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Mass corrections in deep-inelastic scattering*

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The moment sum rules for deep-inelastic lepton scattering are expected for asymptotically free field theories to display a characteristic pattern of logarithmic departures from scaling at large enough Q^2. In the large-Q^2 limit these patterns do not depend on hadron or quark masses m. For modest values of Q^2 one expects corrections at the level of powers of m^2/Q^2. We discuss the question whether these mass effects are accessible in perturbation theory, as applied to the twist-2 Wilson coefficients and more generally. Our conclusion is that some part of the mass effects must arise from a nonperturbative origin. We also discuss the corrections which arise from higher orders in perturbation theory for very large Q^2, where mass effects can perhaps be ignored. The emphasis here is on a characterization of the Q^2, x domain where higher-order corrections are likely to be unimportant.

I. INTRODUCTION

The development of asymptotically free gauge theories[1] as an approach to the strong interactions was prompted, initially, by the experimental observation of approximate Bjorken scaling for the structure functions of deep-inelastic lepton scattering. A renormalization-group analysis of interacting field theories revealed that exact scaling could never be achieved,[2] but in the unique class of non-Abelian gauge theories the departures from exact scaling can be reliably discussed for the limit of very large Q^2. In this limit one finds characteristic and mild (logarithmic) deviations from scaling for the moments of the structure functions. These effects are governed by the leading order in perturbation theory and are independent of physical hadron masses and of quark-mass parameters. The question arises whether one can rely on perturbation methods to deal with corrections to this leading behavior: in particular, "mass" corrections at the level of powers of m^2/Q^2, where m represents characteristic hadrons and/or quark masses. This is clearly important for the present-day regime of experimentation, where m^2/Q^2 is not all that tiny.

We shall offer some general remarks on this issue. In particular, we initially discuss the possibility that mass corrections may be adequately allowed for through use of the ξ-scaling variable introduced in Refs. 3, 4, and 5. On an extreme theoretical picture, ξ scaling would seem to emerge from the notion that one can employ free-field theory (supplemented, perhaps, by low-order perturbative corrections) for the Wilson coefficients in the short-distance expansion of operator products.[6] The authors of Refs. 3, 4, and 5 have in fact carefully qualified the applicability of the free-field approximation, emphasizing its progressive breakdown with increasing order of spin. The theoretical connection of ξ scaling and free-field behavior is therefore subtle and approximate. We shall see this in another way here: The unqualified free-field approximation leads generally to paradoxes that cannot be resolved unless the structure functions vanish identically in a certain *physical* region—surely an unrealistic requirement. In the subsequent discussion we raise the specter that mass corrections may in fact not fully be accessible at all in perturbation theory—that some part of these corrections may arise from nonperturbative effects. In addition, for very large Q^2, we estimate the importance of higher-order perturbative corrections in order to assess how rapidly the structure function approaches its asymptotic behavior in Q^2 for different regions of the x variable.

Let us begin immediately by recalling how one analyzes the structure functions for deep-inelastic scattering on the basis of the short-distance properties of products of currents. We simplify matters by restricting ourselves to the case of scalar currents $j(x)$. This will suffice to bring out all of the qualitative issues that are of concern here. The structure function F is the absorptive part of the amplitude A for forward Compton scattering of the current on a hadron target (a nucleon p, say). The Compton amplitude is given by

$$A = i\int dx\, e^{iq\cdot x}\langle p|T(j(x)j(0))|p\rangle\,, \qquad (1)$$

where q is the four-momentum carried by the current. With M the target-nucleon mass ($p^2=M^2$), we adopt the usual variables $\nu = p\cdot q/M$, $Q^2=-q^2>0$. Following Wilson,[7] one expands the product of currents as a sum of local operators $O^{(n)}_{\mu_1,\mu_2,\ldots,\mu_n}$ of definite spin n:

$$T(j(x)j(0)) = \sum_n C^{(n)}(x^2)\chi^{\mu_1}\cdots\chi^{\mu_n}O^{(n)}_{\mu_1,\ldots,\mu_n}. \qquad (2)$$

There may be several different operators in the expansion with the same spin n; we incorporate the additional index needed to distinguish these in the one index n. Organization of the expansion by angular momentum is convenient because the Wilson coefficients $C^{(n)}$ in this basis obey simple renormalization-group equations. From the Fourier transforms, let us define the moment functions

$$M^{(n)}(Q^2)=(Q^2)^n\left(\frac{\partial}{\partial Q^2}\right)^n\int dx\, e^{iq\cdot x}C^{(n)}(x^2)\,. \qquad (3)$$

The mathematical problem of extracting from F information on the moment functions has been solved by Nachtmann,[8] who shows that

$$\int_0^1 dx\, \xi^n F(x,Q^2)=a_n M^{(n)}(Q^2)\,, \qquad (4)$$

where the a_n are Q^2-independent constants determined by the reduced matrix elements $\langle p\| O^{(n)}\| p\rangle$. Here $x=Q^2/2M\nu$ is the usual Bjorken variable and

$$\xi=\frac{2x}{1+(1+4M^2x^2/Q^2)^{1/2}}\,. \qquad (5)$$

The renormalization-group equation for $M^{(n)}(Q^2)$ is governed by an effective Q^2-dependent coupling constant $\bar g(Q^2)$. In asymptotically free field theories $\bar g\to 0$ as $Q^2\to\infty$, so for large enough Q^2 one can hope to rely on a perturbative treatment of the Wilson coefficients. To leading order as $Q^2\to\infty$ one finds

$$M^{(n)} \underset{Q^2\to\infty}{\propto} \left(\log\frac{Q^2}{\mu^2}\right)^{-d_n}, \qquad (6)$$

where μ is a scale mass and where the exponents d_n—related to the anomalous dimensions of the operators $O^{(n)}$—are fully calculable given the group structure and quark content of the underlying non-Abelian gauge theory. Notice that the Nachtman variable ξ becomes identical to the Bjorken variable x in the limit $Q^2\to\infty$.

The expression in Eq. (6) represents the leading term of an expansion in $\bar g^2$—effectively an expansion in inverse powers of $\log(Q^2/\mu^2)$. For large Q^2 the departures from exact Bjorken scaling are seen to be logarithmic. For more modest values of Q^2 the question arises whether one can say anything about corrections at the level of inverse powers of Q^2, where physical hadron and perhaps quark-mass parameters come into play, i.e., corrections at the level of powers of m^2/Q^2. We discuss this issue in Secs. II and III. In Sec. IV we return to very large Q^2 and consider the corrections arising from the nonleading orders in perturbation theory.

II. THE FREE-FIELD APPROXIMATION FOR MASS CORRECTIONS

Let us suppose that the effective coupling constant $\bar g(Q^2)$ is already small for modest values of Q^2, comparable to, or only somewhat larger than, typical hadron masses. For this situation it has been suggested that one can reasonably ignore all but the leading twist-2 operators, reverting, moreover, to low orders of perturbation theory for the corresponding Wilson coefficients.[3-5] In the extreme version of this sheme one simply uses free-field theory for the Wilson coefficients. For $Q^2\to\infty$ this ignores the logarithmic effects of Eq. (6); but it has been argued[4] that the procedure nevertheless correctly brings out—relevant for modest Q^2—the powers of m^2/Q^2. We shall initially consider this free-field approximation, and under this heading we first deal with the case where the quarks are massless.

The Wilson coefficients, hence the moment functions $M^{(n)}(Q^2)$, are universal quantities, independent of the hadron target. In the present approximation, therefore, they can be computed by studying the scattering of the current off a free, massless quark. But here, clearly, the quark target structure function is just $\delta(1-x)$, hence $M^{(n)}(Q^2)=1$, independent of Q^2. For the physical hadron target, therefore, this approximation leads to the result

$$\int_0^1 dx\,\xi^n F(x,Q^2)=a_n,\quad \text{independent of } Q^2. \qquad (7)$$

Notice that mass parameters (in this case, only the target mass M) enter explicitly here solely through the Nachtmann variable ξ. We want to see whether Eq. (7) makes sense for non-negligible values of M^2/Q^2.

Let us regard F as a function of ξ and Q^2 and rewrite Eq. (7) in the form

$$\int_0^{\xi_{\max}} d\xi\,\xi^n\tilde F(\xi,Q^2)=a_n\,, \qquad (8)$$

where

$$\tilde F=\left(\frac{\partial\xi}{\partial x}\right)^{-1}F \qquad (9)$$

and

$$\xi_{\max}(Q^2)=2/[1+(1+4M^2/Q^2)^{1/2}]\,. \qquad (10)$$

Since $\xi_{\max}$ depends on Q^2 and since it vanishes as $Q^2\to 0$, it is clear that Eq. (8) cannot hold all the way down to $Q^2=0$ unless the structure function vanishes identically everywhere. More reasonably, however, suppose that Eq. (8) is to be believed only for $Q^2>Q_0^2$, where Q_0^2 is some reference value. Let $\tilde F(\xi,Q_0^2)$ be the modified structure function at Q_0^2. For $Q^2=Q_0^2$ we can invert Eq. (8) to find the a_n, then, inverting, find $\tilde F$ for all $Q^2>Q_0^2$. For $Q^2>Q_0^2$ it is clear that we will find

$$\tilde F(\xi,Q^2)=\begin{cases}\tilde F(\xi,Q_0^2), & 0<\xi<\xi_{\max}(Q_0^2)\\ 0, & \xi_{\max}(Q_0^2)<\xi<\xi_{\max}(Q^2)\,.\end{cases} \qquad (11)$$

Thus, in the interval $0-\xi_{max}(Q_0^2)$ $\tilde{F}$ *scales* in ξ, i.e., is independent of Q^2 for fixed ξ.[9] But $\tilde{F}$ must vanish identically in the physically accessible interval from $\xi_{max}(Q_0^2)$ to $\xi_{max}(Q^2)$. Although the mathematics as such would permit this behavior, we must regard the outcome as physically (and experimentally) unreasonable. For example, if we were to believe the model down to $Q_0^2=M^2$, then for $Q^2 \gg M^2$ (where $\xi \approx x$) we would find that F must vanish identically for $0.62<x<1$. Of course, if we take $Q_0^2 \gg M^2$ the problem effectively evaporates, but then we also are not learning anything about mass-correction effects.

We turn next to the situation where the current involves massive quark fields. The meaning of quark mass parameters is, of course, somewhat problematic. Here the meaning is defined by the free-field procedures under discussion. Consider a current formed bilinearly from fields q_i, q_f, with mass parameters m_i, m_f. At the quark level the current scatters off q_i to produce q_f. Adopting again the free-field approximation for the twist-2 Wilson coefficients, one finds

$$M^{(n)}(Q^2)=[\rho(Q^2)]^n, \tag{12}$$

$$\rho(Q^2)=2Q^2\{Q^2+m_f^2-m_i^2 + [(Q^2+m_f^2-m_i^2)^2+4m_i^2Q^2]^{1/2}\}^{-1}. \tag{13}$$

Using this in Eq. (4) we find

$$\int_0^1 dx\, \xi^n F = a_n, \tag{14}$$

where

$$\xi = x\, \frac{1+\dfrac{m_f^2-m_i^2}{Q^2}+\left[\left(1+\dfrac{m_f^2-m_i^2}{Q^2}\right)^2+\dfrac{4m_i^2}{Q^2}\right]^{1/2}}{1+(1+4M^2x^2/Q^2)^{1/2}}. \tag{15}$$

Here we have generalized the definition of the symbol ξ to allow for nonvanishing quark masses: Equations (15) reduces to Eq. (5) when $m_i=m_f=0$.

In Eq. (14) we have formally placed the upper limit of the integral at $x=1$. This limit corresponds to elastic scattering off the nucleon target. However, in contemplating heavy-quark effects we have in view processes involving charm-changing currents, where q_f is a massive charmed quark, and q_i is a light, essentially massless quark. For scattering off an uncharmed (nucleon) target, the physical threshold for charm production lies below $x=1$. Let M_T be the threshold mass—the lightest charm-carrying state that can be reached by action of the current on a nucleon target (clearly $M_T>M$). Then the upper limit on the x variable is given by

$$x_{max}(Q^2)=\frac{Q^2}{Q^2+M_T^2-M^2}. \tag{16}$$

Corresponding to this is an upper limit $\xi_{max}(Q^2)$ on the variable defined in Eq. (15). In general ξ_{max} will depend on Q^2 and we will then find as before that the structure function F must vanish identically for some Q^2-dependent portion of the *physically* accessible range of x (or ξ). The only exception to this pathology occurs when $m_i=M$, $m_f=M_T$. Then $\xi_{max}=1$, independent of Q^2. This is not surprising since, kinematically at least, this corresponds to the situation where the target *is* the quark q_i and the produced state *is* the quark q_f. Under these circumstances, free field theory should indeed be self-consistent.

For practical applications to charm production in neutrino reactions one may reasonably set $m_i \approx 0$ and, at least roughly, set $m_f=M_T$ (perhaps this is to be regarded operationally as a definition of m_f). The pathology still remains, however, unless one also sets $M\approx m_i \approx 0$. When all of this is done, we are in the exceptional case where there is no obvious pathology. Here

$$\xi = x(1+m_f^2/Q^2), \tag{17}$$

$\xi_{max}=1$, and

$$\tilde{F}=\frac{Q^2}{Q^2+m_f^2}F, \tag{18}$$

where $\tilde{F}=\tilde{F}(\xi)$ is independent of Q^2 for fixed ξ and need not vanish in any finite portion of the physical range. It is still worrisome, however, already at this kinematical level, that one cannot incorporate finite-target-mass effects; and that self-consistency for the special case considered above requires the fine tuning involved in setting $m_f=M_T$. For the recently discovered charmed hadrons, with masses in the 2-GeV region, M_T is not very much larger than the nucleon-target mass. It may be that the free-field approximation becomes more reasonable for hadron families composed of still heavier quarks. The possible existence of such families is hinted at by certain anomalies in high-energy antineutrino reactions.[10]

Up to this point, we have been discussing the difficulties that arise when one combines the kinematics of finite-mass corrections with the short-distance approximation based on free-field theory for the Wilson coefficients. As we will next see, these difficulties are not removed when one includes the lowest-order perturbative corrections to the twist-2 Wilson coefficients. We illustrate this for the case of massless quarks. The lowest-order corrections are obtained by adopting Eq. (6) for the moment functions $M^{(n)}(Q^2)$. With this choice, Eq. (4) becomes

$$\int_0^{\xi_{\max}(Q^2)} d\xi\, \xi^n \tilde{F}(\xi, Q^2) = a_n \left(\ln\frac{Q^2}{\mu^2}\right)^{-d_n}. \qquad (19)$$

The asymptotic behavior, as $Q^2 \to \infty$, is now properly incorporated, but we want to see whether Eq. (19) is sensible for modest values of Q^2, down to some reference value Q_0^2 which is comparable to M^2 (but let us suppose that $Q_0^2 > \mu^2$). For $Q^2 = Q_0^2$ the modulus of the left-hand side of Eq. (19) is bounded by $C(Q_0^2)\xi_{\max}^n(Q_0^2)$, where C is a positive, n-independent constant. Therefore

$$|a_n| < C\left(\ln\frac{Q_0^2}{\mu^2}\right)^{d_n} \xi_{\max}^n(Q_0^2). \qquad (20)$$

Let us now consider Eq. (19) for $Q^2 > Q_0^2$, letting the index n become very large and recalling that

$$d_n \underset{n\to\infty}{\simeq} \beta \ln n, \qquad (21)$$

where β is some positive constant. Since $\ln Q^2/\mu^2 > \ln Q_0^2/\mu^2 > 0$, one finds that

$$\left|\int_0^{\xi_{\max}(Q^2)} d\xi\, \xi^n \tilde{F}(\xi, Q^2)\right| < C\xi_{\max}^n(Q_0^2). \qquad (22)$$

We now claim that this implies that $\tilde{F}(\xi, Q^2)$ must vanish identically in the physical interval $\xi_{\max}(Q_0^2) < \xi < \xi_{\max}(Q^2)$. To see this, suppose that $\tilde{F}$ vanishes in the interval $\xi_b \leq \xi \leq \xi_{\max}(Q^2)$ and that there is some finite interval $\Delta\xi$ just below ξ_b for which F has a definite, algebraic sign. For large enough n this interval dominates in the integral of Eq. (22), so that

$$\left|\int_0^{\xi_{\max}(Q^2)} d\xi\, \xi^n \tilde{F}(\xi, Q^2)\right| \underset{n\to\infty}{\geq} D\Delta\xi(\xi_b - \Delta\xi)^n, \qquad (23)$$

where D is some positive constant. Since we can take $\Delta\xi$ as small as we please, Eqs. (22) and (23) are consistent only if $\xi_b \leq \xi_{\max}(Q_0^2)$.

There is an obvious generalization of the above analysis. Return to Eq. (4) and recall that on the right-hand side there is an implied sum over the contributions of all operators of given spin n, so that the right-hand side reads $\sum_i a_n^{(i)} M^{(n,i)}(Q^2)$. For $n \to \infty$ presumably one term dominates for any given Q^2, and we suppose in fact that this one term dominates for some finite range of Q^2, say $Q_0^2 < Q^2 < Q_1^2$. For this interval of Q^2 we can therefore focus on the dominant term; with this understood, we drop the index i again. Let $M^{(n)}(Q^2)$ be some theoretical approximation to the moment function and take $Q_0^2 < Q^2 < Q_1^2$. If the approximation for $M^{(n)}(Q^2)$ is such that

$$\left(\frac{\xi_{\max}(Q_0^2)}{\xi_{\max}(Q^2)}\right)^n \left|\frac{M^{(n)}(Q^2)}{M^{(n)}(Q_0^2)}\right| \underset{n\to\infty}{\longrightarrow} 0,$$

then it will again follow for this approximation that F must vanish identically over a finite, physical interval of ξ, an indication that the approximation does not correctly accommodate finite-target-mass effects. As we have seen, this is just the situation that occurs in the free-field and lowest-order perturbative approximation to the twist-2 Wilson coefficients. In fact, we conjecture that this happens to all finite orders.

It should be clear from the preceding discussion that, kinematically, at least, the low-order approximations must especially fail for large values of spin n. The difficulties would be less severe if the approximations were adopted for a limited range of small spin values, but then the whole scheme would become less predictive and useful. To assess this more modest approach we might illustratively consider a simple scaling form for $\tilde{F}(\xi)$,

$$\tilde{F}(\xi) = A(1-\xi)^B$$

computing the moment functions in order to see, for various spin values, whether these functions are essentially independent of Q^2, as would be predicted by the free field approximation (recall that $\xi_{\max}$ depends on Q^2). The success is very limited if Q^2 is to be permitted to range down to values comparable to M^2. For example, with $B = 2$ we find for $n = 3.5$ that the moment function grows by a *factor of 2* as Q^2 range from M^2 to ∞.

The paradox that we have met here for the strictly-free-field approximation underlines the importance of higher-twist operators, as is recognized and discussed in Ref. 5. We comment further on this issue in the following section.

III. PERTURBATIVE AND NONPERTURBATIVE MASS CORRECTIONS

It is the common belief for asymptotically free field theories that the short-distance behavior of operator products can be calculated reliably by perturbative methods, with the aid of the renormalization group. This is most likely correct for the leading logarithmic terms at large Q^2, but we shall argue that it is probably not correct at the level of mass corrections (powers of m^2/Q^2). Indeed, we expect, in general, that the Wilson coefficients contain nonperturbative terms with a vanishing asymptotic expansion in the coupling constant. Such terms, we believe, will be most important in theories where the origin of hadron masses is dynamical. In particular, if the quark mass parameters are taken to be zero, so that hadronic masses arise from dynamical breaking of chiral symmetry, these nonperturbative mass corrections might well represent the dominant mass-dependent effects in the moment sum rules. We will illustrate this by means of a simple asymptotically free theory which exhibits dynamical breaking of chiral symmetry, the N-component two-dimensional $(\bar{\psi}\psi)^2$ model.[11] Since the real world is believed to possess an approximate chiral symmetry, we must conclude that estimates based on the renormalization group and perturbation theory are unlikely to account for the dominant mass-correction effects in the analysis of short-distance behavior.

There are two obvious sources of mass corrections. First, the Wilson coefficients of the dominant twist-2 operators will in general depend on the mass parameters of the theory. Thus if $O^{(n)}$ is a twist-2 operator of spin n, the Fourier transform of its Wilson coefficient, $M^{(n)}(Q^2,g,m,\mu)$, will depend on the mass parameters (m) of the theory, as well as the coupling constants (g). It is of course possible to render the Wilson coefficients independent of masses by regarding the operators $m^2O^{(n)}$, $m^4O^{(n)}, \ldots$ as independent. However, this is of no value. It is simpler not to make this breakup and to treat the mass dependence of the Wilson coefficients by means of the renormalization group. The moment functions (Fourier transforms of the Wilson coefficients) satisfy the equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_m(g)m\frac{\partial}{\partial m}-\gamma_{O^{(n)}}\right]M^{(n)}(Q^2,g,m,\mu)=0\,, \tag{24}$$

where γ_m is the anomalous dimension of the mass operator in a mass-independent renormalization scheme; $\gamma_{O^{(n)}}$ is the anomalous dimension of $O^{(n)}$ (which we take to be multiplicatively renormalizable). Let $t\equiv\ln(Q^2/Q_0^2)^{1/2}$. Then the solution is

$$M^{(n)}(Q^2,g,m,\mu)=Z^{(n)}(t)M^{(n)}(Q_0^2,\bar{g}(t),e^{-t}\bar{m}(t),\mu)\,, \tag{25}$$

where $\bar{g}(t)$ is the effective coupling constant,

$$\frac{d\bar{g}}{dt}=\beta(\bar{g}),\quad \bar{g}(0)=g\,,$$

and

$$\bar{m}(t)=m\exp\int_0^t dt'\gamma_m(\bar{g}(t'))\,,$$

$$Z^{(n)}(t)=\exp\left\{-\int_0^t dt'\,\gamma_{O^{(n)}}(\bar{g}(t'))\right\}\,.$$

In an asymptotically free theory $\bar{g}^2(t)\to 1/b_0t$ as $t\to\infty$. Thus, if $M^{(n)}(Q_0^2,\bar{g}(t),\ e^{-t}\bar{m}(t),\mu)$ has an asymptotic expansion in powers of $\bar{g}^2(t)$, one can compute the Wilson coefficients order by order in $\bar{g}^2(t)$, and the mass effects will show up in the Q^2-dependent parameter $e^{-t}\bar{m}(t)$.

However, if the quark mass parameters are set equal to zero then no mass terms can show up in perturbation theory for the Wilson coefficients. Since we believe that the mass parameters of the up and down quarks are small (approximate SU_2 chiral symmetry) it seems indeed reasonable to neglect mass dependence of the twist-2 Wilson coefficients for targets composed mainly of up and down quarks (as Witten has shown,[12] the contribution of "heavy"-quark operators is suppressed for such targets).

A second source of mass corrections arises from operators of twist greater than 2 in the operator-product expansion (e.g., $\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma_\mu\psi$, etc). These contribute powers of M^2/Q^2 in the moment sum rules, where M is the target mass parameter that enters from the hadronic matrix elements of the higher-twist operators. The size of these corrections cannot be calculated by perturbative methods, since the ratio of hadronic matrix elements for (say) the twist-4 relative to the twist-2 operators involves questions of the unknown wave functions for hadronic bound states. However, one might argue as in Ref. 4, since the Wilson coefficients of the higher-twist operators are invariably proportional to powers of $\bar{g}^2$, that it might be reasonable to neglect these operators if $\bar{g}^2$ is small, even if M^2/Q^2 is not negligible.

According to this view, it would seem reasonable to neglect *all* mass corrections to the moment functions $M^{(n)}$, provided that $\bar{g}^2(t)$ is small and the quark masses can be disregarded. In this case, mass enters the analysis only through the appearance of the target mass M in the Nachtmann variable ξ. However, as we have already seen these ideas lead to mathematical paradoxes—which suggest that something is amiss with the above reasoning. If the quark mass parameters can indeed be neglected (as for electroproduction) it would seem that the mass corrections must arise from higher-twist operators, even if $\bar{g}^2(t)$ is small. It is in fact likely for large values of the spin n that the higher-twist contributions are much larger than one would naively estimate merely from the size of the effective coupling constant; i.e., for given $\bar{g}^2(t)$ one expects the contributions to grow with n. Notice that the high-spin moments govern the form of the structure function in the threshold-resonance region (x close to unity).

We shall now argue that there is another source of mass corrections even for the massless-quark model, namely, the existence of essential singularities in the Wilson coefficients. Such effects are suggested by the very mechanism of dynamical symmetry breaking itself. In a theory with massless quarks, where hadron masses aries from symmetry breaking, all physical parameters $P(g,\mu)$ depend only on g and μ. They satisfy the renormalization-group equation

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}\right] P(g, \mu) = 0\,, \tag{26}$$

which reflects the fact that a change in the arbitrary renormalization parameter μ corresponds to a related change in the definition of g. This implies that a physical mass m_p is given by

$$m_p(g, \mu) = \mu \exp\left[-\int^g \frac{dx}{\beta(x)}\right]\,. \tag{27}$$

In an asymptotically free theory

$$m_p \underset{g \to 0}{\approx} \mu e^{-b_0/g^2}\,. \tag{28}$$

This has an essential singularity at $g = 0$, hence zero asymptotic expansion in powers of g. If such essential singularities occur in physical mass parameters they could also occur in Wilson coefficients. Thus, if the twist-2 Wilson coefficients contain terms such as

$$\exp\left[-2p \int^g \frac{dx}{\beta(x)}\right],$$

then $M^{(n)}(Q^2, g, \mu)$ will have terms such as

$$\exp\left[-2p \int^{\bar{g}(t)} \frac{dx}{\beta(x)}\right] = \text{const}\left(\frac{m_p^{\,2}}{Q^2}\right)^p\,. \tag{29}$$

Such terms would of course never be seen in a perturbative expansion in powers of $\bar{g}$.

In order to illustrate how such nonperturbative terms can actually arise for the Wilson coefficients, we examine a simple asymptotically free theory in which masses are generated dynamically—the two-dimensional, N component $(\bar{\psi}\psi)^2$ model. The Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i (i\not{\partial}) \psi_i + \frac{\lambda}{N}\left(\sum_{i=1}^{N} \bar{\psi}_i \psi_i\right)^2\,. \tag{30}$$

The theory is soluble for $N \to \infty$, λ fixed. As is well known, chiral symmetry $(\psi \to \gamma_5 \psi)$, which is supposed to prevent the quarks from acquiring mass, is broken dynamically, $\bar{\psi}\psi$ develops a vacuum expectation value, and to lowest order in N^{-1} one ends up with a massive fermion, with mass

$$m_p = \mu e^{-\lambda_0/\lambda}\,, \tag{31}$$

where λ_0 is a constant. Furthermore, to this leading order the fermion is noninteracting (the four-Fermi coupling is of order N^{-1}). The deep-inelastic structure function for scattering of a current off the fermion is therefore trivial. With appropriate normalization, $F = \delta(1 - x)$. Thus

$$\int_0^1 dx\; \xi^n F = \xi^n_{\max}\,,$$

so

$$M^{(n)}(Q^2, \lambda, \mu) = 2^n\left[1 + \left(1 + \frac{4\mu^2}{Q^2} e^{-\lambda_0/\lambda}\right)^{1/2}\right]^{-n}\,. \tag{32}$$

This expression contains precisely the essential singularity of the type discussed above.

It is to be noted for this simple example that the naive sum rule based on the Bjorken scaling variable x *is* analytic in λ. Namely,

$$\int_0^1 dx\, x^n F = 1 \tag{33}$$

is not only independent of Q^2, it is analytic in λ and in fact equal to unity for all n. Since the target has nonvanishing mass m_p, one is *not* with this sum rule projecting out the contributions from operators of definite spin; evidently here the organization by spin [Eq. (32)] only complicates the situation. The simplicity represented by Eq. (33) is, however, peculiar to the leading N^{-1} approximation. Continuum states begin to contribute in higher order and then both the Bjorken and Nachtmann moment functions will be nonanalytic.

The trivial example discussed above illustrates what we may expect in any theory where physical masses have a predominantly dynamical (nonanalytic) origin. Even if the quarks have nonvanishing masses, it is likely that hadron masses will have nonanalytic components. So too the Wilson coefficients can be expected to have nonanalytic components, and these will contribute (perhaps dominantly) to mass corrections in the moment functions. We conclude that these mass corrections cannot be calculated by perturbation methods.

IV. HIGHER-ORDER PERTURBATIVE CORRECTIONS

To any finite order in perturbation theory the twist-2 Wilson coefficients contain only the mass parameters associated with massive quarks. Even in the case of massless quarks, however, mass dependence arises in the Wilson coefficients of higher twist. In addition, we have argued that the Wilson coefficients of all twists, including twist-2, are likely to display nonperturbative mass effects. In this section we discuss a different matter. Suppose for some reason that one can ignore these nonperturbative effects and ignore also the higher-twist contributions—all of this for modest Q^2 comparable, say, to typical hadron masses. So here we are restricting ourselves to a perturbative treatment of the twist-2 Wilson coefficients, and for simplicity we take the case of massless quarks. There is still the question: For given Q^2 how many orders in perturbative theory are needed to capture the dominant contribution to the moment sum rules? For large enough Q^2 only the lowest-order term need be retained [this yields Eq. (6)], but the higher-order terms become increasingly important as Q^2 decreases. How rapidly this sets in can be expected to depend on the spin index n. As we shall see, the higher-order corrections increase in importance as the spin increases. This implies that the higher-order effects are especially significant in the threshold-resonance region, where the Bjorken variable x is close to unity.

For simplicity in the following discussion we drop the target mass and also all quark masses. However, we revert from the scalar currents of the previous sections to the physical case of vector (axial-vector) currents, and we consider in particular the structure function $F_2(x, Q^2)$. Finally, for simplicity we suppose that there is only one twist-2 operator for each spin in the Wilson expansion (as would be the case for the nonsinglet structure function). The Nachtmann formula now reads

$$\int_0^1 dx\, x^{J-2} F_2(x, Q^2) = a_J M^{(J)}(Q^2, g)\,, \qquad (34)$$

where J is the spin index (because of the vector character of the currents the earlier index n is replaced by $J-2$ in the integral).

As indicated in Eq. (25), which we rewrite here, the solution of the renormalization-group equation for the Wilson function $M^J(Q^2, g, \mu)$ is given by

$$M^{(J)}(Q^2, g) = Z^J(\bar{g}(t)) M^J(Q_0^2, \bar{g}(t)) \qquad (35)$$

(recall that quark masses have been set equal to zero and that $t = \frac{1}{2}\ln(Q^2/Q_0^2)$. For large t, $\bar{g}^2(t)$ becomes small, $\bar{g}^2 \to (b_0 t)^{-1}$ as $t \to \infty$, so for large Q^2 we contemplate an asymptotic expansion of M^J in powers of $\bar{g}^2$ (ignoring here the likely possiblity of essential singularities). We want to assess the importance of the nonleading terms in this expansion. The earlier expression for $Z^J(t)$ can be cast into the form

$$Z^J(t) = \exp\left[-\int_{\bar{g}(0)}^{\bar{g}(t)} \frac{\gamma^J(x)dx}{\beta(x)}\right]. \qquad (36)$$

We now make the power-series expansions

$$\gamma^J(g) = \sum_{i=1}^{\infty} \gamma_i^J g^{2i}\,,$$

$$2\beta(g) = -g^3 \sum_{i=0}^{\infty} b_i g^{2i}\,, \qquad (37)$$

$$M^J(Q_0^2, g) = \sum_{i=0}^{\infty} M_i^J(Q_0^2) g^{2i}\,.$$

The expansion coefficients are in principle calculable by standard perturbation methods, and, in fact, γ_1^J, b_0, b_1, M_0^J, and M_1^J are already known.[13] Inserting these expansions into Eqs. (35) and (36), one finds for $M^J(Q^2, g)$ the expansion

$$M^J(Q^2, g) = N^J[\bar{g}^2(t)/\bar{g}^2(0)]^{A^J}\left[\sum_{i=0}^{\infty} M_i^J(Q_0^2)\bar{g}^{2i}(t)\right] \times \exp\left\{\sum_{K=1}^{\infty} K^{-1} Z_K^J[\bar{g}^2(t)]^K\right\}, \qquad (38)$$

where

$$N^J = \exp\int_0^{\bar{g}(0)} dx\left[\frac{\gamma^J(x)}{\beta(x)} - \frac{2\gamma_1^J}{b_0 x}\right] \qquad (39)$$

is a t-independent constant, where

$$A^J = \gamma_1^J/b_0\,, \qquad (40)$$

and where the Z_K^J can be determined from the recursion relation

$$Z_K^J = \frac{\gamma_{K+1}^J}{b_0} - \frac{\gamma_1^J b_K}{b_0^2} - \sum_{i=1}^{K-1} Z_i^J \frac{b_{K-i}}{b_0}\,. \qquad (41)$$

For each spin J the perturbation sum should converge increasingly rapidly as Q^2 increases ($\bar{g}^2 \to 0$ as $Q^2 \to \infty$); so for large enough Q^2, the lowest-order term will dominate. However, there is no reason to expect that the convergence is uniform in J. Indeed, the rate of convergence is highly spin dependent both for $J \to \infty$ and $J \to 0$. This is already suggested by the lowest-order expression for the anomalous dimension. For the nonsinglet operators under discussion here one has

$$\gamma_1^J = G\left[1 - \frac{2}{J(J+1)} + 4\sum_{K=2}^{J} \frac{1}{K}\right], \qquad (42)$$

where G is a pure number determined by the gauge group and quark content of the underlying theory. Analytically continuing the above expression for γ_1^J, we find that it develops a pole at $J=0$:

$$\gamma_1^J \underset{J\to 0}{\sim} -\frac{2G}{J} b_0\,, \qquad (43)$$

whereas for large J

$$\gamma_1^J \underset{J\to\infty}{\sim} 4Gb_0 \ln J\,. \qquad (44)$$

(For the singlet structure function there is a similar logarithmic divergence as $J \to \infty$ and a pole at $J = 1$.)

The strong J dependence encountered already in lowest order suggests that convergence of the perturbation series is likely to be highly nonuniform in J. In turn, this has important implications for any attempt to reconstruct the structure function from the moments, for all the moments enter into the reconstruction. The behavior of the structure function near threshold, $x \approx 1$, is governed chiefly by the large-J moments, whereas for the "Regge" region $x \approx 0$ it is the moment behavior near the pole at $J = 0$ that governs. We shall particularly focus on the threshold region (large J).

For given Q^2 and x (x close to unity) one can roughly estimate the dominant spin, $\bar{J}(Q^2, x)$, that contributes to the structure function, as reconstructed by inverting the moment sum rules (inverse Mellin transformation). In lowest perturbative order, using the large-J approximation of Eq. (44), we find

$$\bar{J}(Q^2,x) \approx 4G \ln \bar{g}^2(t)/\ln x^{-1}. \tag{45}$$

For large Q^2 and x close to unity this becomes

$$\bar{J}(Q^2,x) \approx 4G \frac{\ln\ln(Q^2/Q_0^2)}{1-x}. \tag{46}$$

This is the result in lowest order. We now turn to the higher-order corrections. For all orders it is easy to show[14] that when J is large

$$\gamma_i^J \approx \gamma_i (\ln J)^{2i-1}, \quad M_i^J \approx M_i(\ln J)^{2i}. \tag{47}$$

In turn, this implies for large J that

$$Z_i^J \approx \frac{\gamma_{i+1}^J}{b_0} = \frac{\gamma_{i+1}}{b_0}(\ln J)^{2i+1}. \tag{48}$$

From Eq. (38) we then find

$$M^J(Q^2,g) = N^J M_0^J(Q_0^2)[\bar{g}^2(t)/\bar{g}^2(0)]^{A^J} \times \exp\left[\frac{\gamma_2}{b_0}\bar{g}^2(t)\ln^3 J\right][1+\Delta(\bar{g},J)]. \tag{49}$$

The quantity Δ involves a sum over all orders in perturbation theory; the important point is that it is of order $\bar{g}^2(t)\ln^2 J$ when the latter quantity is small.

We can now distinguish three different regions in the space of J and Q^2 (in all cases, recall that we are supposing that $\ln J$ is large compared to unity):

(a) The first domain to be considered is defined by

$$\ln J \ll [b_0/\gamma_2\bar{g}^2(t)]^{1/3}.$$

In this region $\bar{g}^2(t)\ln^2 J \ll 1$, so the correction term Δ in Eq. (49) can be ignored; also, the exponential factor in this equation can be replaced by unity. Thus, all higher-order effects are small and the moments are dominated by the lowest-order contributions. For the structure function, according to Eq. (45), we are in the domain

$$1-x \gg 4G \ln\{[\bar{g}^2(t)]^{-1}\}\exp[-b_0/\gamma_2\bar{g}^2(t)]^{1/3}. \tag{50}$$

Notice that for large Q^2 this does not include the threshold-resonance region $1-x\approx m^2/Q^2$ (where m is a typical hadron mass). This is so because the right-hand side of Eq. (50) falls only like a power of $\ln Q^2/Q_0^2$ as Q^2 becomes large.

(b) The second domain is defined by

$$\bar{g}^2(t)\ln^2 J \ll 1,$$

but

$$\frac{\gamma_2}{b_0}\bar{g}^2(t)\ln^3 J \gtrsim 1.$$

In this region the correction term Δ is still unimportant, but the exponential factor in Eq. (49) can no longer be approximated by unity. This factor contains the parameter γ_2, which can be determined from a two-loop calculation. It is easy to see that the domain being considered here again does not extend to the threshold-resonance region when Q^2 is large.

(c) The third domain is defined by

$$\bar{g}^2(t)\ln^2 J \gtrsim 1.$$

Now the correction factor Δ in Eq. (49) becomes significant and all orders of perturbation theory have to be taken into account. This is of course prohibitive.

Our conclusion is that in attempting to reconstruct the structure function from the moments, for Q^2 large and x close to unity, one can rely on lowest-order perturbation theory for the moments only if the Q^2-x domain corresponds to the inequality of Eq. (50). In this domain x is precluded from coming *too close* to unity (similar limitations apply for x too close to zero). For fixed large Q^2, as x moves closer to unity one has to begin allowing for higher-order contributions. In region (b) the problem is still tractable—one needs only the second-order parameter γ_2. Finally, for x still closer to unity one enters region (c), and now all orders have to be taken into account. The transition point dividing regions (a) and (b) depends on the size of $\bar{g}^2(t)$ and on the magnitude of γ_2.

The nonsinglet two-loop anomalous dimension γ_2 can be evaluated without great difficulty.[15] The result is

$$\gamma_2 = \alpha^2 \frac{C_2(R)C_2(G)}{24}, \tag{51}$$

where $\alpha = g^2/4\pi^2$, $C_2(G)$ is the quadratic Casimir invariant for the gauge group, and $C_2(R)$ is the same invariant for the fermion representation. The numerical value of γ_2 turns out to be unusually small, reduced by about an order of magnitude relative to naive expectation. This is because of a complete cancellation of the Abelian terms, proportional to $[C_2(R)]^2$; such cancellation is not expected to occur in the coefficient of $(\ln J)^2$ in γ_2^J or for the coefficient of $(\ln J)^{2K-1}$ in γ_K^J. In any case, because γ_2 is small one cannot use Z_1^J alone to determine when higher-order corrections become important. That is, the exponential term in Eq. (49) does not begin to depart appreciably from unity before one has to begin taking into account the $(\alpha\ln^2 J)^K$ corrections contained in term $\Delta(\bar{g},J)$ of Eq. (49). This brings in all the higher orders of perturbation theory. In lieu of a detailed calculation, which would be prohibitive, and assuming no further suppression (or enhancement) effects, we might just guess that Δ is of order $A\alpha(t)\ln^2 J$, with $A\approx 1$. In this case, even for Q^2 such that $\alpha(t)=0.1$, we find that the higher-order effects introduce corrections of about 25% or more for $J \geq 5$, therefore for $x \gtrsim 0.75$.

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