$SU(3)_E$-invariant form factor for pseudoscalar mesons*

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The use of a form factor which is invariant with respect to the spectrum-generating SU(3) leads to a definition for the strangeness-changing suppression factor for K_{l_3} decay which is numerically equal to the strangeness-changing suppression factor for K_{l_2} decay and close to m_π/m_K. It also predicts the electromagnetic radius of the pseudoscalar mesons in terms of λ_+.

The determination of the Cabibbo angle θ from the experimental data leads to different values depending upon the processes considered. From the leptonic decays of mesons one obtains $\tan\theta_A^M = 0.275$, from the semileptonic decays of mesons one obtains $\tan\theta_V^M = 0.224$, and the hyperon decays give values somewhere between these two.[1] This difference is not surprising as the Cabibbo model can only give an approximate description since it uses the Wigner-Eckart theorem for an SU(3) symmetry group, taking SU(3) breaking into account only in the phase-space factors.

In a series of papers[2] we have proposed using SU(3) not as a symmetry group whose predictions are later corrected by symmetry-breaking terms, but as a spectrum-generating group, $SU(3)_E$, which does not commute with the momentum P_μ and mass $M=(P_\mu P^\mu)^{1/2}$ operators but which has the property[3]

$$[\hat{P}_\mu, SU(3)_E]=0, \quad [L_{\mu\nu}, SU(3)_E]=0, \tag{1}$$

where $\hat{P}_\mu = P_\mu M^{-1}$ is the four-velocity operator and $L_{\mu\nu}$ are the generators of the homogeneous Lorentz transformation.

The assumption (1) has been applied previously in conjunction with other assumptions[2,4] and so far the only direct prediction of (1) was the value of the form-factor ratio $\xi = f_-/f_+ = -0.57$ for K_{l_3} decay. In this paper we shall show that the origin of the above-mentioned difference between the experimentally determined suppression ratios $\tan\theta_V^M$ and $\tan\theta_A^M$ is the comparison of the form factors at the wrong values of t. As a consequence of (1) the suppression ratio for K_{l_3} decay should not be determined at $t=0$, i.e., as $\tan\theta_V = [f_+^K(0)/f_+^\pi(0)]\,\tilde{G}_V^{\pi K}/\tilde{G}_V^{\pi\pi}$, where the quantities used here are as usually defined by[5]

$$\langle \pi^0 p_\pi | V_\mu^b | p_\alpha a\rangle = C(a,b,\pi^0)\,\tilde{G}_V^{\pi a}\,[f_+^a(t)\,(p_\alpha + p_\pi)_\mu + f_-^a (p_\alpha - p_\mu)_\mu]\,, \tag{2}$$

with $C(a,b,\pi^0)$ being the Clebsch-Gordan coefficient. But the suppression ratio should be determined at $t=t_{\max}$, i.e., as $[f_+^K(t=t_{\max})/f_+^\pi(t=t_{\max}\approx 0)]\,\tilde{G}_V^{\pi K}/\tilde{G}_V^{\pi\pi}$, and this value agrees with the suppression ratio $\tan\theta_A = G_A^K/G_A^\pi = 0.275$ for K_{l_2} decay.

Equation (2) would be the Wigner-Eckart theorem if SU(3) would commute with the momenta. As this is not the case the factor of the Clebsch-Gordan coefficient at the right-hand side is not a reduced matrix element and $G^{\pi a}$, $f_\pm(t)$, and t are not SU(3) invariants. However, as a consequence of (1) a fully consistent use of the Wigner-Eckart theorem can be safeguarded if one expresses the matrix elements in terms of four-velocities and factorizes out the mass-dependent factors. Using (1) and the conserved-vector-current (CVC) hypothesis [which renders the factor of $(\hat{p}-\hat{p}')$ to zero[2]] we obtain for the matrix elements of V_ρ^b between velocity eigenstates

$$\langle \hat{p}'\pi | V_\rho^b | \hat{p} a\rangle = C(ab\pi)\, g^{\pi a} F(\hat{t})(\hat{p}+\hat{p}')_\rho\ , \tag{3}$$

where

$$\hat{t} = (\hat{p}-\hat{p}')^2 = (m_a m_\pi)^{-1}[t^{(a\pi)} - (m_a - m_\pi)^2]\,, \quad \hat{p} = p/m\,. \tag{4}$$

Because of (1) the form factor $F(\hat{t})$ is now a true $SU(3)_E$-invariant function of the $SU(3)_E$-invariant parameter $\hat{t}$ and is the same for all vector currents and all transitions between pseudoscalar mesons from the same octet.

We shall make for F the linear ansatz

$$F(\hat{t}) = F(0)\,(1 + b\hat{t})\ , \tag{5}$$

where

$$F(0) = F(\hat{t}=0) = F(t_{\max})\ . \tag{6}$$

SU(3)-invariant F linear in $\hat{t}$ implies linear dependence of $f_\pm^a(t)$ upon t:

$$f_\pm^a(t)=f_\pm^a(0)\left(1+\lambda_\pm^a \frac{t}{m_\pi^2}\right), \tag{7}$$

with the following relation between the parameters b and λ:

$$\lambda_+^a=\frac{m_\pi}{m_a} b\left[1-b\frac{(m_a-m_\pi)^2}{m_a m_\pi}\right]^{-1}, \quad \lambda_-^a=\lambda_+^a . \tag{8}$$

The relation

$$\xi^a=f_-^a(t)/f_+^a(t)=\frac{m_\pi-m_a}{m_\pi+m_a} \tag{9}$$

(or $\lambda_0=\lambda_+-0.046$), which was already derived in Ref. 2, follows from comparison of (3) and (2).

A look at (3) leads to the conclusion that the suppression ratio should be defined from the matrix elements for $a=K$ and $a=\pi$ at the same, and in fact any value of $\hat{t}$. However, as the π_{l_3} phase space restricts us to the value $t^{\pi\pi}\approx m_\pi^2\hat{t}\approx 0$, we have to choose also for the matrix element of K_{l_3} decay the value $\hat{t}=0$, i.e., $t^{K\pi}=t_{\max}$. This then confirms our statement that $[f_+^K(t=t_{\max})/f_+^\pi(0)]\,\tilde{G}_V^{\pi K}/\tilde{G}_V^{\pi\pi}$ should be the suppression factor, which is experimentally very close to $0.27\cdots 0.28$.

We shall now make a more detailed comparison with the experimental data without using the SU(3)-noninvariant form factors $f_+(t)$ as auxiliary quantities.

The decay rate for a_{l_3} decay is calculated[2] from (3) to be

$$\Gamma(a\to\pi l\nu)=|C(ab\pi)|^2|G_V^{\pi a}|^2\int\frac{d^3p_l}{2E_l}\frac{d^3p_\nu}{2E_\nu}\frac{d^3p_\pi}{2E_\pi}(2E_a)^{-1}\delta^4(p_a-p_\pi-p_l-p_\nu)|F(\hat{t})|^2$$
$$\times\sum_{\text{pol}}|[(p_a+p_\pi)_\rho+\xi(p_a-p_\pi)_\rho]\bar{u}(p_\nu)\gamma^\rho(1-\gamma_5)v(p_l)|^2 , \tag{10}$$

where we have used the abbreviation

$$G_V^{\pi a}=\sqrt{2\pi}\,\tfrac{1}{2}g(m_a+m_\pi)(m_a m_\pi)^{-2}g_V^{\pi a}, \tag{11}$$

with g an overall constant proportional to the weak-interaction constant. Inserting (5) into this and performing the integration we obtain

$$\Gamma(a\to\pi l\nu)=\tfrac{4}{3}\pi^2|C(ab\pi)|^2\,|G_V^{\pi a}|^2\,|F(0)|^2$$
$$\times\left(\frac{m_a+m_\pi}{2m_a}\right)^3(m_a-m_\pi)^5 I(a\to\pi l\nu) , \tag{12}$$

where

$$\begin{aligned} I(\pi^\pm\to\pi^0 e\nu)&=0.753 ,\\ I(K^\pm\to\pi^0 e\nu)&=0.693-1.975b+1.514b^2 ,\\ I(K^\pm\to\pi^0\mu\nu)&=0.401-0.924b+0.584b^2 ,\\ I(K_L^0\to\pi^\pm e\nu)&=0.696-1.895b+1.388b^2 ,\\ I(K_L^0\to\pi^\pm\mu\nu)&=0.404-0.887b+0.535b^2 . \end{aligned} \tag{13}$$

The expressions (13) have been calculated using the mass values of the 1974 Particle Data Table.[6]

One can now determine the experimental values for

$$S_{l_3}^{+,0}=\frac{G_V^{\pi K^{+,0}}}{G_V^{\pi\pi}} \tag{14}$$

from the experimental ratios

$$r=\frac{\Gamma(K\to\pi e\nu)}{\Gamma(\pi\to\pi e\nu)} \tag{15}$$

for various values of b. Table I gives the results for $S_{l_3}^+$ obtained for the 1974 Particle Data value of the world average

$$r^+=\frac{\Gamma(K^+e_3)}{\Gamma(\pi^+e_3)}=0.994\times10^7 , \tag{16}$$

and two values differing from it by one standard deviation. Table II presents the results of an analogous calculation of $S_{l_3}^0$ obtained for the world

TABLE I. $S_{l_3}^+$ for various values of λ_+ and r^+. b and λ_+ are connected by (8). $S_{l_3}^+$ in columns 3, 4, and 5 are then calculated from (14), (12), (13) using for r^+ the value on top of each column. Column 4 uses the 1974 world-average value for r^+.

b	λ_+	$r^+=1.063\times10^7$	$S_{l_3}^+$ for the values $r^+=0.994\times10^7$	$r^+=0.925\times10^7$
0.0509	0.0154	0.2626	0.2540	0.2450
0.0751	0.0240	0.2727	0.2637	0.2544
0.0868	0.0285	0.2778	0.2687	0.2592
0.0974	0.0328	0.2827	0.2733	0.2637
0.1177	0.0417	0.2923	0.2827	0.2727
0.1388	0.0518	0.3031	0.2930	0.2827

TABLE II. $S^0_{l_3}$ for various values of λ_+ and r^0.

b	λ_+	$S^0_{l_3}$ for the values $r^0=2.06\times10^7$	1.92×10^7	1.78×10^7
0.0509	0.0157	0.2560	0.2472	0.2380
0.0739	0.0240	0.2649	0.2557	0.2462
0.0784	0.0257	0.2667	0.2574	0.2479
0.0893	0.0300	0.2711	0.2618	0.2520
0.1120	0.0396	0.2809	0.2712	0.2611
0.1356	0.0507	0.2917	0.2816	0.2712

average

$$r^0=\frac{\Gamma(K^0_L e_3)}{\Gamma(\pi^+ e_3)}=1.92\times10^7\ . \tag{17}$$

Performing the above calculation of the rates for the a_{l_2} decay, $a=K,\pi$ is much simpler. Instead of (2) one has

$$\langle\sigma|A^b_\mu|\hat{p}_a a\rangle=C(a,b,\sigma)\,\tilde{G}^a_A p^{(a)}_\mu\ , \tag{2$'$}$$

where $|\sigma\rangle$ denotes a state with SU(3)-singlet quantum numbers. Instead of (3) one has

$$\langle\sigma|A^b_\mu|p_a a\rangle=C(a,b,\sigma)\,g^a_A\hat{p}^{(a)}_\mu\ , \tag{3$'$}$$

and calculating from this the decay rate along the lines of Ref. 2 one obtains instead of (10) and (12)

$$\Gamma=|C(a,b,\sigma)|^2|G^a_A|^2 \times\int\frac{d^3p_l}{2E_l}\frac{d^3p_\nu}{2E_\nu}\,\delta^4(p_a-p_l-p_\nu) \times\sum_{\text{pol}}|u_\nu(p_\nu)\gamma^\mu(1-\gamma_5)v_l(p_l)p^{(a)}_\mu|^2 \tag{10$'$}$$

and

$$\Gamma(a\to l\nu)=\pi|C(ab\sigma)|^2|G^a_A|^2m_l{}^2m_a\left(1-\frac{m_l{}^2}{m_a{}^2}\right)^2\ , \tag{12$'$}$$

where we have used the abbreviation

$$G^a_A=\sqrt{2\pi}\,gm_a{}^{-2}g^a_A\ . \tag{11$'$}$$

From the experimental values of $\Gamma(K\to\mu\nu)$ and $\Gamma(\pi\to\mu\nu)$ we can obtain the experimental value for

$$S_{l_2}=G^K_A/G^\pi_A=0.276\ . \tag{14$'$}$$

Comparing (10′) with (10) one observes that the quantities $G^{\pi a}_V$ serve for a_{l_3} decay the same purpose as the quantities G^a_A for a_{l_2} decay, namely they describe the dependence of the decay rate upon the hadron quantum numbers in addition to the SU(3) dependence given by the Clebsch-Gordan coefficients. The quantity $S_{l_2}=G^K_A/G^\pi_A$ is clearly identical to the conventional strangeness-changing suppression factor for axial-vector decay, $\tan\theta^M_A$. The quantity S_{l_3} is not identical to the conventional strangeness-changing suppression factor for vector decay, $\tan\theta^M_V$. However, one sees from comparison of (10) and (10′) that $S_{l_3}=G^{\pi K}_V/G^{\pi\pi}_V$ is defined in complete analogy to S_{l_2} if one uses the SU(3)$_E$-invariant form factor $F(\hat{t})$ instead of the SU(3)-noninvariant form factors $f^K_\pm(t)$ and $f^\pi_+(t)$. Therefore under the assumption (1) S_{l_3} defined by (14) should be used to describe the strangeness-changing suppression instead of the conventional $\tan\theta^M_V$. One easily convinces oneself that expressed in terms of the conventional quantities defined in (2) and (2′) S_{l_2} and S_{l_3} are given by

$$S_{l_2}=\frac{\tilde{G}^K_A}{\tilde{G}^\pi_A}\ ,\qquad S_{l_3}=\frac{f^K_+(t=t_{\max})\,\tilde{G}^{\pi K}_V}{f^\pi(t=t_{\max}\approx0)\,\tilde{G}^{\pi\pi}_V}\ . \tag{18}$$

The agreement between the experimental values for the axial-vector strangeness suppression factor S_{l_2} given by (14′) and the vector strangeness suppression factor S_{l_3} given in Tables I and II is striking. It even appears tempting to use the condition $S_{l_3}=S_{l_2}$ as the requirement for the determination of the value of λ_+ from the experimental value of r.

As the equality of axial-vector and vector suppression factors is a physically attractive feature, because it reduces the number of arbitrary parameters, and this equality is obtained if the suppression factors are defined consistent with assumption (1), we consider these experimental data as a further support of the assumption (1).

One can now go one step further and make assumptions which allow to compute the values of S_{l_2} and S_{l_3}. This has been discussed in detail in Ref. 2 where, however, we ignored the t dependence of the form factors because we had underestimated its effect. A possible assumption of this kind is

$$\hat{V}^b_\mu=\{M^{-1},V^b_\mu\},\quad \hat{A}^b_\mu=\{M^{-1},A^b_\mu\} \tag{19}$$

are octet operators with respect to the spectrum-

TABLE III. $\langle r_\pi^2\rangle$ in 10^{-13} cm for various values of λ_+ or b calculated from (27).

b	λ_+	$\langle r_\pi^2\rangle$
0.039	0.012	0.47
0.051	0.015	0.61
0.075	0.024	0.90
0.087	0.029	1.04
0.118	0.042	1.41

generating $SU(3)_E$. From (19) one calculates, with the help of the Wigner-Eckart theorem for $\hat{V}^b_\mu$,

$$\langle \hat{p}'\pi|\{M_1^{-1}V^b\}|\hat{p}a\rangle = C(ab\pi)F(\hat{t})(\hat{p}+\hat{p}')_\mu\,; \quad (20)$$

comparing this with (3) $g^{\pi a}$ is obtained as

$$g^{\pi a} = \frac{m_a m_\pi}{m_a + m_\pi}\langle \underline{8}\|V\|\underline{8}\rangle \ . \quad (21)$$

Inserting this into (11) and absorbing the reduced matrix element $\langle \underline{8}\|V\|\underline{8}\rangle$ into g gives

$$G_V^{\pi a} = \sqrt{2\pi}\,\tfrac{1}{2}g\,\frac{1}{m_\pi m_a} \ . \quad (22)$$

Thus (19) predicts, for S_{l_3} defined by (14),

$$S^+_{l_3} = m_{\pi^+}/m_{K^+} = 0.283 \ ,$$
$$S^0_{l_3} = m_{\pi^0}/m_{K^0} = 0.271 \ . \quad (23)$$

In the same way one calculates, with the help of the Wigner-Eckart theorem for $\hat{A}^b_\mu$, that

$$g^a_A = m_a\langle \underline{0}\|A\|\underline{8}\rangle \ . \quad (24)$$

$\langle \underline{0}\|A\|\underline{8}\rangle$ in (24) and $\langle \underline{8}\|V\|\underline{8}\rangle$ in (21) are the singlet-octet and F-type octet-octet reduced matrix elements, respectively. Inserting (24) into (11′) yields

$$G^a_A = \sqrt{2\pi}\,g\,\frac{\langle \underline{0}\|A\|\underline{8}\rangle}{\langle \underline{8}\|V\|\underline{8}\rangle}\frac{1}{m_a} \ . \quad (25)$$

Thus (19) leads to the prediction

$$S_{l_2} = m_{\pi^+}/m_{K^+} = 0.283 \ . \quad (26)$$

In spite of the agreement between the predictions (26) and (23) and the experimental data (14′) and Tables I, II, we do not wish to put too much emphasis upon the assumption (19); we use them only as an illustration, because one can easily think of other assumptions to replace (19). The main result of this paper is the equality of the strangeness-changing suppression factor for axial-vector and vector decay. The effect of this result upon the fit of the baryon semileptonic decay data is under investigation.

Concluding, we remark that with the use of the $SU(3)_E$-invariant form factor $F(\hat{t})$ one may also predict the following relation between λ_+ and the electromagnetic radius of the pseudoscalar meson a:

$$\langle r_a^2\rangle = \frac{6}{m_a^2}\,b = \frac{6}{m_a^2}\,\frac{m_\pi m_K \lambda_+}{m_\pi^2 + (m_K - m_\pi)^2\lambda_+} \ . \quad (27)$$

The numerical relation between $\langle r_\pi^2\rangle$ and λ_+ given by (27) is displayed in Table III; the experimental values for $\langle r_\pi^2\rangle$ are discussed in Refs. 7 and 8. The only direct measurement of the pion electromagnetic radius[7] gives values between $\langle r_\pi^2\rangle = (0.46 \pm 0.03)\,\mathrm{fm}^2$ and $\langle r_\pi^2\rangle = (1.03 \pm 0.35)\,\mathrm{fm}^2$.

One consequence of (27),

$$m_\pi^2\langle r_\pi^2\rangle = m_K^2\langle r_K^2\rangle\,, \quad (28)$$

is such a drastic deviation from the vector-dominance prediction that experimental data should soon be able to discriminate between these and (28).

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