(\bar{x}+\bar{x}'),\end{aligned} \tag{61}$$

where the function $\tilde{\Phi}$ is the Laplace transform of the function $z^4\psi(z)$. The inclusive distribution does *not* factorize [except in the trivial case $\psi(z)=\delta(z-1)$] but becomes a function of the single variable $\bar{x}+\bar{x}'$. This is an example of correlations created by interference between the single-particle semi-inclusive distributions.[22]

We could go on to assume that scaling in the mean holds for the k-particle inclusive distribution,

$$\frac{\langle p_L\rangle_n^k}{n(n-1)\cdots(n-k)\sigma_n}\frac{d^k\sigma_n}{dp_Ldp_L'\cdots dp_L^{(k)}}=\phi_k\left(\frac{p_L}{\langle p_L\rangle_n},\frac{p_L'}{\langle p_L\rangle_n}\cdots\frac{p_L^{(k)}}{\langle p_L\rangle_n}\right), \tag{62}$$

but we shall leave the derivation of any further results as an exercise for the reader.

V. DISCUSSION

In Sec. II we considered the single-particle semi-inclusive distributions and argued that the data could be fitted with a simple exponential form. We showed that scaling in the mean implies Feynman scaling for the semi-inclusive distributions as a consequence of which each semi-inclusive distribution must vanish at $x=0$ at very high energies.

In Sec. III we showed that scaling in the mean for the semi-inclusive distributions, together with KNO scaling for the multiplicity distribution, implies scaling in the mean for the inclusive distribution. However, since the average value of the longitudinal momentum in the inclusive reaction $\langle p_L \rangle$ depends on the average multiplicity $\langle n \rangle$ which, in turn, depends on s, Feynman scaling will be violated. However, since $\langle n \rangle$ is a slowly varying function of s, one cannot say for certain, on the basis of presently available data, whether such a scaling violation is present in the real world or not.

For any reasonable choice of the KNO function, the inclusive distribution will also vanish at $x=0$ at asymptotic energies. If $\langle n \rangle$ rises slower than $s^{1/4}$, there is a rapidity plateau in the central region but its height is zero. If $\langle n \rangle$ rises like $s^{1/4}$ or faster, there will be no plateau but a more complicated structure in the central region. One cannot say anything about the rise in the inclusive cross section over the ISR energy range at 90° in the c.m. frame ($y=0$) without making additional independent assumptions about the behavior of $\langle n \rangle$. The explanation of the rise by Ernst and Schmitt depends critically on the additional assumption that $\langle n \rangle \propto s^{1/3}$.

We have obtained the function $\bar{x}\Phi(\bar{x})$ that determines the shape of the inclusive distribution for three different forms for the KNO function and have shown that its form does not depend very much on the particular form of the KNO function. We have also obtained the inclusive distribution in the case where the KNO function is given by Slattery's fit for various values of $\langle n \rangle$ to see explicitly how Feynman scaling is broken.

In Sec. IV we investigated the consequences of the assumption that scaling in the mean is valid for the two-particle semi-inclusive distributions. We obtained a set of constraints that the scaling function must satisfy and obtained an expression for the two-body inclusive distribution. We have shown that, even if the two-particle semi-inclusive distributions factorize, the two-particle inclusive distribution will not.

We must re-emphasize that the results that we obtained are for asymptotic energies and cannot be expected to hold at Fermilab energies and conventional accelerators where the data supporting the scaling-in-the-mean hypothesis was obtained. In fact, it is not really possible to distinguish conclusively between scaling in the mean and conventional Feynman scaling on the basis of data at these energies alone. More higher-energy data are needed.

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[9]See, for example, F. T. Dao, in *Particles and Fields—1974*, proceedings of the 1974 Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by Carl E. Carlson (A.I.P., New York, 1975).

[10]There is also evidence that, asymptotically, each particle species will obtain a constant fraction of the total energy, thus making the application of the energy sum rule straightforward, even in the real world [D. Sivers, Phys. Rev. D 8, 4004 (1973)].

[11]The variable t can be $p_L/\langle p_L \rangle_n$ for an arbitrary n.

[12]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971); K. J. Biebl and J. Wolf, Phys. Lett. 37B, 197 (1971); A. Ballestrero and

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$$\int dz\, \psi(z) = \sum_n \sigma_n/\sigma = 1 .$$

In Slattery's fit to the charged multiplicity only even-prong cross sections, σ_n, are nonvanishing since charged particles must be produced in pairs owing to charge conservation. The sum over probabilities that must equal unity would involve only even-prong cross sections and we could write

$$\int dz\, \psi(z) = \sum_n \sigma_n/\sigma \cong 2 \sum_{n \text{ even}} \sigma_n/\sigma = 2 .$$

The change of normalization is a simple matter of dividing the coefficients by 2. Although there have been other fits to the KNO function [H. Weisberg, Phys. Rev. D 8, 331 (1973); D. Weingarten, Nucl. Phys. B70, 501 (1974)], there is no need to further complicate matters by considering them as well since, as we shall see, the inclusive cross section is not very strongly dependent on the shape of the KNO function.

[16]D. R. O. Morrison, in *Proceedings of the Fifth Hawaii Topical Conference on Particle Physics*, edited by P. N. Dobson, Jr., V. Z. Peterson, and S. F. Tuan (Univ. of Hawaii Press, Honolulu, 1974), section 6.

[17]It could be possible to add up an infinite number of terms, each of which vanishes at $x=0$, and still fill the hole so that the sum is nonzero at $x=0$. A concrete example where this occurs is in the integral equation for the inclusive cross section in the bootstrap model [A. Krzywicki and B. Petersson, Phys. Rev. D 6, 924 (1972); J. Finkelstein and R. D. Peccei, *ibid.* 6, 2606 (1972)]. In the simple soluble example that we have considered [R. J. Yaes, Phys. Rev. D 10, 941 (1974); 12, 805 (1975)] the driving term and each finite iteration of the integral equation vanishes at $x=0$, but the solution is obviously nonvanishing at $x=0$.

[18]K. Guettler, B. G. Duff, M. N. Prentice, S. J. Sharrock, W. M. Gibson, A. Duane, L. Leistam, H. Newman, H. Ogren, H. Bøggild, S. Henning, G. Jarlskog, U. Becker, R. Little, T. Sanford, and S. L. Wu, Phys. Lett. 64B, 111 (1976). See also L. Foá, Ref. 8, above.

[19]While an $s^{1/3}$ fit to the average multiplicity may be good at low energies, a logarithmic function seems to give a better fit at higher energies. This can be explained by the fact that phase space is three-dimensional at lower energies but becomes one-dimensional at high energies because of the transverse-momentum cutoff (see D. R. O. Morrison, Ref. 16, above).

[20]This result is not as trivial as it might seem. We have previously discussed the use of the energy-momentum-conservation sum rules to determine the average values of certain quantities in inclusive reactions and have shown that the average value of a certain quantity in a single-particle inclusive reaction is not necessarily the same as the average value of the same quantity in a two-particle inclusive reaction [R. J. Yaes, Lett. Nuovo Cimento 4, 611 (1972); Phys. Rev. D 7, 2161 (1973)].

[21]See, for example, R. J. Yaes, Lett. Nuovo Cimento 8, 365 (1973).

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