Bethe-Salpeter equation for $I = 1$ nucleon-nucleon scattering with one-boson exchange

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We extend our earlier work on the Bethe-Salpeter equation for $J = 0$ by considering in addition the isospin $I = 1$ phase shifts $^3P_1$, $^3P_2$, $^3F_2$, ϵ_2, and $^1D_2$. These require the solution of a system of eight (instead of four for $J = 0$) coupled integral equations which have again been solved by iteration and application of the Padé method. The kernel is the same as before; i.e., the same particles are exchanged with the same analytic form of the cutoff. However, in order to find a fair agreement of our phase shifts with the experimental data, it was necessary to lower the cutoff mass. The phase shifts obtained show the general deficiency of dropping too fast at higher energies. In addition, we have studied the operator Padé approach and find that relatively few off-shell states are indeed sufficient to obtain reasonable accuracy for the summation of the Born series in a one-loop approximation.

I. INTRODUCTION

A common starting point in the investigation of the nuclear-force problem within the off-energy-shell approach has been traditionally the Bethe-Salpeter equation (BSE). In order to obtain more tractable equations, one is led for practical reasons to discuss certain ways of reducing the BSE to generalized Lippmann-Schwinger–type equations, which also include some relativistic aspects. In a previous paper[1] (to be referred to as I) the full BSE for spinors with one-boson exchanges has been employed as a model to study for isospin $I=1$ and total angular momentum $J=0$ possible off-mass-shell effects. Since the one-pion-exchange graph with pseudoscalar interaction couples the positive- and negative-energy states strongly, it is expected that these effects are important. In particular, it was shown in I that the BSE introduces considerably more attraction in these partial waves as compared to the Blankenbecler-Sugar equation. The latter equation, one of the possible relativistic Lippmann-Schwinger–type equations, has been used in an actual detailed analysis of the nucleon-nucleon interaction within a one-boson-exchange model.[2]

In the present paper we extend the above BSE calculations to higher partial waves with $I=1$. Changing some parameters of the kernel as introduced in I, the $I=1$ partial waves can be reproduced fairly well. However, a fundamental deficiency of this "potential" manifests itself in a drop of the phase shifts at higher energies. The lack of agreement appears to be even stronger for the $I=0$ channels.

The purpose of the present paper is twofold: First the numerical solutions of the BSE for $J>0$ are given for the one-boson-exchange model and, assuming that this is a field-theoretical approach of some relevance to the actual physical problem, we secondly study the operator Padé approximation (OPA) in this framework. It has been observed by various authors[3] that the OPA seems to be the most natural generalization of the Schrödinger equation to field theory. For NN scattering it has been applied by Gammel *et al.*,[4] taking into account pion exchange only and studying the pole structure of the $^1S_0$ wave as a function of the coupling constant. Our work yields further evidence that below the one-pion-production threshold the one-loop ([1/1]) OPA with a few off-shell states is a very reasonable approximation to the solution of the BSE.

In Sec. II the BSE for $J>0$ is described. In particular we present a general description of the kernel for one-boson exchange and give some technical points which deviate from I. In Sec. III we briefly discuss the OPA and Sec. IV contains numerical results and their discussion.

II. THE BETHE-SALPETER EQUATION FOR $J>0$

The BSE is formally the same as (2.9) of I:

$$\phi(p, p_0, \alpha) = G(p, p_0, \alpha; \hat{p}, 0, \kappa) - \frac{i}{2\pi^2} \int dq dq_0 \sum_{\beta,\gamma} G(p, p_0, \alpha; q, q_0, \beta) S(q, q_0, \beta, \gamma) \phi(q, q_0, \gamma), \tag{2.1}$$

with $\hat{p}=(E^2-m^2)^{1/2}$, except that for $J>0$ eight channels couple and the indices therefore take the values $\alpha,\beta,\gamma=1,2,\ldots,8$ (for the meaning of these labels, see Ref. 5). For $L=J$, the indices $1,\ldots,4$ refer to states even in the relative energy and $5,\ldots,8$ refer to states odd in relative energy. Similarly, for the coupled triplet states with $L=J\pm1$, the indices $1,\ldots,6$ refer to even states and 7, 8 refer to odd states. In the latter case there is a mixing between the ${}^3(L=J-1)_J$ and ${}^3(L=J+1)_J$ states, thus $\alpha=1,2$ in (2.1) are physical states and therefore κ takes the values $\kappa=1,2$ as well. From the physical (symmetric) 2×2 K matrix the nuclear bar phase shifts are then calculated.[6] In the present work we have performed the iteration and summation of the perturbation series with Padé approximants for each of the three independent matrix elements separately.

The kernel $G(p,p_0,\alpha;q,q_0,\beta)$ of the BSE contains in the present work the same exchanged particles as in I. Its general structure, as given by Kubis[5] for pseudoscalar and scalar exchanges, can be generalized to the case of vector and tensor exchanges, which have also matrix elements odd in the relative energy. This structure specifies various equalities between matrix elements, which we have attributed to CP in I. The labeling of the matrices is as in Ref. 5. Their general structure is for singlet $L=J$

	1	2	3	4	[5]	[6]	[7]	[8]
1						(1,5)		
2	(1,2)	(1,1)	−(1,3)	−(1,4)	−(1,5)	−(1,5)	−(1,7)	−(1,8)
3		−(3,1)				(3,5)		
4		−(4,1)				(4,5)		
[5]	−(1,5)	(1,5)						
[6]	−(1,5)	(1,5)	(5,3)	(5,4)	(5,6)	(5,5)	(5,7)	(5,8)
[7]		−(7,1)				(7,5)		
[8]		−(8,1)				(8,5)		

and for coupled triplet $L=J\pm1$

	1	2	3	4	5	6	[7]	[8]
1								
2								
3	(1,3)	(1,4)	(1,1)	(1,2)	−(1,5)	(1,6)	−(1,7)	(1,8)
4	(2,3)	(2,4)	(2,1)	(2,2)	−(2,5)	(2,6)	−(2,7)	(2,8)
5			−(5,1)	−(5,2)		0		0
6			(6,1)	(6,2)	0		0	
[7]			−(7,1)	−(7,2)		0		0
[8]			(8,1)	(8,2)	0		0	

where the explicitly labeled elements can be expressed by unlabeled ones in the indicated manner. The off-diagonal matrices (odd in the relative energy) connect bracketed (unbracketed) rows and unbracketed (bracketed) columns. The triplet $L=J$ are obtained from the singlet $L=J$ ones by a transposition (with the 4×4 submatrices as entities) as described in Ref. 5. However, there is an extra isospin factor -3 in the isospin-0 channel for isospin-1 exchange. The analytic expressions of the kernel matrices for the various exchanges have been determined by means of REDUCE.[7,8]

The basic elements (not explicitly labeled above) are given in the following general form: They are expressed in terms of some momentum-dependent factors, Clebsch-Gordan coefficients, and coefficients $C(i)$; $i=1,2,3,\ldots$. Only these coefficients $C(i)$ depend on the specific type of

exchanged particle. They are normalized such that for scalar, pseudoscalar, and "photon" exchange they simply are Legendre functions of the second kind (Q_J) or combinations thereof.

As in I the elements listed stand for

$$2E(p)E(q)G(p, p_0, \alpha;\ q, q_0, \beta).$$

Taking into account the symmetry property

$$G(q, q_0, \beta; p, p_0, \alpha) = G(p, p_0, \alpha; q, q_0, \beta)$$

renders it sufficient to give only the elements with $\alpha \leq \beta$.

For singlet $L=J$ we introduce the following abbreviations:

$$\begin{aligned} x_1 &= E(p)E(q)C(1), \quad x_2 = pqC(2)+C(3), \\ x_3 &= E(p)E(q)C(11), \quad x_4 = pqC(12)+C(13), \\ x_5 &= E(p)E(q)C(21), \quad x_6 = pqC(22)+C(23), \\ x_7 &= E(p)E(q)C(31), \quad x_8 = pqC(32)+C(33). \end{aligned} \tag{2.2}$$

The general form of the basic elements is then given by ($k_0 = p_0 - q_0$)

$$\begin{aligned}
&(1,1) = x_1 + x_2, \\
&(1,2) = x_1 - x_2, \\
&(1,3) = \sqrt{2}\,\{c_J[pC(4)+qC(5)] + c_{J+1}pE(q)C(6)\}, \\
&(1,4) = \mathcal{E}(1,3), \\
&(1,5) = C(7)k_0, \\
&(1,7) = \sqrt{2}\,\{c_J E(q)[pC(8)+qC(9)] + c_{J+1}pC(10)\}k_0, \\
&(1,8) = \mathcal{E}(1,7), \\
&(3,3) = 2\{c_J^2 x_4 + c_J c_{J+1}[E(p)+E(q)]C(14) + c_{J+1}^2 x_3\}, \\
&(3,4) = 2\{c_J^2 E(q)C(6) + c_J c_{J+1}(x_3 - x_4) + c_{J+1}^2 E(p)C(15) + [c_J^2 E(p) - c_{J+1}^2 E(q)]C(16)\}, \\
&(3,5) = \sqrt{2}\,\{c_J qC(17) + c_{J+1}E(p)[qC(18) + pC(19)]\}k_0, \\
&(3,7) = 2\{c_J^2 E(q)C(20) + c_J c_{J+1}[C(24)+C(25)] + c_{J+1}^2 E(p)C(26)\}k_0, \\
&(3,8) = 2\{c_J^2 C(24) - c_J c_{J+1}[E(q)C(20) - E(p)C(26)] - c_{J+1}^2 C(25)\}k_0, \\
&(4,4) = \mathcal{E}(3,3), \\
&(4,5) = \mathcal{E}(3,5), \\
&(4,7) = -\mathcal{E}(3,8), \\
&(4,8) = \mathcal{E}(3,7), \\
&(5,5) = x_5 + x_6, \\
&(5,6) = -x_5 + x_6, \\
&(5,7) = \sqrt{2}\,\{c_J E(q)[pC(27)+qC(28)] + c_{J+1}[pC(29)+qC(30)]\}, \\
&(5,8) = \mathcal{E}(5,7), \\
&(7,7) = 2\{c_J^2 x_7 + c_J c_{J+1}[E(p)+E(q)]C(34) + c_{J+1}^2 x_8\}, \\
&(7,8) = 2\{c_J^2 E(p)C(35) + c_J c_{J+1}(x_8 - x_7) + c_{J+1}^2 E(q)C(36) + [c_J^2 E(p)q^2 - c_{J+1}^2 E(q)p^2]C(37)\}, \\
&(8,8) = \mathcal{E}(7,7).
\end{aligned} \tag{2.3}$$

The representation of the basic elements has been simplified by taking into account that various elements are obtained from others by just exchanging the Clebsch-Gordan coefficient

$$c_J = \left(\frac{J}{2J+1}\right)^{1/2} \text{ and } c_{J+1} = \left(\frac{J+1}{2J+1}\right)^{1/2}. \tag{2.4}$$

In equation (2.3) the symbol $\mathcal{E}(\kappa, \lambda)$ has been

introduced to represent the expression obtained from the (κ, λ) matrix element by simply replacing c_J by $-c_{J+1}$ and c_{J+1} by c_J in this matrix element.

Similarly we define for coupled triplet $L=J\pm1$

$$
\begin{aligned}
&y_1=E(p)E(q)C(1), \quad y_2=pqC(2)+C(3),\\
&y_3=E(p)E(q)C(4), \quad y_4=pqC(5)+C(6),\\
&y_5=E(p)C(7), \quad y_6=-E(q)C(8),\\
&y_7=E(p)C(9), \quad y_8=-E(q)C(10),\\
&z_1=c_J{}^2y_1+c_Jc_{J+1}(y_5-y_6)+c_{J+1}{}^2y_2,\\
&z_2=c_J{}^2y_4-c_Jc_{J+1}(y_7-y_8)+c_{J+1}{}^2y_3,\\
&z_3=c_J{}^2y_5-c_Jc_{J+1}(y_1-y_2)+c_{J+1}{}^2y_6,\\
&z_4=c_J{}^2y_8+c_Jc_{J+1}(y_3-y_4)+c_{J+1}{}^2y_7,\\
&z_5=\mathcal{E}z_1, \quad z_6=\mathcal{E}z_2,\\
&z_7=\mathcal{E}z_3, \quad z_8=\mathcal{E}z_4
\end{aligned}
\tag{2.5}
$$

and the general form is

$$
\begin{aligned}
&(1,1)=z_1+z_2,\\
&(1,2)=z_3+z_4,\\
&(1,3)=z_1-z_2,\\
&(1,4)=z_3-z_4,\\
&(1,5)=\sqrt{2}\{c_J[pC(11)+qC(12)]+c_{J+1}E(p)qC(13)\},\\
&(1,6)=\sqrt{2}\{c_JE(p)qC(14)+c_{J+1}[pC(15)+qC(16)]\},\\
&(1,7)=\sqrt{2}\{c_J[pC(17)+qC(18)]\\
&\qquad +c_{J+1}E(p)qC(19)\}E(q)k_0,\\
&(1,8)=\sqrt{2}\{c_JE(p)qC(20)\\
&\qquad +c_{J+1}[pC(21)+qC(22)]\}E(q)k_0,\\
&(2,1)=-z_7-z_8,\\
&(2,2)=z_5+z_6,\\
&(2,3)=-z_7+z_8,\\
&(2,4)=z_5-z_6,\\
&(2,5)=\mathcal{E}(1,5),\\
&(2,6)=\mathcal{E}(1,6),\\
&(2,7)=\mathcal{E}(1,7),\\
&(2,8)=\mathcal{E}(1,8),\\
&(5,5)=2[pqC(23)+C(24)],\\
&(5,7)=2C(25)E(q)k_0,\\
&(6,6)=2[pqC(26)+C(27)],\\
&(6,8)=2C(28)E(q)k_0,\\
&(7,7)=2E(p)E(q)C(29),\\
&(8,8)=2E(p)E(q)C(30).
\end{aligned}
\tag{2.6}
$$

p, q, $E(p)$, $E(q)$, and k_0 are the three-momentum, off-shell energy, and relative energy as defined in I. The C's are given in Tables I and II for the various exchanged particles.

Besides the Legendre functions of second kind and index J [$Q_J(z)$, z as in I], we have introduced the following combinations (in slight deviation from Ref. 5):

$$
\begin{aligned}
&Z_J=c_{J+1}{}^2Q_{J+1}+c_J{}^2Q_{J-1},\\
&R_J=c_J{}^2Q_{J+1}+c_{J+1}{}^2Q_{J-1},\\
&ZQ_J=c_{J+1}{}^2Z_{J+1}+c_J{}^2Z_{J-1},\\
&ZR_J=c_J{}^2Z_{J+1}+c_{J+1}{}^2Z_{J-1},\\
&S_J=c_Jc_{J+1}(Q_{J+1}-Q_{J-1}),\\
&ZS_J=c_Jc_{J+1}(Z_{J+1}-Z_{J-1}).
\end{aligned}
$$

It should be noted that we have here used the phase convention of Jacob and Wick as described in Ref. 5, Appendix B. This means that the singlet-triplet elements have opposite sign compared to the ones in I.

After the Wick-rotation the real K's [see (4.7)–(4.10) of I] are obtained in the same manner as before. One only has to observe that in those formulas (see I) the latin indices stand for even and "4" stands for odd states.

The two-nucleon propagator S is given by

$$
S=\begin{bmatrix}
S_{++} & 0 & & & & & & \\
0 & S_{--} & & & & & & \\
 & & S_{ee} & 0 & & & S_{e0} & 0 \\
 & & 0 & S_{ee} & & & 0 & S_{e0} \\
 & & & & S_{++} & 0 & & \\
 & & & & 0 & S_{--} & & \\
 & & S_{0e} & 0 & & & S_{00} & 0 \\
 & & 0 & S_{0e} & & & 0 & S_{00}
\end{bmatrix}
\tag{2.8}
$$

for $L=J$ and

$$
S=\begin{bmatrix}
S_{++} & 0 & & & & & & \\
0 & S_{++} & & & & & & \\
 & & S_{--} & 0 & & & & \\
 & & 0 & S_{--} & & & & \\
 & & & & S_{ee} & 0 & S_{e0} & 0 \\
 & & & & 0 & S_{00} & 0 & S_{0e} \\
 & & & & S_{0e} & 0 & S_{00} & 0 \\
 & & & & 0 & S_{e0} & 0 & S_{ee}
\end{bmatrix}
\tag{2.9}
$$

for the coupled-triplet states $L=J\pm1$.

A final point of interest is the introduction of a new subtraction function. Instead of $f(q, q_0)$ given by (3.8) of I we now use $g(q, q_0)=qf(q, q_0)$. In (3.10) of I $f(q, q_0)$ has to be replaced by $g(q, q_0)$, which

TABLE I. Coefficients $C(i)$ for singlet $L=J$ and the various exchanged particles. These are specified by P: pseudoscalar, S: scalar, V_1: $-\gamma_\mu^{(1)}\gamma^{\mu(2)}$ ("photon exchange"), V_2: $\not{k}^{(1)}\not{k}^{(2)}$, M: mixed coupling. $E_1, E_2, \ldots$ are the same as in I.

i	P	S	V_1	$\mu_B{}^2V_2$	$4T$	M
1	$-Q_J$	Q_J	$-4Q_J$	$-E_6Q_J+2pqZ_J$	$-3(E_6Q_J-2pqZ_J)$	
2	Z_J	$-Z_J$		$-[E_1+2E(q)^2]Z_J+2pqQ_J$	$(2-E_6)Z_J+2pqZQ_J$	$2Z_J$
3	Q_J	Q_J	$2Q_J$	$-[E_1-2p^2]Q_J$	$-(E_5+2k_0{}^2)Q_J$	$-(p^2+q^2)Q_J$
4	Z_J	$-Z_J$		$-E_1Z_J$	$-(2q^2+E_6)Z_J+2pqZQ_J$	$(1-q^2)Z_J$
5	$-Q_J$	$-Q_J$	$-2Q_J$	E_1Q_J	$-C(3)$	$-(1-p^2)Q_J$
6	$-S_J$	S_J		E_1S_J	$E_6S_J-2pqZS_J$	$-S_J$
7					$4pqS_J$	pqS_J
8				$2Z_J$	$-4Z_J$	$-Z_J$
9				$-2Q_J$	$4Q_J$	
10				$-2E(q)^2S_J$	$4S_J$	S_J
11	R_J	$-R_J$		$E_5R_J-2pqQ_J$	$-E_6R_J+2pqZR_J$	
12	Q_J	Q_J	$2Q_J$	E_2Q_J	$C(3)$	$(1-q^2)Q_J$
13	Z_J	$-Z_J$		$E_2Z_J-2p(pZ_J-qQ_J)E(q)^2$	$(2p^2q^2-E_6)Z_J+2pqZQ_J$	$pqE(q)^2Q_J-(p^2+q^2)Z_J$
14	$-S_J$	S_J		$-\{E_5+2[1-E(p)E(q)]\}S_J$	$C(6)$	$-[1-E(p)E(q)]S_J$
15	S_J	$-S_J$		E_2S_J	$-C(6)$	S_J
16						$E(p)E(q)S_J$
17				$2E(p)^2S_J$	$-4p^2S_J$	S_J
18				$-2R_J$		$-R_J$
19				$2Q_J$		
20				$-2p(pZ_J-qQ_J)$	$4p(pZ_J-qQ_J)$	$-Z_J$
21	Q_J	Q_J	$-2Q_J$	$-E_6Q_J+2pqR_J$	$-E_6Q_J-2pq(Z_J-2R_J)$	
22	$-R_J$	$-R_J$	$-2R_J$	$-(2+E_6)R_J$	$E_6R_J+2pq(ZR_J-2Q_J)$	$-2R_J$
23	$-Q_J$	Q_J		$(2p^2q^2+E_5)Q_J$	$E_6Q_J-2pqZ_J$	$(p^2+q^2)Q_J$
24				$2[E(p)^2-E(q)^2]S_J$	$C(17)$	S_J
25						$E(p)E(q)S_J$
26				$-2q(pQ_J-qR_J)$		$-R_J$
27	S_J	S_J	$2S_J$	E_1S_J	$-E_6S_J-2pqZS_J$	S_J
28					$C(7)$	
29	$-R_J$	$-R_J$	$-2R_J$	$-E_1R_J$	$C(22)$	$-(1-q^2)R_J$
30	Q_J	$-Q_J$		E_1Q_J	$-C(23)$	$(1-p^2)Q_J$
31	$-Z_J$	$-Z_J$	$-2Z_J$	$E_5Z_J-2pqQ_J$	$-(3E_5-2k_0{}^2)Z_J+2pq(ZQ_J+2Q_J)$	
32	$-Q_J$	Q_J		$(2+E_5)Q_J$	$C(23)+2(ZR_J-2Q_J)$	$-2Q_J$
33	$-R_J$	$-R_J$	$-2R_J$	$-(2p^2q^2+E_6)R_J$	E_6R_J	$(p^2+q^2)R_J$
34	S_J	S_J	$2S_J$	$C(14)$	$\{E_5+2[E(p)-E(q)]^2\}S_J-2pqZS_J$	$[1-E(p)E(q)]S_J$
35	S_J	S_J	$2S_J$	$-E_2S_J=-C(15)$	$4p^2S_J+C(27)$	
36	$-S_J$	$-S_J$	$-2S_J$	$-E_1S_J$	$-4q^2S_J-C(27)$	
37						$-S_J$

yields

$$\text{subt} = \frac{1}{8E^2} \,. \tag{2.10}$$

This subtraction is better suited for lower energies (below 10 MeV) because of its finiteness at the elastic threshold ($\hat{p}=0$) and gives more stability to the Padé approximations in this energy range.

III. OPERATOR PADÉ APPROXIMATIONS

Referring to the literature on OPA's in general[3] we present here a more heuristic argument that these approximants may be useful to solve the BSE. Assuming a mesh fine enough for the numerical integration we can formally write (2.1) as

$$\phi = G + GDG + GDGDG + \cdots \,, \tag{3.1}$$

where each term in (3.1) is assumed to be a matrix in the product space of the spin variables and discretized external off-shell momenta. The D's represent the propagator S together with the weight factors arising from the discretization of the integrations over the intermediate momenta.

TABLE II. Coefficients $C(i)$ for coupled triplet $L=J\pm1$ and the various exchanged particles. See caption of Table I.

i	P	S	V_1	$\mu_B{}^2V_2$	$4T$	M
1	$-Z_J$	Z_J		$-E_6Z_J+2pqQ_J$	$E_6Z_J-2pqZQ_J$	
2	$-Q_J$	$-Q_J$	$-2Q_J$	$-(2+E_6)Q_J$	$E_6Q_J+2pqZ_J-2(2pqR_J+ZR_J)$	$-2Q_J$
3	$-R_J$	R_J		$(2p^2q^2+E_5)R_J$	E_6R_J	$(p^2+q^2)R_J$
4	R_J	R_J	$-2R_J$	$-E_6R_J+2pqQ_J$	$-E_6R_J+2pq(2Q_J-ZR_J)$	
5	Q_J	$-Q_J$	$-4Q_J$	$C(2)$	$3(E_6Q_J-2pqZ_J)-2(2Q_J+ZQ_J)$	$-6Q_J$
6	Z_J	Z_J	$-2Z_J$	$(2p^2q^2+E_5)Z_J$	$(3E_5-2k_0{}^2)Z_J$	$3(p^2+q^2)Z_J$
7	S_J	$-S_J$		E_1S_J	$-E_6S_J+2pqZS_J$	$-q^2S_J$
8	S_J	$-S_J$		$-E_2S_J$	$C(7)$	$-p^2S_J$
9	S_J	S_J	$-2S_J$	$-E_1S_J$	$-(E_6-4q^2)S_J-2pqZS_J$	$3q^2S_J$
10	S_J	S_J	$-2S_J$	E_2S_J	$-(E_6-4p^2)S_J-2pqZS_J$	$3p^2S_J$
11	Q_J	$-Q_J$	$-4Q_J$	$-E_1Q_J$	$(3E_6+4q^2)Q_J+2q^2ZQ_J$	$-3(1-q^2)Q_J$
12	$-Z_J$	$-Z_J$	$2Z_J$	E_1Z_J	$-(E_5+8p^2+2q^2)Z_J$	$3(1-p^2)Z_J$
13	S_J	S_J	$-2S_J$	$-E_1S_J$	$-(E_1-2q^2)S_J-2pqZS_J$	$-3S_J$
14	$-S_J$	S_J		$-E_1S_J$	$E_6S_J-2pqZS_J$	$-S_J$
15	$-Q_J$	$-Q_J$	$-2Q_J$	$-E_1Q_J$	$E_6Q_J+2pqZ_J$	$-(1-q^2)Q_J$
16	R_J	$-R_J$		E_1R_J	$-(E_6+4p^2)R_J+2pqZR_J$	$(1-p^2)R_J$
17				$2Q_J$	$4Q_J$	$3Q_J$
18				$-2Z_J$	$-4Z_J$	
19				$2S_J$	$4S_J$	
20				$2S_J$		
21				$2Q_J$		Q_J
22				$-2R_J$		
23	Z_J	Z_J	$-2Z_J$	$(2+E_5)Z_J$	$C(6)$	$-6Z_J$
24	Q_J	$-Q_J$	$-4Q_J$	$-(2p^2q^2+E_6)Q_J$	$3(E_6Q_J-2pqZ_J)-2p^2q^2(2Q_J+ZQ_J)$	$3(p^2+q^2)Q_J$
25				$-2p(pQ_J-qZ_J)$	$-4p(pQ_J-qZ_J)$	$3Q_J$
26	$-R_J$	R_J		$(2+E_5)R_J$	$(E_6-4)R_J-2pqZR_J$	$-2R_J$
27	$-Q_J$	$-Q_J$	$-2Q_J$	$C(24)$	$E_6Q_J+2pqZ_J$	$(p^2+q^2)Q_J$
28				$-2p(pQ_J-qR_J)$		Q_J
29	$-Q_J$	$-Q_J$	$2Q_J$	$E_5Q_J-2pqZ_J$	$(E_5+2k_0{}^2)Q_J-2pqZ_J$	
30	Q_J	$-Q_J$		$E_5Q_J-2pqR_J$	$-E_6Q_J+2pqZ_J$	

Equation (3.1) can be formally summed to

$$\phi = G[G - GDG]^{-1}G, \tag{3.2}$$

which is precisely the [1/1] OPA.[3] From this we see that if we take for the external momenta the same mesh points as in the integrations, this would actually solve the BSE. However, we may as well start with just a few mesh points in the external momenta. By enlarging the dimensions of the matrices one must in principle obtain convergence towards the solution of the BSE. Ultimately, also the choice of the external points is arbitrary and need not coincide with the mesh points of the integrations.

Clearly, all of these considerations do in fact make sense only if convergence is obtained quickly, otherwise the dimension of the matrices is getting so large that there is no advantage as compared to the matrix-inversion method for solving the BSE, which is even slower than the Padé method used in I. However, in NN scattering the method indeed turns out to work reasonably well for the energies under consideration.

Finally a technical comment is in order. For the calculation of the OPA, the Born and box graphs are needed. The BSE may formally be written

$$\phi = \pm K + \frac{1}{\pi^2} KS\phi, \tag{3.3}$$

where the K's are the same as in I. Following Gammel and Menzel,[4] an additional minus sign has been introduced in the Born term for the odd-states columns in order to obtain a symmetric tangent matrix. It is clear that the physical matrix elements are not changed by this modification.

IV. RESULTS

With the BSE we have calculated the following $I=1$ phase shifts: $^1S_0$, $^3P_0$, $^3P_1$, $^3P_2$, $^3F_2$, ϵ_2, and $^1D_2$. These were calculated at first with the same parameters as given in table 1 of I. The results are represented in Figs. 1–5 (full lines). Although the energy dependence is qualitatively reasonable, we see from these figures that the phase shifts drop too fast at higher energies. Also the $^3P_1$, $^3P_2$, and $^1D_2$ channels are too repulsive for this set of parameters. Trials to obtain a better fit to the phase shifts essentially indicated that in any case a lower cutoff mass was needed. The dashed lines in Figs. 1–5 correspond to the following parameter set: $\Lambda^2 = 0.9$, $g_\epsilon{}^2/4\pi = 7.8$, with all other parameters as in Table 1 of I. The new phase shifts are better than the former ones, but still indicate the same deficiency at higher energies. Varying the ω and ρ coupling constants did not cure this behavior.

Though these results indicate a deficiency at higher energies we do assume, however, that the underlying mechanism is closely enough related to physics to allow a relevant study of the OPA which has now to be compared with the solution of the BSE.

The simplest approximation which can be considered is the one with only the on-shell momentum in the external states. As can be seen from Figs. 1–5, this approximation is already very reasonable for the $^3P_2$ and $^1D_2$ waves. For the other ones it gives by far not enough repulsion at higher energies (~100–200 MeV).

However, introducing off-shell momenta in the external states improves the situation drastically. For convenience we have taken as off-shell momenta the mesh points of the BSE as described in I, Sect. 5 (q and iq_4, not the points in the single integral which one might choose as well[4]). In the present work we have chosen for one $^1S_0$ $C_p = C_{p4} = 2$ and $= 1$ for the higher waves (see I).

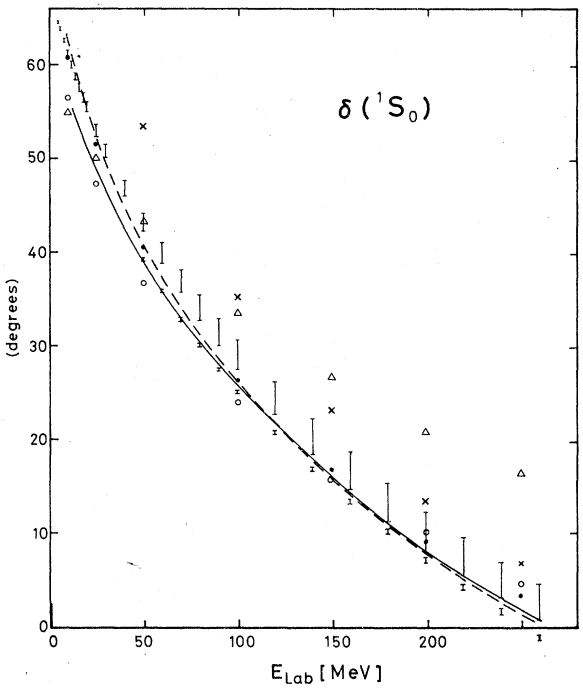

FIG. 1. Results for the $^1S_0$ phase shift. The solid line is the former result of Ref. 1. With $\Lambda^2 = 0.9$, $g_\epsilon{}^2/4\pi = 7.8$ and the other parameters as in Table 1 of Ref. 1 we obtain the dashed line. The results obtained by means of the OPA's are also indicated. The external off-shell states are labeled by their number as Gaussian integration point in the (q, iq_4) plane as described in the text. Δ corresponds to only the on-shell momentum as external state; ○ has (10, 5) as additional external point, × has (8, 3) and (12, 7); ● has (8, 3), (10, 5), and (12, 7). The experimental points are taken from M. H. Mac Gregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 182, 1714 (1969).

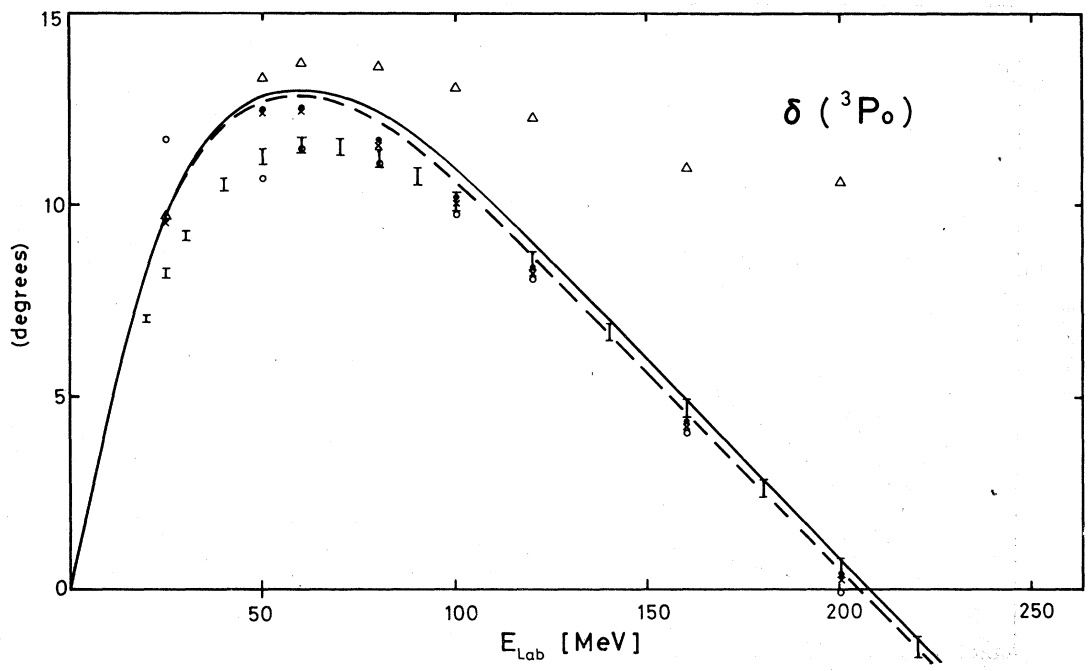

FIG. 2. Results for the $^3P_0$ phase shift as in Fig. 1.

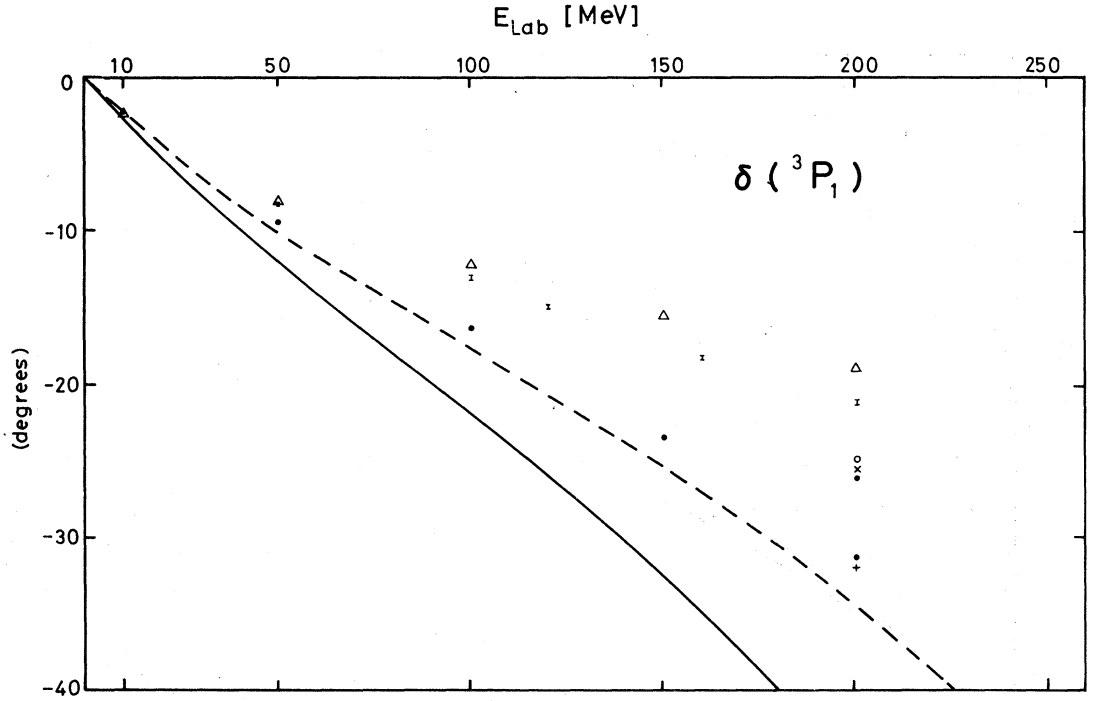

FIG. 3. Results for the $^3P_1$ phase shift. The solid line is calculated with the parameters of Ref. 1, the dashed line with the new ones as in Fig. 1. For the OPA's we have used, in addition to the on-shell momentum (Δ), three low external momenta (2, 3), (7, 4), and (8, 5) indicated by ●. At 200 MeV we show the slow convergence with the large external momenta as used for $J=0$. Also at 200 MeV we have added to the above low-external-momentum states the large one (10, 6) indicated by +.

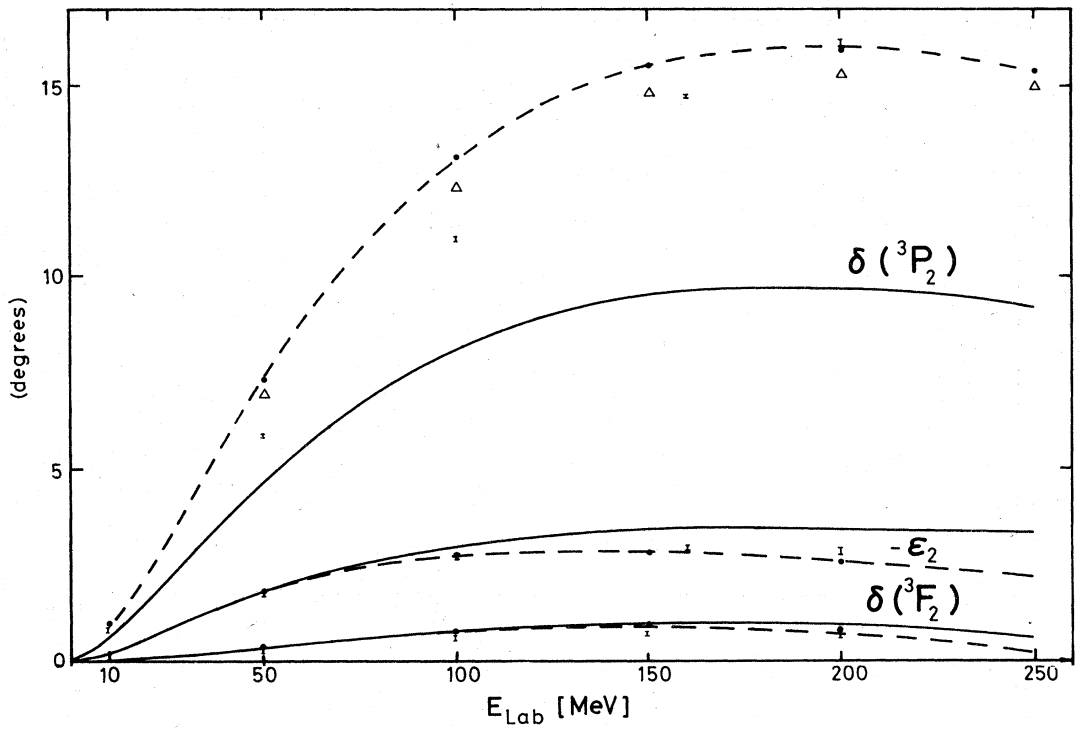

FIG. 4. Results for the $^3P_2$ phase shift as in Fig. 3 with only three low-momentum states (●) in addition to the on-shell state (△).

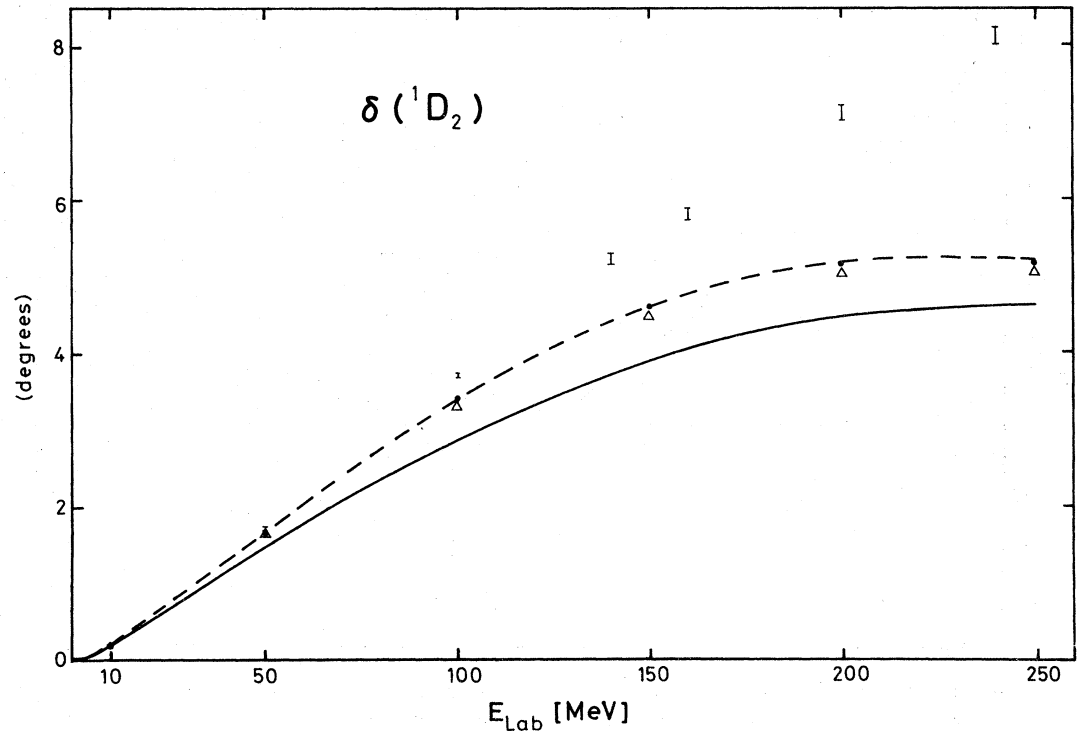

FIG. 5. Results for the $^1D_2$ phase shift as in Fig. 3 with only three low-momentum states (●) in addition to the on-shell state (△).

Thus each point is given by its number as Gaussian integration point. The choice of the BSE points is suggested also because they have been adjusted already to cover the off-shell domain, where the most important contributions come from. The total number of points in the BSE was 16 in the q integration (where the first 8 cover the range $0 \leq q \leq 2\hat{p}$) and 12 in the q_4 integral.

Introducing in addition to the on-shell momentum three off-shell momenta with $0 \leq q \leq 2\hat{p}$ improves the waves with $J \geq 1$ considerably: The $^3P_2$ and $^1D_2$ are practically exact while the $^3P_1$ would need some more states to obtain complete agreement. The example of the $^3P_1$ indicates that the choice of the low momenta is crucial for the high waves. For 200 MeV the result of introducing a number of points with high momenta ($q \gtrsim 2\hat{p}$) is given for the $^3P_1$ in Fig. 3. It shows that convergence toward the right answer is not achieved though the same number of points is used.

In contrast to that, the low momenta are not so important for $J = 0$. With three additional momenta $q \gtrsim 2\hat{p}$ the $^3P_0$ is described within 0.3 degrees (Fig. 2). This is an important improvement compared to the approximation with only the on-shell momentum in the external states. Similarly for the $^1S_0$. The introduction of additional three high off-shell momenta reproduces almost the right energy dependence.

Concerning the stability of the OPA's, in general we observe that best stability is obtained for high partial waves and large energies. Experimenting with additional off-shell momenta (4 and 5) and taking different ones for fixed numbers as well, shows that for $E_{\text{lab}} \gtrsim 50$ MeV the OPA's for the $^1S_0$ are stable within $\sim 1^0$. It should be observed that in general the stability will also depend on the particular model under consideration.

In conclusion one may say that the OPA works surprisingly well and will be useful in several respects. Clearly, it will simplify the fitting of the BSE kernel to the experimental phase shifts because it is not so time consuming with only a few external off-shell momenta. As a result we may hope that the BSE can form a practical basis for a fully relativistic treatment of the nuclear-force problem. In particular, since three-particle unitarity can readily be incorporated in this set of equations.[9] They can be used in principle to describe two-nucleon scattering above the one-pion-production threshold.

Beyond that the OPA approach may provide a firm basis for perturbative calculations in strong interactions. Taking into account within this framework all one-loop graphs of a Lagrangian describing NN interaction, e.g., one may hope to get a good approximation to the complete perturbation series. Attempts taking into account only the on-shell momentum have already been performed.[10, 11] In these calculations it has always been a severe deficiency that the inversion of the $^1S_0$ and $^3P_0$ could not be obtained properly. Our results show that the introduction of further off-shell states is certainly very helpful in that respect.

ACKNOWLEDGMENT

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