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Violation of unitarity in a finite quantum electrodynamics*

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By explicit calculations, we show the violation of unitarity in a recently proposed theory of finite quantum electrodynamics on the basis of a limiting procedure. The failure can be traced back to the mathematical nonexistence of the limit upon which the theory is built.

Recently, a theory of finite quantum electrodynamics was formulated on the basis of a limiting procedure.[1] One starts with a Lagrangian involving a real parameter ϵ and a new set of positive-metric massive vector fields and ends up with a finite theory involving negative-metric fields in the limit $\epsilon \to 0$. We examine this strange situation in this paper.

Usually, the attempt to reduce severe divergent difficulties inevitably leads to some unusual properties in a field theory. For example, Heitler tried to construct a finite quantum electrodynamics by introducing form factors, but his theory is not Lorentz-invariant.[2] Sudarshan and collaborators introduced shadow states in field theories and the S matrix becomes piecewise analytic.[3] Lee and Wick employed negative-metric particles with complex masses to construct a finite and unitary field theory, yet the theory is not Lorentz-invariant.[4] Hsu introduced an aoraton, a massless physical entity that carries spin angular momentum but not 4-momentum, and a heavy scalar boson to mediate the $V-A$ weak interactions[5] so that the theory contains only mild divergences; the theory becomes CPT-noninvariant at high energies.[6]

The strange situation of the limit $\epsilon \to 0$ in Ref. 1 compels us to look into the theory and its limiting procedure. We find that the theory violates unitarity at 1-loop level already. The violation can be traced back to the very origin, namely, the limit $\epsilon \to 0$ does not exist mathematically.

To check unitarity, we calculate the electron-positron scattering in fourth order according to the Feynman rules given in Ref. 1. The exact propagator (to all orders in ϵ^{-1}) for the vector field with the mass M is given by[1]

$$\frac{-i(\delta_{\mu\nu} - \alpha(k^2)k_\mu k_\nu/k^2)}{k^2+M^2} \xrightarrow[\epsilon\to 0]{} \frac{-i(\delta_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2+M^2} \equiv \frac{-iG_{\mu\nu}}{k^2+M^2}\,, \tag{1}$$

where $\alpha(k^2) = (1+\epsilon^2 k^2)/(1-\epsilon^2 M^2)$. In the limit $\epsilon \to 0$,[1] (1) can be decomposed into a physical part $P_{\mu\nu}$ with a pole at $k^2 = -M^2$ and an unphysical part $U_{\mu\nu}$ with a pole at $k^2 = 0$:

$$\frac{-i(\delta_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2+M^2} = P_{\mu\nu}(k) + U_{\mu\nu}(k),$$
$$P_{\mu\nu}(k) = \frac{-i(\delta_{\mu\nu} + k_\mu k_\nu/M^2)}{k^2+M^2}\,, \qquad U_{\mu\nu}(k) = \frac{ik_\mu k_\nu}{k^2 M^2}\,. \tag{2}$$

For this theory to be unitary, the unphysical pole at $k^2 = 0$ in (2) must not contribute to the imaginary part of the e^--e^+ scattering amplitude, $\mathrm{Im}A_s$. For example, the imaginary part of the amplitude for the diagram with two massive vector bosons a in the intermediate state is

$$\mathrm{Im}A_1(e(p_1)\bar{e}(p_2) \to a(k)a(q) \to e(p_3)\bar{e}(p_4))$$
$$= \mathrm{Im}(ig)^4 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p_3)\gamma_\alpha\gamma_5 G_{\alpha\lambda}(k) Q(p_3-k) G_{\xi\beta}(q)\gamma_\beta\gamma_5 v(p_4)$$
$$\times \bar{v}(p_2)\gamma_\nu\gamma_5 G_{\rho\nu}(q) Q(p_1-k)\gamma_\mu\gamma_5 G_{\delta\mu}(k) u(p_1) \left\{ \frac{[-iG_{\lambda\delta}(k)]}{(k^2-M^2)} \frac{[-iG_{\xi\rho}(q)]}{(q^2-M^2)} \right\}$$
$$= \mathrm{Im}(-g^4) \int \frac{d^4k}{(2\pi)^4} \bar{u}(p_3)\gamma_\alpha\gamma_5 Q(p_3-k)\gamma_\beta\gamma_5 v(p_4)\bar{v}(p_2)\gamma_\nu\gamma_5\, Q(p_1-k)\gamma_\mu\gamma_5 u(p_1)$$
$$\times [(P_{\alpha\mu}(k) + U_{\alpha\mu}(k))(P_{\beta\nu}(q) + U_{\beta\nu}(q))]\,, \tag{3}$$
$$q = p_1 + p_2 - k, \quad Q(p) = (-\gamma\cdot p - im)/(p^2+m^2), \quad p = p_3 - k,\ p_1 - k.$$

The contributions to $\mathrm{Im}A_1$ due to the two unphysical poles, i.e., $U_{\alpha\mu}(k)\,U_{\beta\nu}(q)$, *cannot* be canceled by other terms in (3) because of the phase space. Thus, for this theory to be unitary, the net contribution to $\mathrm{Im}A_s$ due to two unphysical poles must vanish, similar to those in the unitarity check in non-Abelian gauge theories.[7]

We have the following results in the limit $\epsilon\to 0$:

$$\mathrm{Im}[A_1(i\to a(k)a(q)\to f)+A_2\ (\text{crossed diagram})]$$
$$=\frac{g^4(2\pi)^2}{M^4}\int d^4k\Big\{\bar u(p_3)\Big[2im_e+\gamma\cdot k\Big(1+\frac{2m_e^2}{p_3\cdot k}\Big)\Big]v(p_4)\bar v(p_2)\Big[2im_e+\gamma\cdot k\Big(1+\frac{2m_e^2}{p_1\cdot k}\Big)\Big]u(p_1)$$
$$+\bar u(p_3)\Big[2im_e-\gamma\cdot k\Big(1+\frac{2m_e^2}{p_4\cdot k}\Big)\Big]v(p_4)\bar v(p_2)\Big[2im_e+\gamma\cdot k\Big(1+\frac{2m_e^2}{p_1\cdot k}\Big)\Big]u(p_1)\Big\}\delta(k^2)\delta(q^2)\,, \tag{4}$$

$$\mathrm{Im}A_3(i\to A^{(-)}(k)A^{(+)}(q)\to f)=\frac{g^4(2\pi)^2}{M_V^4}\int d^4k\Big\{\bar u(p_3)\Big[\gamma\cdot k+i(m_\mu-m_e)\frac{2p_3\cdot k+i(m_\mu-m_e)\gamma\cdot k}{m_\mu^2-m_e^2-2p_3\cdot k}\Big]v(p_4)$$
$$\times\bar v(p_2)\Big[\gamma\cdot k+i(m_\mu-m_e)\frac{2p_1\cdot k+i(m_\mu-m_e)\gamma\cdot k}{m_\mu^2-m_e^2-2p_1\cdot k}\Big]u(p_1)\Big\}\delta(k^2)\delta(q^2)\,, \tag{5}$$

$$\mathrm{Im}A_4(i\to a^{(-)}(k)a^{(+)}(q)\to f)=\frac{g^4(2\pi)^2}{M_A^4}\int d^4k\Big\{\bar u(p_3)\Big[\gamma\cdot k-i(m_\mu+m_e)\frac{2p_3\cdot k-i(m_\mu+m_e)\gamma\cdot k}{m_\mu^2-m_e^2-2p_3\cdot k}\Big]v(p_4)$$
$$\times\bar v(p_2)\Big[\gamma\cdot k-i(m_\mu+m_e)\frac{2p_1\cdot k-i(m_\mu+m_e)\gamma\cdot k}{m_\mu^2-m_e^2-2p_1\cdot k}\Big]u(p_1)\Big\}\delta(k^2)\delta(q^2)\,, \tag{6}$$

$$\mathrm{Im}[A_5(i\to\phi(k)\phi(q)\to f)+A_6(\text{crossed diagram})]=0, \tag{7}$$

$$\mathrm{Im}A_7(i\to\phi(k)a(q)\to f)=0\,, \tag{8}$$

$$\mathrm{Im}A_8(i\to\phi(q)a(k)\to f)=0\,, \tag{9}$$

where $i\equiv e(p_1)\bar e(p_2)$, $f\equiv e(p_3)\bar e(p_4)$, and $q\equiv p_1+p_2-k$.

As we can easily see, the sum of all these contributions due to two unphysical poles does not add up to zero,

$$\sum_{i=1}^{8}\mathrm{Im}A_i\neq 0\,. \tag{10}$$

Thus, the theory violates unitarity in the limit $\epsilon\to 0$. We also note that the contributions to $\mathrm{Im}A_s$ due to one unphysical pole and one physical pole are not zero.

The trouble in this theory can be traced back to the improper use of the $\epsilon\to 0$ limit. This limit cannot be taken because the exact propagators (to all orders of ϵ^{-1}) of the fields $a_\mu(x)$ and $\phi(x)$ have been calculated with the restriction that $\epsilon^2>1/M^2$. To see this more clearly, let us calculate the exact propagator for the field $a_\mu(x)$: Letting $M_{\mu\nu}=\delta_{\mu\nu}+k_\mu k_\nu/M^2$, we obtain

$$\frac{-iM_{\mu\nu}}{k^2+M^2}+\frac{(-i)M_{\mu\lambda}}{k^2+M^2}\Big(\frac{-ik_\lambda}{\epsilon}\Big)\Big(\frac{-i}{k^2}\Big)\Big(\frac{ik_{\lambda'}}{\epsilon}\Big)\frac{(-i)M_{\lambda'\nu}}{k^2+M^2}$$
$$+\frac{(-i)M_{\mu\lambda}}{k^2+M^2}\Big(\frac{-ik_\lambda}{\epsilon}\Big)\Big(\frac{-i}{k^2}\Big)\Big(\frac{ik_{\lambda'}}{\epsilon}\Big)\frac{(-i)M_{\lambda'\lambda''}}{k^2+M^2}\Big(\frac{-ik_{\lambda''}}{\epsilon}\Big)\Big(\frac{-i}{k^2}\Big)\Big(\frac{ik_\sigma}{\epsilon}\Big)\frac{(-iM_{\sigma\nu})}{k^2+M^2}+\cdots$$
$$=\frac{-iM_{\mu\nu}}{k^2+M^2}+\frac{(-i)k_\mu k_\nu}{\epsilon^2M^4k^2}\Big[1+\frac{1}{\epsilon^2M^2}+\frac{1}{(\epsilon^2M^2)^2}+\cdots\Big]\,, \tag{11}$$

where the factors $\pm i k_\mu/\epsilon$ are due to the contact coupling $-\epsilon^{-1}a_\lambda\partial_\lambda\phi$ in the Lagrangian.[1] We can see that the series between brackets can be summed up *if and only if* $\epsilon^2M^2>1$; then we obtain

$$-i\frac{\delta_{\mu\nu}-\dfrac{1+\epsilon^2k^2}{1-\epsilon^2M^2}\dfrac{k_\mu k_\nu}{k^2}}{k^2+M^2}\quad\text{with } \epsilon^2>\frac{1}{M^2}, \tag{12}$$

as given by (1). Also, the ϕ propagator $-i\beta(k^2)/k^2$ with $\beta(k^2)=-\epsilon^2M^2/(1-\epsilon^2M^2)$ and $\epsilon^2>1/M^2$ cannot be decoupled from the theory because the limit $\epsilon\to0$ cannot be taken. This also explains the inconsistency in the appearance of negative-metric fields when in the original Lagrangian there were none: Equation (12) can be written as

$$\frac{-i(\delta_{\mu\nu}+k_\mu k_\nu/M^2)}{k^2+M^2}-\left(\frac{1}{\epsilon^2M^2-1}\right)\frac{ik_\mu k_\nu}{M^2k^2}. \tag{13}$$

Thus, we see that the metric of the pole at $k^2=0$ in the second term is always positive because $\epsilon^2M^2>1$.

The restriction $\epsilon^2>1/M^2$ for the validity of the exact propagator (12) escaped the attention of the authors in Ref. 1 because they used a formal method to determine $\alpha(k^2)$ in (1) and an infinite series in (11) did not appear explicitly. We may remark that once the exact propagator (to all orders in ϵ^{-1}) is used, it is really not necessary to have the vertex corrected to all orders in $1/\epsilon$ in the Feynman rule. Also, the theory is not renormalizable because of the restriction $\epsilon^2>1/M^2$, as can be seen from the propagator (13). There are other minor mistakes in Ref. 1; however, their corrections will not change our conclusions that (i) the theory proposed in Ref. 1 violates unitarity and (ii) the limit $\epsilon\to0$ cannot be taken mathematically and therefore all the statements based on the limit $\epsilon\to0$ are wrong.

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