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Dynamical equations for a Regge theory with crossing symmetry and unitarity. II. The case of strong coupling, and elimination of ghost poles*

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Equations for the construction of a crossing-symmetric unitary Regge theory of meson-meson scattering are described. In the case of strong coupling, Regge trajectories are to be generated dynamically as zeros of the D function in a nonlinear N/D system. This paper is concerned mainly with writing the inputs to the N/D system in such a way that a convergent theory with exact crossing symmetry is defined. The scheme demands elimination of ghosts, i.e., bound-state poles at energies below threshold where trajectories pass through zero. A method for ghost elimination is proposed which entails an s-wave subtraction constant, and allows the physical s wave to be different from the l-analytic amplitude evaluated at $l = 0$. A dynamical model is suggested in which the subtraction constant alone generates the meson-meson interaction. An alternative ghost-elimination scheme proposed by Gell-Mann, in which only l-analytic amplitudes are involved, can be discussed in a formalism including channels with spin.

I. INTRODUCTION

In part I of this series of papers we proposed a program for construction of a crossing-symmetric unitary Regge theory of meson-meson scattering.[1] The construction is based on the solution of a nonlinear functional equation, $\psi = G(\psi)$, for certain partial-wave scattering functions ψ. In part I, the operator G was defined for the weak-coupling case, in which no Regge poles enter the right half of the l plane. Our present task is to define G for the strong-coupling case, with Regge poles in the right half plane. The meaning of ψ and G was explained schematically in the Introduction to part I [see Eqs. (I1.8)–(I1.10)]. The explicit expressions for the functionals A and B of (I1.7) were given in (I2.53)–(I2.54) for the case of weak coupling. In the present work, A and B have the same physical significance as before; A is related to the elasticity, and B is the force function (a left-cut term plus an inelastic term) in the N/D equation. The main new feature of the strong-coupling case is that the formal expressions for A and B, which may be read off from the Froissart-Gribov representation of the partial-wave amplitude, are not obviously well defined. A rearrangement of the expressions, involving contour distortions and changes of integration order, reveals that they are well behaved after all.

In Sec. II we work out the "raw" formulas for A and B, by direct calculation from the Froissart-Gribov amplitude. We introduce certain essential assumptions about Regge trajectories $\alpha(s)$; for instance, $\alpha(s)$ is required to be analytic and invertible in a particular finite region of the s plane, and $\mathrm{Im}\,\alpha(s_+)$ must have the proper sign. We cannot predict the validity of these assumptions in general, but we hope that they will be valid at least for certain choices of physical inputs (intermeson forces and inelastic effects).

Section III is devoted to the problem of rearranging the expressions for A and B. Some of the problems encountered were discussed previously by Omnès[2] and Squires,[3] who noticed that the discontinuity of the partial wave $a(l,s)$ over the left cut in the s plane appears on first sight to grow as a power of s at $s=-\infty$. Omnès and Squires suggested certain contour distortions as a means of regularizing this discontinuity. The specific suggestions of Refs. 2 and 3 were not sufficient for our purposes, but by making refinements we found that the method of contour distortions is indeed effective in regularizing integrals. The analyticity of $\alpha(s)$ mentioned above is a necessary condition for success of the method. We find it hard to imagine existence of a Regge theory without some kind of analyticity of α, since we see no chance of continuing the Froissart-Gribov partial wave to small values of $\mathrm{Re}\,l$ if analyticity is not available.

An appealing feature of our scheme is that it involves only the half l plane $\mathrm{Re}\,l \geq -\epsilon$, where $0<\epsilon<\frac{1}{2}$. Thus, we need not investigate the whole Regge trajectory $\alpha(s)$; only the part of it with $\mathrm{Re}\,\alpha \geq -\epsilon$ enters the equations. In particular, we can ignore the Gribov-Pomeranchuk singularities[4] and other complicated phenomena that may occur in the left l plane (see R. G. Newton, Ref. I6).

The behavior of A and B at large $|l|$ is closely related to that at large s. There are two aspects of large $|l|$ behavior: First, one must guarantee exponential decrease of partial waves at large $\mathrm{Re}\,l$, which is required for the correct support of double-spectral functions; second, one must have a power decrease at large Iml, to ensure convergence of the Watson-Sommerfeld integral that occurs in the partial-wave development of double-spectral functions. The first point is handled by a modification of the method of part I. The second point requires an analysis similar to that of Frederiksen *et al.* (Ref. I21), which is summarized in Sec. III.

If there are Regge poles in the right plane, we cannot get good behavior at large Iml and large s simultaneously. The contour deformations that make explicit the good behavior of A and B at large s have the effect of giving bounds with exponential increase in Iml, rather than the desired power decrease. This situation is caused by the t-channel Regge-pole terms. A closer investigation suggests that we are confronting the partial-wave version of the old difficulty of catastrophic powers of s in the Mandelstam iteration (Chew *et al.*, Refs. I3, I9, and I10). We are led to deal with it in the way proposed by Chew *et al.*, namely, to multiply the s-channel elastic part of the double-spectral function, $\rho^{\text{el}}(s,t)$, by a cutoff factor $h(s)$. For convenience we have introduced the cutoff already in part I, although it was not necessary in the weak-coupling case. When the cutoff is present, it is no longer necessary to distort contours of the t-channel Regge pole terms; one can solve the N/D equation even if the force function B has polynomial growth at large s. With this observation alone, however, the situation does not look better than it was in the Mandelstam iteration. The great advantage of the partial-wave approach is that after the equations with cutoff are solved, one can perform the contour deformation to show that the partial waves of the solution are actually bounded at large s for fixed l.

The presence of the cutoff is in effect a restriction on the form of the central spectral function, and is just one aspect of our semiphenomenological treatment of inelasticity. With ρ^{el} defined so as to include the cutoff, we have

$$\rho(s,t)=\rho^{\text{el}}(s,t)+\rho^{\text{el}}(t,s)+v(s,t), \tag{1.1}$$

where the central spectral function v is regarded as given. All we know of $\rho(s,t)$ independently of models is that it is equal to $\rho^{\text{el}}(s,t)$ for $s<16$ and $\rho^{\text{el}}(t,s)$ for $t<16$, with $\rho^{\text{el}}(s,t)$ being given in terms of two-body absorptive parts for $s<16$. The requirement of a cutoff suggests that s-channel two-particle states should not be present at full strength simultaneously with t-channel two-particle states at full strength, if s and t are simultaneously large. Thus, the cutoff forbids a kind of "double-counting." In a complete crossing-symmetric theory, ρ would have the form

$$\rho(s,t)=\rho^{\text{el}}(s,t)+\rho^{\text{inel}}(s,t), \tag{1.2}$$

where $\rho^{\text{inel}}(s,t)$ would arise from all s-channel inelastic states, and would reduce to $\rho^{\text{el}}(t,s)$ for $t<16$. It is very difficult to make a model, however, in which $\rho^{\text{inel}}(s,t)$ is manifestly a sum over inelastic absorptive parts and is also equal to $\rho^{\text{el}}(t,s)$ for small t.

In discussions of the strip model, the term $\rho^{\text{el}}(t,s)$ in (1.1) is associated with the picture shown in Fig. 1. This term corresponds to peripheral processes of two-meson exchange, with four and more mesons in the s channel. Such an interpretation cannot be taken quite literally, so that the corresponding part of the overlap function is not necessarily positive. We make a few additional remarks on this topic in part IV. Our scheme is essentially different from the bootstrap strip model, in that we have nonperipheral contributions to inelasticity and short-range cross-channel driving forces in v. We shall see presently that it is possible to make a nontrivial model with $v=0$, but even this model is very different from the strip model, since it contains a short-range "contact interaction" of mesons, described by an s-wave subtraction constant.

A long-standing problem in Regge theory is the question of ghost poles[5] (see Ref. I7). If the even- (odd-) signature partial wave has a Regge trajectory $\alpha(s)$ which passes through an even (odd) integer at $s=s_*<4$, there will be a bound-state pole at s_* unless the Regge residue function $\beta(s)$ happens to vanish at that point. Such "ghost" poles are inconsistent with cross-channel unitarity for $s_*\leq 0$, and inconsistent with observation (in 0^-0^- channels such as $\pi\pi$, πK, ...) for $s_*<4$.

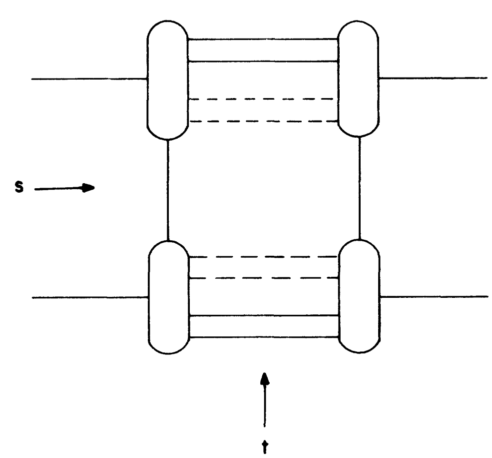

FIG. 1. An heuristic view of the contribution of $\rho^{\text{el}}(t,s)$ to the overlap function. The dashed lines represent states containing an arbitrarily large (even) number of particles.

We expect a ghost problem to occur only in even-signature amplitudes at $l=0$ (see Sec. IV). Gell-Mann[5] suggested that a ghost at $l=0$, $I=0$ in $\pi\pi$ scattering might be eliminated by β having a zero at $s=s_*$. In his view, the trajectory would be associated with a set of coupled channels, including channels with spin ($\pi\pi, N\bar{N}$, etc.). At $l=0$, channels with spin contain "nonsense" (negative integer) values of orbital angular momentum L. If the trajectory "chooses nonsense" at $s=s_*$ (i.e., is associated only with nonsense channels in a channel labeling based on L), then the Regge pole will not appear in the $\pi\pi$ channel which is a "sense" channel at $l=0$. To our knowledge, this mechanism has not been verified in a concrete model, and it leaves one without a method of ghost elimination in a theory without spin, such as $\lambda\phi^4$ field theory. Perhaps Gell-Mann's idea could be realized in a phenomenological way in our scheme, through constraints on v. That would be a technically difficult feat, if possible at all. We suggest a more direct way of eliminating the ghost, which does not involve spin, and which has some interesting consequences.

Our plan, as described in Sec. IV, is to let the bound-state pole occur in the l-analytic amplitude $a(l,s)$ at $l=0$, but construct the total amplitude $A(s,t)$ so that its s wave differs from $a(0,s)$. The actual s wave, denoted by $a_0(s)$, is determined through a separate N/D equation which differs from that for $a(0,s)$ by having a subtraction constant, or Castillejo-Dalitz-Dyson (CDD) poles, or both. The extra parameters of the N/D equation for a_0 are to be chosen so that a_0 does not have a bound-state pole or ghost poles. For real π-π scattering, s-wave resonance poles would be desired, and these might be associated with the CDD poles. Crossing symmetry and unitarity are achieved through single-spectral integrals in $A(s,t)$. A similar idea for ghost elimination was suggested by Saito,[6] who considered the difference between $a_0(s)$ and $a(0,s)$ to be due to one CDD pole, but did not show that the proposal was consistent with a crossing-symmetric Regge theory. Also, there are hints of the idea in a paper of Mandelstam[7] about theories with spin. Mandelstam considers a possible difference between physical and l-analytic partial waves, but supposes that any difference is due to an elementary particle in the channel at hand. Since we allow a subtraction constant in a_0 without CDD poles (at least if that choice gives no ghost in a_0), our proposal is more general, i.e., we do not necessarily require an elementary particle.

The subtraction term in a_0 may be understood as corresponding to a "contact interaction" of mesons such as one has in a $\lambda\phi^4$ field theory, with the subtraction constant being analogous to the renormalized coupling constant. One can take the subtraction term as the sole driving force of the theory by putting $v=0$. If this is done without CDD poles, the resulting system looks as though it could be an approximation to the $\lambda\phi^4$ theory. The separate s wave seems quite natural from the viewpoint of field theory, even if decidedly foreign to bootstrap theory.

A theory with $v=0$ has, in some sense, the simplest inelasticity allowed by crossing symmetry, namely, that illustrated in Fig. 1. It will be interesting to see if such a theory has a Pomeranchuk trajectory when the coupling is sufficiently large. That is, it is interesting to ask whether peripheral inelastic effects can "cast a shadow" in such a way as to cause a constant asymptotic total cross section.

We do not insist on the ghost-elimination mechanism of Sec. IV as the only possibility. We intend to discuss elsewhere the Gell-Mann mechanism in an extended formalism including channels with spin. The Gell-Mann scheme, if realized, would be more consonant with the philosophy of maximal analyticity and with dual resonance models. Furthermore, our scheme of Sec. IV has certain consequences for high-energy behavior of amplitudes, which might in principle allow it to be ruled out in a realistic model of π-π scattering. This matter will be examined in part III.

In Sec. V we collect the results of the paper to state the equation $\psi=G(\psi)$ in regularized form. We defer to part III the problem of showing that a solution of that equation actually provides a crossing-symmetric unitary amplitude.

Our theory as currently formulated does not have "infinitely rising" Regge trajectories, although we do allow the maximum value of $\mathrm{Re}\,\alpha$ to be arbitrarily large. Also, the theory does not have Regge branch points in the right half of the l plane, thanks to the cutoff and restrictions on the behavior of the central spectral function. In ruling out infinitely rising trajectories and Regge branch points we ignore aspects of Regge theory which are considered important by many authors. It seems premature to allow such complications in a program for actual construction of a Regge theory, however, especially since the experimental and theoretical arguments in favor of rising trajectories and branch points are not conclusive. Some of the techniques developed for our program, such as our methods of handling large-l behavior, might well prove to be useful in approaching more difficult versions of Regge dynamics.

For simplicity we restrict the discussion to equal-mass isoscalar 0^- meson-meson scattering. Extensions to account for pion isospin would be straightforward. The generalization to treat several coupled channels with complete crossing symmetry will be treated in part IV of this series.

We use notation from part I, and give a summary of notation in Sec. V.

II. THE FUNCTIONALS A AND B IN PRESENCE OF REGGE POLES

As in part I, we work with the Froissart-Gribov representation of the reduced amplitude $c(l,s)$, namely

$$c(l,s)=\left[\frac{p(s)}{s-4}\right]^l a(l,s)$$
$$=\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\,Q_l(z_{st})A_t(s,t)\,. \qquad (2.1)$$

Our notation is the same as in part I, except for a modified definition of $p(s)$. We define

$$p(s)=[(s+a^2)^{1/2}+2]^2,\quad a^2>0\,, \qquad (2.2)$$

$$p_0(s)=(s^{1/2}+2)^2\,. \qquad (2.3)$$

The function p_0 was called p in part I. The purpose of the term a^2 in (2.2) is to move the branch point of p from $s=0$ to $s=-a^2$, in order to put it out of the way of a contour deformation which will be performed later. The cut of p runs from $s=-a^2$ to $s=-\infty$, with $(s+a^2)^{1/2}>0$ for $s>-a^2$.

The t-channel elastic part of A_t, which is equal to A_t itself for $4\leqslant t\leqslant 16$, has the Legendre development

$$A_t^{\mathrm{el}}(s,t)=\sum_{l=0}^\infty \zeta(l,t)P_l^{(e)}(z_{ts})\,, \qquad (2.4)$$

$$\zeta(l,t)=(2l+1)q(t)h(t)a(l,t_+)a(l,t_-)\,. \qquad (2.5)$$

We make a Watson-Sommerfeld transformation of that part of A_t^{el} with l values lying to the right of all Regge poles. Let

$$\alpha_m=\max[\mathrm{Re}\,\alpha(s_+)]\,, \qquad (2.6)$$

where the maximum is taken over all trajectories α and over all $s\geqslant 4$. Let the integer L and the number L_0 satisfy the relations $L\leqslant\alpha_m<L_0<L+1$. Then a Watson-Sommerfeld transformation yields

$$A_t^{\mathrm{el}}(s,t)=\frac{i}{2}\int_{L_0} dl\,\frac{\zeta(l,t)}{\sin\pi l}P_l^{(e)}(z_{ts})$$
$$+\sum_{l=0}^{L}\zeta(l,t)P_l^{(e)}(z_{ts})\,. \qquad (2.7)$$

The polynomial in (2.7) does not contribute to the discontinuity of A_t^{el} over the s cut, which is equal to the t-channel elastic part of the double-spectral function,

$$\rho^{\mathrm{el}}(t,s)=\frac{1}{4i}\int_{L_0} dl\,\zeta(l,t)P_l(z_{ts})\,. \qquad (2.8)$$

The complete expression for A_t, valid at all t, is obtained by adding in the t-channel inelastic terms provided by $\rho^{\mathrm{el}}(s,t)+v(s,t)$:

$$A_t(s,t)=A_t^{\mathrm{el}}(s,t)$$
$$+\frac{1}{\pi}\int_{\sigma(t)}^\infty ds'[\rho^{\mathrm{el}}(s't)+v(s',t)]$$
$$\times\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right). \qquad (2.9)$$

If A_t^{el} is represented as in (2.7), this expression for A_t is well defined even in the presence of Regge poles, provided that $v(s,t)$ is such that the s' integral in (2.9) converges.

An estimate of the large-t dependence of $\rho^{\mathrm{el}}(s,t)$ from (2.8) indicates a behavior t^{L_0}. If we put (2.9) into (2.1) we can then be sure of convergence of the t integral only for $\mathrm{Re}\,l>L_0$, since $Q_l(z)$ behaves as z^{-l-1} at large z. In order to define $c(l,s)$ for $\mathrm{Re}\,l\leqslant L_0$, we must take account of Regge poles. Before doing so, we have to explain our assumptions about general properties of Regge trajectories.

To keep the notation short, we shall suppose that there is only one Regge trajectory $\alpha(s)$ in the half plane $\mathrm{Re}\,l\geqslant-\epsilon$. We fix ϵ at the outset, with $0<\epsilon<\frac{1}{2}$, and refer to the region $\mathrm{Re}\,l\geqslant-\epsilon$ as the "right half plane." The extension to account for a finite number of (nonintersecting) trajectories is immediate. The trajectory is defined as the zero of the denominator function D of (I2.52):

$$D(\alpha(s),s)=1-\frac{1}{\pi}\int_4^\infty \frac{r(\alpha(s),s')n(\alpha(s),s')}{s'-s}ds'$$
$$=0\,. \qquad (2.10)$$

Since $n(l,s)$ will be known only for $\mathrm{Re}\,l\geqslant-\epsilon$, $\alpha(s)$ can be determined only for s such that $\mathrm{Re}\,\alpha(s)\geqslant-\epsilon$. In fact, $\alpha(s)$ is defined only in a finite region of the s plane, since for sufficiently large $|s|$, Eq. (2.10) cannot be satisfied. The factor $r(l,s)$, given in (I2.35), contains the cutoff, and the function space in which we seek solutions is such that $r(l,s)n(l,s)$ is small for all l when $|s|$ is large. Thus, the trajectory leaves the right half plane at large $|s|$, since the integral in (2.10) becomes small compared to 1.

Let Ω denote the set of all points in the cut s plane (the cut being $[4,\infty)$) such that $\alpha(s)$ is defined and $\mathrm{Re}\,\alpha(s)>-\epsilon$. Also, let $\bar\Omega$ be the closure of Ω, i.e., Ω plus its boundary points. Our first assumption concerning α is that

$$\text{(i)}\ D_l(\alpha(s),s)\neq 0,\quad s\in\bar\Omega\,, \qquad (2.11)$$

where $D_l(l,s)=\partial D(l,s)/\partial l$. Since $D(l,s)$ is analytic in l for $\text{Re}\,l > -\epsilon$, and analytic in s in the cut plane, it follows from (2.11) and the implicit function theorem on analytic functions[8] that $\alpha(s)$ is analytic in the finite region Ω.

The assumption (i) is not sufficient to ensure that the boundary $\partial\Omega$ of the domain Ω is smooth enough for our purposes, and also is not sufficient to rule out discontinuities of α on $\partial\Omega$.[9] Accordingly, we shall assume that the inverse of the function $\alpha(s)$ exists. The inverse, denoted by $\omega(l)$, takes the domain $\alpha(\Omega)$ into Ω; we have

$$\alpha(\omega(l)) = l, \quad \omega(\alpha(s)) = s\,. \tag{2.12}$$

According to the inverse function theorem,[10] a sufficient condition for the existence of the inverse is that

$$\alpha'(s) \neq 0, \quad s \in \Omega\,. \tag{2.13}$$

In view of the identity

$$D_l(\alpha(s), s)\,\alpha'(s) + D_s(\alpha(s), s) = 0 \tag{2.14}$$

and assumption (i), an equivalent requirement is that D_s not vanish. Our second assumption, therefore, is that

$$\text{(ii)}\ D_s(\alpha(s), s) \neq 0, \quad s \in \bar{\Omega}\,. \tag{2.15}$$

The third essential assumption is that the imaginary part of the boundary value on the cut, $\text{Im}\,\alpha(s_+)$, has the proper sign; namely

$$\text{(iii)}\ \text{Im}\,\alpha(s_+) > 0, \quad s > 4\,. \tag{2.16}$$

Assumptions (i)–(iii) can be, and must be, checked numerically in any application of the scheme. For a discussion of analyticity and invertibility of $\alpha(s)$ in potential theory, see R. G. Newton, Ref. I6, Chap. 14. According to the implicit-function theorem, the condition (i) excludes crossing or bifurcation of Regge trajectories. In other words, if $D(\alpha_0, s_0) = 0$ and $D_l(\alpha_0, s_0) \neq 0$, then there exists a unique $\alpha(s)$ such that $\alpha(s_0) = \alpha_0$ and $D(\alpha(s), s) = 0$, for s in a sufficiently small neighborhood of s_0.

For simplicity we suppose also that $\text{Re}\,\alpha(s_+)$ has only one relative maximum, and that the trajectory can have $\text{Re}\,\alpha > -\epsilon$ for some interval of s only if $\alpha(4) > -\epsilon$ (see Fig. 2). These assumptions are suggested by results from potential scattering (see Ref. I6),[11] and are made only for notational convenience. Questions of notation aside, the shape of the trajectory is unimportant, as long as properties (i)–(iii) are verified. Our assumptions imply that the trajectory enters the right half plane at some energy $s_2 \leq 4$ and leaves it at an energy $s_1 \geq 4$. The boundary values of $a(l,s)$ may be written as

$$a(l, s_\pm) = \theta(s_1 - s)\frac{\beta(s_\pm)}{l - \alpha(s_\pm)} + \bar{a}(l, s_\pm), \quad s \geq 4 \tag{2.17}$$

where $\bar{a}(l, s_\pm)$ is analytic in l in the right half plane (but not necessarily continuous in s at $s = s_1$).

The boundary $\partial\Omega$ consists of the cut $[4, s_1]$ plus the map by $\omega(l)$ of the straight-line segment with end points $(-\epsilon, \text{Im}\,\alpha(s_{1-}))$, $(-\epsilon, \text{Im}\,\alpha(s_{1+}))$. A graph of this boundary is shown in Fig. 3, for the case of a simple model trajectory.

The residue of the Regge pole in the reduced amplitude $c(l,s)$ will be denoted by $\tilde{\beta}(s)$. By Eq. (I2.51), it has the representation

$$\tilde{\beta}(s) = \left[\frac{1}{D_l(l,s)} \times \frac{1}{\pi}\int_4^\infty \frac{B(l,s')r(l,s')n(l,s')}{s'-s}\,ds'\right]_{l=\alpha(s)}. \tag{2.18}$$

It follows from assumption (i) that $\tilde{\beta}(s)$ is analytic in Ω, since the expression in square brackets in (2.18) is analytic in l in the right half plane, and analytic in s in the plane with cut $[4,\infty)$. The residue $\beta(s)$ of the pole in $a(l,s)$ is related to $\tilde{\beta}(s)$ by the equation

$$\beta(s) = \left[\frac{s-4}{p(s)}\right]^{\alpha(s)} \tilde{\beta}(s)\,. \tag{2.19}$$

Thus, $\beta(s)$ is analytic only in the upper and lower halves of Ω separately, because of the cut of the first factor in $\beta(s)$ which goes to the left from $s = 4$. The functions α, β, and $\tilde{\beta}$ have the reflection property $f(s) = f^*(s^*)$.

To proceed with the continuation of (2.1) to the region $\text{Re}\,l \leq L_0$, the first step is to move the contour in (2.8) from the line $\text{Re}\,l = L_0$ to $\text{Re}\,l = -\epsilon$. In

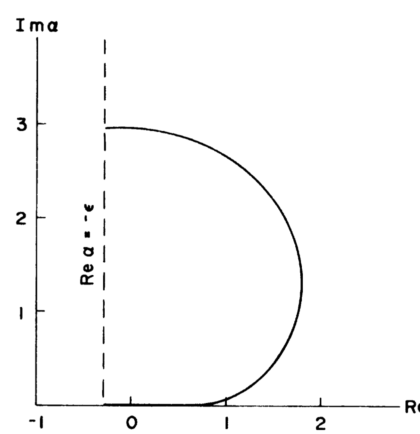

FIG. 2. A Regge trajectory of the type described in Sec. II.

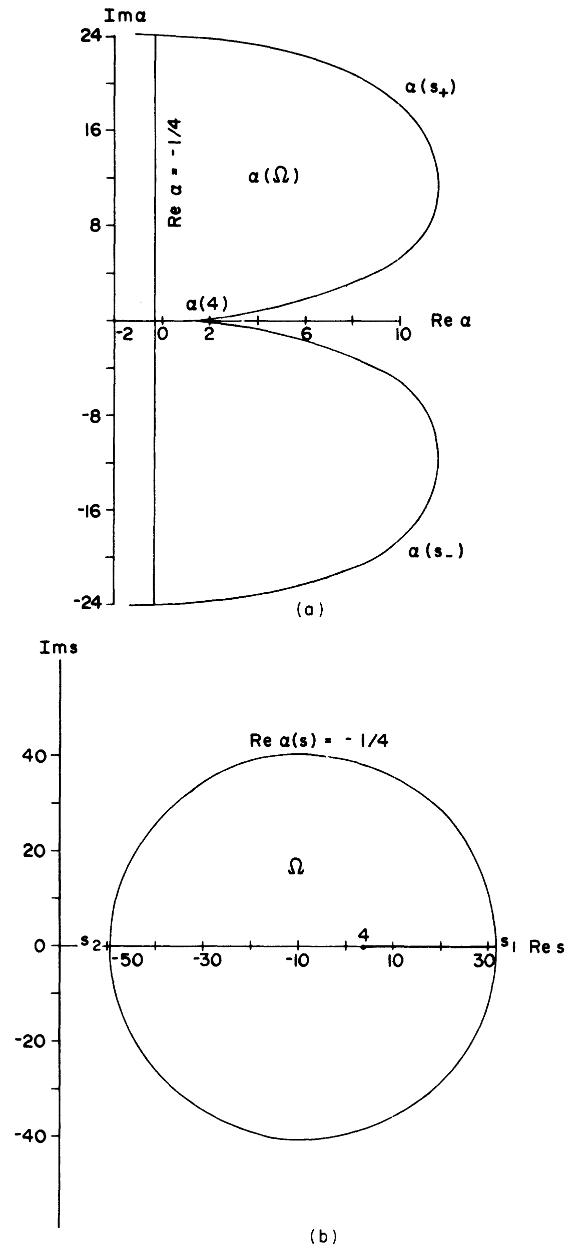

FIG. 3. (a) A model of a Regge trajectory given by the formula

$$\alpha(s) = -1 - \frac{a}{s - s_0 + ib(s-4)^{1/2}}.$$

The graph is for $s_0 = 32$, $a = 64$, $b = 0.5$. In this example, the curve $\alpha(t)$ leaves the real axis at $t = 4$ with infinite slope, but the region of large slope is so small that it does not show up in the graph. (b) A domain Ω of analyticity of the model trajectory $\alpha(s)$ of (a). The boundary of the domain consists of the cut $[4, s_1]$ and the curve $\mathrm{Re}\,\alpha(s) = -\frac{1}{4}$, which is nearly a circle. The inverse of $\alpha(s)$ is

$$\omega(\alpha) = s_0 - \frac{a}{\alpha+1} - \frac{b^2}{2} + \frac{b}{2}\left(\frac{4a}{\alpha+1} + b^2 + 16 - 4s_0\right)^{1/2}.$$

so doing, one picks up contributions from the two Regge poles at $l = \alpha(t_\pm)$. The residues of the poles entail the factors $a(\alpha(t_\pm), t_\mp)$, which we evaluate with the help of the unitarity condition (I2.37). The latter condition may be written as

$$c(l, t_\mp) = \frac{c(l, t_\pm) \mp 2iI(l, t)}{1 \pm 2ir(l, t)c(l, t_\pm)}, \tag{2.20}$$

$$I(l, t) = \frac{1 - \hat{\eta}^2(l, t)}{4r(l, t)}. \tag{2.21}$$

In evaluating (2.20) at $l = \alpha(t_\pm)$, we make use of the fact that $I(l, t)$, defined in (I2.36), is analytic in l for $\mathrm{Re}\,l > -\epsilon$, continuous for $\mathrm{Re}\,l = -\epsilon$, and hence bounded for l near $\alpha(t_\pm)$. The poles in numerator and denominator then cancel to yield

$$a(\alpha(t_\pm), t_\mp) = \pm[2iq(t)h(t)]^{-1}. \tag{2.22}$$

With the help of (2.22) one can move the contour in (2.8) to the left to obtain

$$\rho^{\mathrm{el}}(t, s) = \frac{1}{4i}\int_{-\epsilon} dl\, \xi(l, t) P_l(z_{ts}) + \frac{\pi}{2}\,\theta(s_1 - t)\Delta f(t, s). \tag{2.23}$$

where Δf is the t discontinuity of f,

$$f(t, s) = [2\alpha(t) + 1]\beta(t) P_{\alpha(t)}(z_{ts}), \tag{2.24}$$

$$\Delta f(t, s) = \frac{1}{2i}[f(t_+, s) - f(t_-, s)]. \tag{2.25}$$

Similarly, one may move the contour in A_t^{el}, after first including the polynomial in (2.7) as part of the Watson-Sommerfeld integral. The result is

$$A_t^{\mathrm{el}}(s, t) = \frac{i}{2}\int_{-\epsilon} dl\, \frac{\xi(l,t)}{\sin \pi l} P_l^{(e)}(z_{ts}) - \pi\theta(s_1 - t)\Delta g(t, s), \tag{2.26}$$

$$g(t, s) = \frac{[2\alpha(t) + 1]\beta(t)}{\sin \pi\alpha(t)} P_{\alpha(t)}^{(e)}(z_{ts}). \tag{2.27}$$

Through the introduction of (2.9), (2.23), and (2.26) in the Froissart-Gribov integral (2.1) we obtain

$$c(l, s) = V(l, s) + \sum_{i=0}^{3} c_i(l, s), \tag{2.28}$$

where V is the central-spectral-function term defined in (I3.7), and

$$c_i(l, s) = \frac{p(s)^l}{(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st})\xi_i(s, t), \tag{2.29}$$

$$\xi_1(s, t) = -4\theta(s_1 - t)\Delta g(t, s) + \frac{2i}{\pi}\int_{-\epsilon} dl'\, \frac{\xi(l', t)}{\sin \pi l'} P_{l'}^{(e)}(z_{ts}), \tag{2.30}$$

$$\xi_2(s,t) = \theta(s_1-\sigma(t))\frac{2}{\pi}\int_{\sigma(t)}^{s_1} ds'\Delta f(s',t)\left(\frac{1}{s'-u}+\frac{1}{s'-s}\right), \tag{2.31}$$

$$\xi_3(s,t)+\xi_0(s,t) = \theta(t-16)\frac{1}{\pi}\int_{\sigma(t)}^{\infty} ds'\left(\frac{1}{s'-u}+\frac{1}{s'-s}\right) \times \frac{1}{\pi i}\int_{-\epsilon} dl'\zeta(l',t)P_{l'}(z_{s't}). \tag{2.32}$$

Since $\Delta g(t,s)$ behaves as $s^{\alpha(t)}$ at large s, a superficial estimate would indicate that c_1 behaves as $s^{\alpha_m-1}\ln s$. Since we wish to allow an arbitrarily large value of α_m, we cannot allow such a behavior; it would violate unitarity. The difficulty is only apparent, however, as we shall demonstrate later by means of a contour deformation. The t integral in c_2 appears not to converge for small values of Rel, since $\Delta f(s',t)$ behaves as $t^{\alpha(s')}$ at large t. Again, the cure for this apparent obstacle is a contour distortion.

To derive the force function B for the N/D equation, we must separate the elastic unitarity term, which is part of c_0. We change the order of integrations to obtain

$$c_0 = \frac{1}{\pi}\int_4^{\infty}\frac{ds'}{s'-s}\frac{1}{2\pi i}\int_{-\epsilon} dl'\zeta(l',s')\Sigma(l,l',s,s'), \tag{2.33}$$

$$\Sigma = \frac{2p(s)^l}{(s-4)^{l+1}}\int_{\tau(s')}^{\infty} dt\, Q_l(z_{st})P_{l'}(z_{s't}) = \Sigma_1+\Sigma_2, \tag{2.34}$$

$$\Sigma_1 = \frac{2p(s')^l}{(s'-4)^{l+1}}\int_{\tau(s')}^{\infty} dt\, Q_l(z_{s't})P_{l'}(z_{s't}) = \left[\frac{p(s')}{s'-4}\right]^l \frac{W(l,l',z_{s'\tau(s')})}{(l'-l)(l'+l+1)}, \tag{2.35}$$

$$\Sigma_2 = \int_{\tau(s')}^{\infty} dt\, P_l(z_{s't})\left[\frac{2p(s)^l}{(s-4)^{l+1}}Q_l(z_{st}) - \frac{2p(s')^l}{(s'-4)^{l+1}}Q_l(z_{s't})\right]. \tag{2.36}$$

The function W was defined in (I2.32). The l' integral of Σ_1 may be evaluated by closing the contour on the right. There are contributions from the Regge poles as well as from the pole at $l'=l$. The residues are evaluated with the help of (2.22) and (I2.33), and the result of the calculation is that

$$c_0(l,s) = c_4(l,s)+c_5(l,s) + \frac{1}{\pi}\int_4^{\infty}\frac{ds'r(l,s')c(l,s'_+)c(l,s'_-)}{s'-s}, \tag{2.37}$$

$$c_4 = -\frac{1}{\pi}\int_4^{s_1}\frac{ds'}{s'-s}\left[\frac{p(s')}{s'-4}\right]^l \times\Delta\left[\frac{(2\alpha+1)\beta(s')W(l,\alpha,z_{s'\tau(s')})}{(\alpha-l)(\alpha+l+1)}\right]_{\alpha=\alpha(s')}, \tag{2.38}$$

$$c_5 = \frac{1}{\pi}\int_4^{\infty}\frac{ds'}{s'-s} \times\frac{1}{2\pi i}\int_{-\epsilon} dl'\zeta(l',s')\Sigma_2(l,l',s,s'). \tag{2.39}$$

The third term of (2.37) is the desired elastic unitarity term. Notice that $\Delta c_4 = -\Delta c_2$, so that c_2+c_4 has only a left cut. Also, $\Sigma_2=0$ for $s=s'$, so that c_5 has only a left cut. The total left-cut contribution to c is then

$$c_L = (c_1+V)_L + \sum_{i=2}^{5} c_i, \tag{2.40}$$

where the subscript L means "left-cut part." The force function B that we seek is c_L plus the principal-value integral of $(1-\hat\eta)/2r$, as shown in Eq. (I1.5). The sum of that integral and $(c_1+V)_L$ may be represented neatly through an application of the identities (see Ref. I10)

$$\tfrac{1}{2}[c_1(l,s_+)+c_1(l,s_-)+V(l,s_+)+V(l,s_-)] = [c_1(l,s)+V(l,s)]_L + \frac{\mathrm{P}}{\pi}\int_{16}^{\infty}\frac{[1-\hat\eta^2(l,s')]ds'}{4r(l,s')(s'-s)}, \tag{2.41}$$

$$\frac{1-\hat\eta}{2r} = \frac{1-\hat\eta^2}{4r}+\frac{(1-\hat\eta)^2}{4r}. \tag{2.42}$$

We define

$$c_6(l,s) = \tfrac{1}{2}[c_1(l,s_+)+c_1(l,s_-)+V(l,s_+)+V(l,s_-)], \tag{2.43}$$

$$c_7(l,s) = \frac{\mathrm{P}}{\pi}\int_{16}^{\infty}\frac{[1-\hat\eta(l,s')]^2ds'}{4r(l,s')(s'-s)}. \tag{2.44}$$

The complete formula for B is then written as

$$B[a,v;l,s] = \sum_{i=2}^{7} c_i(l,s). \tag{2.45}$$

For construction of the amplitude $c(l,s)$ from the N/D method, we also need $C(l,s)$, which differs from B by containing the complete Cauchy integral, rather than the principal-value integral, of $(1-\hat\eta)/2r$. The function C is analytic in the cut s plane, and is given by

$$C(l,s)=V(l,s)+\sum_{i=1}^{5} c_i(l,s) + \frac{1}{\pi}\int_{16}^{\infty} \frac{[1-\hat{\eta}(l,s')]^2}{4r(l,s')(s'-s)} . \qquad (2.46)$$

The other functional A, which is identified with the function $\hat{\eta}$, is

$$A[a,v;l,s]=[1-4r(l,s)I(l,s)]^{1/2} , \qquad (2.47)$$

where

$$I(l,s)=\Delta[c_1(l,s)+V(l,s)] . \qquad (2.48)$$

The expression for c_6 may be put into a more useful form by applying the well-known identities[12]

$$P_\nu(-z)=e^{\pm i\pi\nu}P_\nu(z)-\frac{2}{\pi}\sin\pi\nu\, Q_\nu(z) , \quad \mathrm{Im}z \gtrless 0. \qquad (2.49)$$

With s, $t>4$ these identities give

$$\tfrac{1}{2}[P_\nu^{(e)}(z_{ts_+})+P_\nu^{(e)}(z_{ts_-})] = \tfrac{1}{2}(1+\cos\pi\nu)P_\nu(z_{ts})-\frac{1}{\pi}\sin\pi\nu\, Q_\nu(z_{ts}) . \qquad (2.50)$$

In using this result to evaluate c_6 through (2.30), we find Legendre functions having only right-hand cuts in the s plane, whereas there were inconvenient left cuts before.

III. THE FUNCTIONALS A AND B IN REGULAR FORM

In this section we transform the N/D inputs A and B, defined in (2.47) and (2.45), so that good behavior in all regions of l and s will be evident. We begin with a discussion of c_6, which dominates the large-s behavior of B. The input terms from V in (2.43) are well behaved by hypothesis. Of the remaining term in (2.43) from c_1, only the Regge-pole term, the first term in (2.30), is problematical. Furthermore, if we evaluate that term with the help of identity (2.50), we find that the piece from $Q_\alpha(z_{ts})$ has suitable behavior at large s as it stands. We need discuss only the piece involving P_α,

$$d_1(l,s)=\frac{-2p(s)^l}{(s-4)^{l+1}}\int_4^{s_1} dt\, Q_l(z_{st}) \times\Delta\left[(2\alpha+1)\beta(t)\,\frac{1+\cos\pi\alpha}{\sin\pi\alpha}P_\alpha(z_{ts})\right]_{\alpha=\alpha(t)} . \qquad (3.1)$$

First, suppose that $\alpha(0)>-\epsilon$. Because of the analyticity properties of α and β assumed in Sec. II, the function in square brackets in (3.1) is analytic in t in the upper or lower half of the region Ω. Consequently, we may distort the contour of the term with $P_{\alpha(t_+)}$ to follow the course shown in Fig. 4(b), i.e., the integration will follow the upper side of the real axis from 4 to $-s_0$, and return to s_1 along a path for which $\mathrm{Re}\alpha(t)<\alpha(-s_0)$. Here s_0 is chosen so that $-\epsilon<\alpha(-s_0)$, and $0<s_0<a^2$, where a^2 is the constant that appears in $p(s)$, Eq. (2.2). The part of the path from $-s_0$ to s_1 will be denoted by $\omega(\Gamma_+)$. It is the image under the mapping ω (the inverse of α) of the path Γ_+ in the α plane, shown in Fig. 4(a). There is a similar distortion of the integral with $P_{\alpha(t_-)}$ into the lower t plane. The union of the curves Γ_+ and Γ_- is called Γ.

The contribution of $\omega(\Gamma)$ to d_1 is of order $s^{\alpha(-s_0)-1}\ln s$ at large s. The remainder, the integral from 4 to $-s_0$, is

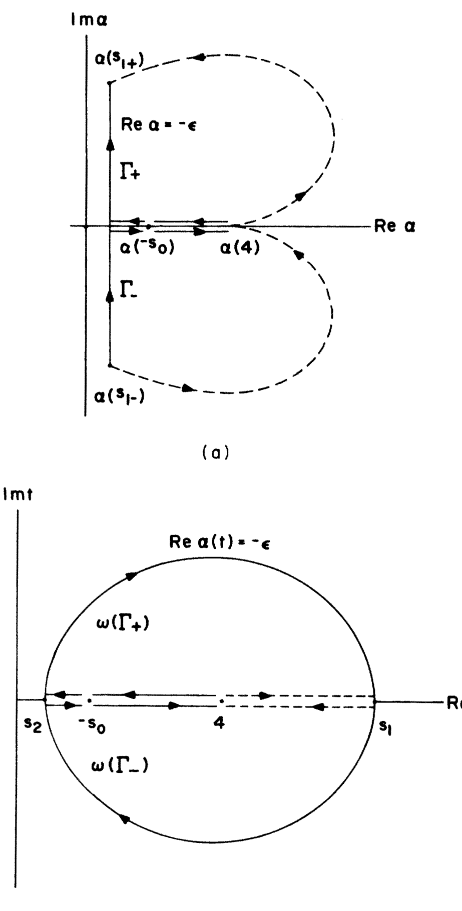

FIG. 4. The dashed curves represent the original paths of integration for certain Regge-pole terms (for instance c_6, Δc_1) in the α plane [Fig. 4(a)] and in the t plane [Fig. 4(b)]. The solid curves are the deformed paths used in the discussion of Sec. III. In (a), the paths $\Gamma_\pm$ have end points $\alpha(-s_0)$ and $\alpha(s_{1\pm})$. The union of Γ_+ and Γ_- is called Γ in the text.

$$\frac{p(s)^l}{i(s-4)^{l+1}}\int_{-s_0}^{4} dt\,\frac{(2\alpha+1)(1+\cos\pi\alpha)\,\tilde{\beta}(t)}{\sin\pi\alpha\, p^{\alpha}(t)}\left[Q_l(z_{st+})(t_+-4)^{\alpha}P_{\alpha}(z_{t+s})-Q_l(z_{st\,-})(t_- -4)^{\alpha}P_{\alpha}(z_{t-s})\right]\Big|_{\alpha=\alpha(t)}\,. \tag{3.2}$$

We have applied the relation (2.19) between β and $\tilde{\beta}$. If $\alpha(t_*)=0$ for some $t_*<4$, then (3.2) applies as written only if $\beta(t_*)=0$, as in Gell-Mann's ghost-elimination mechanism. If $\beta(t_*)\neq 0$, then (3.2) is to be replaced by a modified expression to realize the ghost-elimination scheme of Sec. IV. In the modified expression, given later in Eq. (5.17), there is no pole at $\alpha=0$ and the present method of analysis still works. The function (3.2) may be analyzed by applying the identities (2.49), which lead to the relations

$$(t_+-4)^{\alpha(t)}P_{\alpha(t)}(z_{t+s})-(t_- -4)^{\alpha(t)}P_{\alpha(t)}(z_{t-s})=\frac{4}{\pi i}(4-t)^{\alpha(t)}\sin^2\pi\alpha(t)\,Q_{\alpha(t)}(-z_{ts})\,,\quad s>4+s_0 \tag{3.3}$$

$$Q_l(z_{st_+})-Q_l(z_{st\_})=\begin{cases}0, & 0\leqslant t\leqslant 4\,,\\ -\pi i P_l(z_{st}), & -s_0\leqslant t\leqslant 0,\ s>4+s_0.\end{cases} \tag{3.4}$$

The integral (3.2) then takes the form

$$\frac{-4p(s)^l}{\pi(s-4)^{l+1}}\int_{-s_0}^{4} dt\,Q_l(z_{st_+})(2\alpha+1)(1+\cos\pi\alpha)\sin\pi\alpha\,\tilde{\beta}(t)\left[\frac{4-t}{p(t)}\right]^{\alpha}Q_{\alpha}(-z_{ts})\Big|_{\alpha=\alpha(t)}$$
$$-\frac{\pi p(s)^l}{(s-4)^{l+1}}\int_{-s_0}^{0} dt\,P_l(z_{st})(2\alpha+1)\left(\frac{1+\cos\pi\alpha}{\sin\pi\alpha}\right)e^{-i\pi\alpha}\tilde{\beta}(t)\left[\frac{4-t}{p(t)}\right]^{\alpha}P_{\alpha}(z_{t-s})\Big|_{\alpha=\alpha(t)}\,. \tag{3.5}$$

The first term in (3.5) is of order $s^{-\alpha(-s_0)-2}\ln s$ at large s. The second term is the dominant part of d_1, and its behavior is determined by an arbitrarily small neighborhood of $t=0$, since $\alpha'(t)>0$ by assumption. If $\alpha(0)<1$, the integral behaves as

$$s^{-1}\int_{-\delta}^{0} dt\,e^{\ln s\,[\alpha(0)+\alpha'(0)t]}=O\left(s^{\alpha(0)-1}(\ln s)^{-1}\right). \tag{3.6}$$

If $\alpha(0)=1$, then

$$\frac{1+\cos\pi\alpha(t)}{\sin\pi\alpha(t)}=O(t),\quad t\to 0\,, \tag{3.7}$$

and the integral behaves as

$$s^{-1}\int_{-\delta}^{0} t\,dt\,e^{\ln s\,[1+\alpha'(0)t]}=O\left((\ln s)^{-2}\right). \tag{3.8}$$

Heretofore, $\alpha(0)>-\epsilon$; if $\alpha(\tau)=-\epsilon$ for some $\tau>0$, we move the contour only to $-s_0>0$ such that $\alpha(-s_0)=-\epsilon$, and find that d_1 is of order $s^{-\epsilon-1}\ln s$. The integral Δc_1 that appears in the functional A in (2.47) may be treated in a similar way.

To summarize,

$$c_6(l,s)=\begin{cases}O\left((\ln s)^{-2}\right), & \alpha(0)=1\\ O\left(s^{\alpha(0)-1}(\ln s)^{-1}\right), & -\epsilon<\alpha(0)<1\\ O(s^{-\epsilon-1}\ln s), & \alpha(\tau)=-\epsilon,\ \tau>0\end{cases} \tag{3.9}$$

$$\Delta c_1(l,s)=\begin{cases}O\left(s^{\alpha(0)-1}(\ln s)^{-1}\right), & -\epsilon<\alpha(0)\leqslant 1\\ O(s^{-\epsilon-1}\ln s), & \alpha(\tau)=-\epsilon,\ \tau>0\,.\end{cases} \tag{3.10}$$

It is possible to obtain the exact asymptotes of c_6 and Δc_1 by applying the method of Ref. I25. From those asymptotes one can work out the asymptote of the partial-wave amplitude itself.

The contour distortion in (3.1) has the effect of bringing in $Q_l(z)$ at complex z. Since $Q_l(z)$ grows exponentially as a function of Iml for Im$l\to\pm\infty$ when Im$z\gtrless 0$, respectively, it appears that c_6 will not satisfy the bounds at large Iml required for convergence of Watson-Sommerfeld integrals such as that in (2.23). Fortunately, we need not employ the distorted form of the integral in solving the N/D system, thanks to the presence of the cutoff function $h(s)$. We know from the undistorted form (2.43) that c_6 is at least bounded as

$$|c_6(l,s)|\leqslant\frac{\kappa s^{\gamma}}{l_+^{3/2}},\quad s\geqslant 4,\quad \mathrm{Re}\,l\geqslant-\epsilon\,, \tag{3.11}$$

for some $\gamma>0$. The cutoff can compensate this potential growth of c_6 at large s.

Accordingly, we set up a Banach space T, which is similar to the space S of part I, Sec. III, except that the asymptotic behaviors of B and $\hat{\eta}$ are different. In place of (I3.2) and (I3.3), the conditions met by elements $(1-\hat{\eta},B)$ of T are

$$l_+|B(l,s)|\,,s\,|\partial_s B(l,s)|\leqslant\frac{\kappa s^{\gamma}}{l_+^{1/2}}\left[\frac{p(s)}{p_0(s)}\right]^{\mathrm{Re}\,l}, \tag{3.12}$$

$$l_+|1-\hat{\eta}(l,s)|\,,s\,|\partial_s\hat{\eta}(l,s)|\leq\frac{\kappa|h(s)|s^{\gamma}}{l_+^{1/2}}\left[\frac{s-4}{p_1(s)}\right]^{\mathrm{Re}\,l}. \tag{3.13}$$

Here p and p_0 are defined in (2.2) and (2.3), and

$$p_1(s)=u(s-4,\sigma(s))$$
$$=\frac{1}{s-16}(s+2s^{1/2}-8)^2, \tag{3.14}$$

$$u(x,t)=[t^{1/2}+(x+t)^{1/2}]^2. \tag{3.15}$$

We shall find that the operator G, defined by the N/D system as in (I1.8)–(I1.10), maps a subspace U of T into T, provided that (a) a trajectory α generated from any element of U satisfies conditions (i)–(iii) of Sec. II, (b) functions $\hat{\eta}$ from U have no zeros, and (c) the N/D kernels obtained from elements of U do not have unit eigenvalues. Now suppose that a solution of the dynamical equation in such a subspace U is known. One may bound the high-energy behavior of the partial wave $a(l,s_+)$ constructed from that solution with the help of the contour distortion in Fig. 4. At fixed l (in particular, at a fixed physical value of l) we shall have

$$|a(l,s_+)|\le \kappa s^{\alpha(0)-1}(\ln s)^{-1}. \tag{3.16}$$

That is, the behavior at large s and fixed l is much better than is required for membership in the space T. This bound follows from formulas (I2.51) and (I2.49), in view of the fact (to emerge in the following) that c_6 determines the large-s behavior of B.

The next important question is the large-t behavior of ξ_2, the integrand of c_2 in (2.29). We discuss the second term in ξ_2, which involves the integral

$$\int_{\sigma(t)}^{s_1}\frac{ds'}{s'-s}\Delta f(s',t)=-\int_4^{\sigma(t)}\frac{ds'}{s'-s}\Delta f(s',t)+\int_4^{s_1}\frac{ds'}{s'-s}\Delta\left\{(2\alpha+1)\beta(s')\left[P_\alpha(z_{s't})-\left(\frac{s-4}{s'-4}\right)^\alpha P_\alpha(z_{st})\right]\right\}_{\alpha=\alpha(s')}$$
$$+\int_4^{s_1}\frac{ds'}{s'-s}\Delta\left\{(2\alpha+1)\beta(s')\left(\frac{s-4}{s'-4}\right)^\alpha P_\alpha(z_{st})\right\}_{\alpha=\alpha(s')}. \tag{3.17}$$

The contributions of the three integrals in (3.17) to c_2 will be called d_2,d_3,d_4.

We evaluate the t integral in d_4 with the help of (I2.32), for $\mathrm{Re}\,l>\alpha_m$. We then combine d_2+d_4 with c_4 from (2.38) to obtain

$$d_5(l,s)=\frac{1}{\pi}\int_4^{s_1}\frac{ds'}{s'-s}\Delta\left[(2\alpha+1)\beta(s')\left\{\frac{1}{\alpha-l}\frac{1}{\alpha+l+1}\left[\left(\frac{p(s)}{s-4}\right)^l\left(\frac{s-4}{s'-4}\right)^\alpha W(l,\alpha,z_{s\tau(s_1)})-\left(\frac{p(s')}{s'-4}\right)^l W(l,\alpha,z_{s'\tau(s')})\right]\right.\right.$$
$$\left.\left.+\frac{2p(s)^l}{(s-4)^{l+1}}\int_{\tau(s')}^{\tau(s_1)}dt\,Q_l(z_{st})P_\alpha(z_{s't})\right\}\right]_{\alpha=\alpha(s')}. \tag{3.18}$$

The integrand of d_5 is regular at $s=s'$, as one sees by evaluating its third term at that point through the use of (I2.32). The expression (3.18) was obtained for $\mathrm{Re}\,l>\alpha_m$, but it may be continued analytically into the entire half plane $\mathrm{Re}\,l>-\epsilon$. Because of the denominator $\alpha(s')-l$, it might seem at first sight that d_5 would have a cut in the l plane following the trajectories $\alpha(s_+)$, frpm $l=\alpha(4)$ to $l=\alpha(s_{1\pm})$. Actually, d_5 has no such cut, as may be verified by moving the s' contour in the manner of our treatment of d_1. We prefer not to move the contour, however, since we shall find later another term to add to d_5, and the sum of the two will be manifestly analytic for $\mathrm{Re}\,l>-\epsilon$.

The remaining term d_3 from (3.17) is subjected to a contour deformation such as we used for d_1. The result is

$$d_3(l,s)=\frac{p(s)^l}{\pi i(s-4)^{l+1}}\int_{\tau(s_1)}^\infty dt\,Q_l(z_{st})\left\{\int_{\omega(\Gamma)}\frac{ds'}{s'-s}\frac{(2\alpha+1)\tilde\beta(s')}{p^\alpha(s')}\left[(s'-4)^\alpha P_\alpha(z_{s't})-(s-4)^\alpha P_\alpha(z_{st})\right]\right.$$
$$\left.+\frac{4i}{\pi}\int_{-s_0}^4\frac{ds'}{s'-s}(2\alpha+1)\tilde\beta(s')\left[\frac{4-s'}{p(s')}\right]^\alpha\sin^2\pi\alpha\,Q_\alpha(-z_{s't})\right\}_{\alpha=\alpha(s')}. \tag{3.19}$$

Here we require that $s_0<12$, so that $-z_{s't}>1$.

It is now clear that the t integral in (3.19) converges absolutely for $\mathrm{Re}\,l\ge-\epsilon$, if we make an appropriate choice of ϵ and s_0. The two terms involving P_α cancel at large t, so that the expression in square brackets behaves as $t^{\alpha-1}$. The corresponding part of the t integral converges absolutely for $\mathrm{Re}\,l\ge-\epsilon$, provided $\alpha(-s_0)<1-\epsilon$. Since $\epsilon>0$ may be chosen to be arbitrarily small, and since $\alpha(0)\le 1$ is required by unitarity, our assumption of monotonicity of $\alpha(s)$ for $s<4$ ensures that we can always meet the condition $\alpha(-s_0)<1-\epsilon$. The term in Q_α is even less restrictive regarding α, and the corresponding t integral converges absolutely.

The first term in d_5 has a factor $s^{\alpha(s')}$, which looks dangerous at large s. A contour deformation again shows that there is actually no difficulty. The integrand has no cut for $0\le s'\le 4$, so the deformed integral has only a contribution from the path $\omega(\Gamma)$. The deformation should be done only for large s, since it obscures the situation at $s=s'$.

The other term in c_2, from the $s'-u$ denominator of (2.31), may be treated in much the same way. The analysis is easier, since we need not subtract and add a term analogous to d_4 as we did in the decomposition (3.17). The subtraction was made to secure the convergence of the t integral in the first term of (3.19). The denominator $s'-u=s'+s+t-4$ contains a t, which gives convergence of the t integral without subtraction.

The integral defining the function Σ_2 of (2.36), which appears in c_5, does not converge absolutely if Re$l=-\epsilon$. As was noticed in part I, Eq. (I3.10), the integral may be decomposed into an absolutely convergent integral and the known integral (I2.32):

$$\Sigma_2=2\int_{\tau(s')}^{\infty}dt\,P_{l'}(z_{s't})\left\{\frac{p(s)^l}{(s-4)^{l+1}}Q_l(z_{st})-\frac{p(s)^l}{(s'-4)^{l+1}}Q_l(z_{s't})+\frac{1}{(s'-4)^{l+1}}[p(s)^l-p(s')^l]Q_l(z_{s't})\right\}$$
$$=2\pi i\left[\frac{p(s)}{s-4}\right]^l\Lambda_2(l,l',s,s')+\frac{p(s)^l-p(s')^l}{(s'-4)^l}\frac{1}{l'-l}\frac{1}{l'+l+1}W(l,l',z_{s'\tau(s')}). \tag{3.20}$$

When the l' integral of (2.39) is carried out, residues being evaluated with the help of (I2.33) and (2.22), we find that

$$c_5(l,s)=\left[\frac{p(s)}{s-4}\right]^l\frac{1}{\pi}\int_4^{\infty}\frac{ds'}{s'-s}\int_{-\epsilon}dl'\zeta(l',s')\Lambda_2(l,l',s,s')+\frac{1}{\pi}\int_4^{\infty}ds'\frac{p(s)^l-p(s')^l}{s-s'}\frac{\zeta(l,s')}{(2l+1)(s'-4)^l}$$
$$-\frac{1}{\pi}\int_4^{s_1}\frac{ds'}{s'-s}\left[\frac{p(s)^l-p(s')^l}{(s'-4)^l}\right]\Delta\left[\frac{(2\alpha+1)\beta(s')}{(\alpha-l)(\alpha+l+1)}W(l,\alpha,z_{s'\tau(s')})\right]_{\alpha=\alpha(s')}. \tag{3.21}$$

Now c_5 is expressed in terms of integrals which converge absolutely at Re$l=-\epsilon$. This expression is convenient only for small Rel, say Re$l<L_0$, where L_0 is the number defined after (2.6). At large Rel the original form (2.39) is better suited for demonstrating the bounds (3.12).

For small Rel, the denominators $l-\alpha(s_\pm')$ in the third term of (3.21) may vanish, so one must be careful to get the proper continuation in l through the Regge trajectories $\alpha(s_\pm')$. If we add this third term to d_5, however, we see that the sum is obviously analytic for Re$l>-\epsilon$. Because the W's are equal to -1 at $l=\alpha(s')$, the numerator of the sum has a zero at that point. Thus, it is convenient to combine d_5 with the third term of (3.21) for Re$l<L_0$, but to leave c_5 in its original form (2.39) for Re$l>L_0$. The second term in (3.21) is analytic in l if α is restricted appropriately; see footnote 13.

This completes our discussion of large-s and large-t behavior. The final formulas for A and B, and a summary of notations, will be found in Sec. V.

The second and third terms of d_5 have a questionable appearance regarding integrability at $s'=4$. For instance, if one applies the definition (I2.32) of W and the known asymptotic behaviors of Legendre functions it appears that the second term in the integrand of (3.18) behaves as $(s'-4)^{l-\alpha(s')}$ at $s'=4$. Since $\alpha(4)$ is allowed to be large, this could spoil convergence of the s' integral at $s'=4$. On closer examination we find that the actual behavior is no worse than $(s'-4)^{\mathrm{Re}l+1/2}|\ln(s'-4)|$, because the terms in $\alpha(s_+')$ cancel those in $\alpha(s_-')$. To demonstrate the cancellation, one invokes the bounds

$$|\alpha(s_+)-\alpha(s_-)|,\ |\tilde\beta(s_+)-\tilde\beta(s_-)|\ |\ln(s-4)|^{-1}\le\kappa(s-4)^{\mathrm{Re}\alpha(s_+)+1/2}. \tag{3.22}$$

The inequalities (3.22) hold when $(1-\hat\eta,B)\in U$, as may be seen by appealing to the definitions (2.10) and (2.18) of α and $\tilde\beta$. For example, to bound $\Delta\alpha$, take the first terms in the Taylor expansion of the difference

$$D(\alpha(s_+),s_+)-D(\alpha(s_-),s_-)=0, \tag{3.23}$$

and note the assumption (i), Eq. (2.11). One may show that the higher terms in the Taylor expansion are negligible for s near 4, by applying the contraction mapping theorem.

In the remainder of this section, we shall indicate the main steps in the demonstration that the operator G maps the subspace U into T. We must show that $(\hat\eta',B')=(A[a;v],B[a;v])$ satisfies inequalities (3.12) and (3.13), when A and B are evaluated at a partial wave $a(l,s)$ which is constructed through the N/D equation from $(1-\hat\eta,B)\in U$. The demonstration depends on the behavior of Q_l at large l_+. The Laplace representation of Q_l (Ref. 12, formula 8.822.2) gives

$$Q_l(z)=[z+(z^2-1)^{1/2}]^{-l-1}R_l(x), \tag{3.24}$$

$$R_l(x)=\int_0^{\infty}[1+x(\cosh\theta-1)]^{-l-1}d\theta, \tag{3.25}$$

$$x=\frac{(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}, \tag{3.26}$$

where $z>1$ and $0<x<\frac{1}{2}$. An essential property of R_l, not shared by Q_l, is that it has the same bound that its derivatives have at large l_+ and fixed x.

The following bounds are proved in Appendix A of Ref. I21:

$$|R_l(x)|\,,\ |xR_l'(x)|\,,\ |x^2R_l''(x)| \le \kappa(l_+x)^{-1/2}, \tag{3.27}$$

$$|R_l(x)| \le \kappa\ln(1/x),\ |xR_l'(x)|\,,\ |x^2R_l''(x)| \le \kappa. \tag{3.28}$$

These results hold uniformly for $\mathrm{Re}\,l \ge -1+\delta$, any $\delta>0$, and $0<x\le\frac{1}{2}$. It is convenient to state (3.24) in terms of the variable u of (3.15):

$$(s-4)^{-l-1}Q_l(z_{st}) = u(s-4,t)^{-l-1}R_l(x), \tag{3.29}$$

$$x = \frac{2[t(t+s-4)]^{1/2}}{u(s-4,t)}, \tag{3.30}$$

$$u(s-4,t) = [t^{1/2}+(t+s-4)^{1/2}]^2. \tag{3.31}$$

In proving the bounds (3.12), (3.13), there are two main points that require attention. First, many of the terms in A and B entail a factor $p(s)^l$ or $p(s')^l$. To ensure proper behavior at large s and large $\mathrm{Re}\,l$, these factors are compensated by factors from Q_l functions. The required factor from Q_l is u^{-l} in (3.29); indeed,

$$p_0(s) = u(s-4,4). \tag{3.32}$$

Second, we must get a factor $l_+^{-3/2}$ in the bounds of B' and $1-\hat{\eta}'$, whereas only $l_+^{-1/2}$ comes directly from Q_l. The extra factor of l_+^{-1} has to be obtained by partial integration. The necessity of partial integration is the source of requirements on derivatives in the definition of our function space T.

To show the general pattern of the partial integration, we quote lemmas on an arbitrary integral of the form

$$\phi(l,s) = \frac{p(s)^l}{(s-4)^{l+1}}\int_{t_1(s)}^{t_2} dt\, Q_l(z_{st})\psi(t,s), \tag{3.33}$$

where $t_1(s)\ge 4$, $t_2>t_1$ may be finite or infinite, and the partial derivatives $\psi_1(t,s), \psi_2(t,s)$ are continuous. To estimate ϕ, notice that

$$u(s-4,t)^{-l-1} = -\frac{[t(t+s-4)]^{1/2}}{l+1}\,\frac{\partial u(s-4,t)^{-l-1}}{\partial t}. \tag{3.34}$$

We substitute (3.34) and (3.29) in (3.33), and do a partial integration in which $\partial u^{-l-1}/\partial t$ is integrated, and its coefficient differentiated. After some calculation and an application of the bounds (3.27) one finds the following:

Lemma 1:

$$\left|\frac{\phi(l,s)}{p(s)^l}\right| < \frac{\kappa}{l_+^{3/2}}\left[u(s-4,t)^{-\mathrm{Re}\,l}\left(\frac{t}{s+t}\right)^{1/4}|\psi(t,s)|\right]_{t_1,t_2} + \frac{\kappa}{l_+^{3/2}}\int_{t_1}^{t_2} dt\,\frac{u(s-4,t)^{-\mathrm{Re}\,l}}{(s+t)^{1/4}}[t^{-3/4}|\psi(t,s)|+t^{1/4}|\psi_1(t,s)|]. \tag{3.35}$$

In a similar way we obtain a result for the s derivative,

Lemma 2:

$$\left|\partial_s\left[\frac{\phi(l,s)}{p(s)^l}\right]\right| < \frac{\kappa}{l_+^{1/2}}\left[u(s-4,t)^{-\mathrm{Re}\,l}\,\frac{t^{1/4}}{s+t}|\psi(t,s)|\right]_{t_1,t_2} + \frac{\kappa}{l_+^{1/2}}u(s-4,t_1)^{-\mathrm{Re}\,l}\,\frac{|t_1'(s)\psi(t_1,s)|}{t_1^{1/4}(s+t_1)^{3/4}}$$
$$+\frac{\kappa}{l_+^{1/2}}\int_{t_1}^{t_2}\frac{dt\,u(s-4,t)^{-\mathrm{Re}\,l}}{t^{3/4}(s+t)^{5/4}}[\,|\psi(t,s)|+t\,|\psi_1(t,s)|+t^{1/2}(s+t)^{1/2}|\psi_2(t,s)|\,]. \tag{3.36}$$

Several of the terms in B' and $\hat{\eta}'$ have exactly the form (3.33), and for those terms the lemmas lead directly to bounds such as (3.12), (3.13). Other terms have slightly different forms, but variations on the theme of the lemmas lead to the required bounds. Full details will be reported elsewhere; here we shall mention only some of the salient points of the analysis.

(a) The first two terms of (3.18) are not of the form considered in the lemmas, but no partial integration is required for these terms; there are already extra powers of l in the denominator $(\alpha-l)(\alpha+l+1)$.

(b) The method of the lemmas must be elaborated to study c_5, as given in (2.39), at large $\mathrm{Re}\,l$ and large s. A closely similar term was analyzed in Ref. I21, Appendix C. Only a small part of the work of Appendix C is needed, however, since our partial waves decrease more rapidly at large l_+ than those considered there; the bounds (C8) and (C21) are relevant in the present case. The second term in Σ_2, defined in (2.36), is majorized as follows at large $\mathrm{Re}\,l$; we have

$$\left|\frac{p(s')^l}{(s'-4)^{l+1}}Q_l(z_{s't})\right| \le \left[\frac{p(s')}{u(s'-4,\tau(s'))}\right]^{\mathrm{Re}\,l}|R_l(x')|\,. \tag{3.37}$$

A calculation shows that

$$\frac{p(s')}{u(s'-4,\tau(s'))} = \frac{(s'-4)p(s')}{p_0^2(s')}\,. \tag{3.38}$$

A numerical evaluation of the function (3.38) shows that it is less than 1 for all s' provided that $a^2 < a_{\max}{}^2 \approx 50.5$. Henceforth, we shall take $a^2 = 50$. Now note that $p(s)/p_0(s)$ has it minimum value of 1 at $s=\infty$. Hence, at all s, s',

$$\frac{(s'-4)p(s')}{p_0{}^2(s')} \leqslant \frac{p(s)}{p_0(s)}\,. \tag{3.39}$$

The required factor $[p(s)/p_0(s)]^{\mathrm{Re}\,l}$ in the bound of Σ_2 is obtained from (3.39).

(c) Similar considerations of monotonicity lead to the required bound at large Rel of c_7, defined in (2.44). The integrand of c_7 obeys the following inequality, as a direct consequence of (3.13):

$$\frac{[1-\hat{\eta}(l,s)]^2}{r(l,s)} \leqslant \frac{\kappa|h(s)|s^{2\gamma}}{l_+{}^3}\left[\frac{(s-4)p(s)}{p_1{}^2(s)}\right]^{\mathrm{Re}\,l}. \tag{3.40}$$

The function

$$(s-4)p(s)/p_1{}^2(s) \tag{3.41}$$

is less than 1 for $a^2 = 50$. Consequently,

$$\frac{(s'-4)p(s')}{p_1{}^2(s')} \leqslant \frac{p(s)}{p_0(s)}\,, \tag{3.42}$$

for all s, s'; it follows that our bound for c_7 contains an appropriate factor $[p(s)/p_0(s)]^{\mathrm{Re}\,l}$. One would like to remove the restriction on a^2, but the outlook for doing so is not encouraging; see footnote 14.

(d) The discussion of the terms involving A_t^{el} (namely c_6 in B, and Δc_1 in A) is simplified by going back to our original representation (2.7) of A_t^{el}. Because of the cutoff, the modified representation in which Regge poles are exhibited is not needed for definition of the mapping G. It is only needed to demonstrate finally that the partial waves of the solution are bounded at large s and physical l. We actually prefer the modified form for numerical calculations, but for derivation of bounds the form (2.7) is better. From (2.7), (2.43), and (2.50), the part of c_6 of interest is $\phi_1+\phi_2$, with

$$\phi_1(l,s) = \frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st}) \sum_{l'=0}^{L} \zeta(l',t) P_{l'}^{(e)}(z_{ts})\,, \tag{3.43}$$

$$\phi_2(l,s) = \frac{2ip(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st}) \times \int_{L_0} dl'\,\zeta(l',t)\left[\frac{1+\cos\pi l'}{2\sin\pi l'}P_{l'}(z_{ts}) - \frac{1}{\pi}Q_{l'}(z_{ts})\right]. \tag{3.44}$$

Similarly, the term that contributes to A is

$$\Delta c_1(l,s) = \phi_3(l,s) = \frac{p(s)^l}{\pi i(s-4)^{l+1}}\int_{\sigma(s)}^\infty dt\, Q_l(z_{st})\int_{L_0} dl'\,\zeta(l',t)P_{l'}(z_{ts})\,. \tag{3.45}$$

We apply lemmas 1 and 2 to estimate $\phi_{1,2,3}$ and their derivatives. To that end, we note that all Regge poles lie to the left of Re$l = L_0$, so that the partial waves which appear through ζ in ϕ_2 and ϕ_3 obey inequalities similar to (3.12); namely, for Re$l = L_0$,

$$l_+|a(l,t_+)|\,,\ t|\partial_t a(l,t_+)| \leqslant \frac{\kappa t^\gamma}{l_+{}^{1/2}}\left[\frac{t-4}{p_0(t)}\right]^{\mathrm{Re}\,l}. \tag{3.46}$$

In order to majorize the functions ψ, ψ_1, ψ_2 that appear in the lemmas, we employ (3.46) and the following bounds of Legendre functions[15]:

$$\begin{aligned}
&|P_l(z)| \leqslant \kappa(l_+x)^{-\delta}\, y^{\mathrm{Re}\,l}\,,\\
&|P_l'(z)| \leqslant \kappa l_+{}^{1-\delta}x^{-\delta}(z^2-1)^{-1/2}y^{\mathrm{Re}\,l}\,,\\
&|Q_l(z)| \leqslant \kappa(l_+x)^{-\delta}[\ln(1/x)]^{1-2\delta}y^{-\mathrm{Re}\,l-1}\,,\\
&|Q_l'(z)| < \kappa l_+{}^{1/2}(z^2-1)^{-1}y^{-\mathrm{Re}\,l}\,,\\
&z>1,\quad x=\frac{(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}},\quad y=z+(z^2-1)^{1/2},\\
&0\leqslant\delta\leqslant\tfrac{1}{2},\quad \mathrm{Re}\,l \geqslant -\epsilon > -\tfrac{1}{2}\,.
\end{aligned} \tag{3.47}$$

After some calculation and easy estimates, one sees that the lemmas yield the inequalities

$$l_+|\phi_{2,3}(l,s)|\,,\ s|\partial_s\phi_{2,3}(l,s)| \leqslant \frac{\kappa s^{L_0-1/4}}{l_+{}^{1/2}}\left[\frac{p(s)}{p_0(s)}\right]^{\mathrm{Re}\,l} \tag{3.48}$$

provided that the cutoff $h(t)$ satisfies suitable conditions. For the latter it is sufficient to assume that h has a continuous derivative and that

$$|h(t)|\,,\ t|h'(t)| \leqslant \kappa t^{-2\gamma-1}\,. \tag{3.49}$$

To estimate ϕ_1 we note that the bounds (3.46) are valid for the partial waves of even integer l that appear in ϕ_1, with the possible exception of the s wave near $t=4$. That is, the inequalities (3.46) can fail only if Regge trajectories pass through the integers in question. Since Im$\alpha(t_\pm)$ $\neq 0$ for $t>4$, and since one may argue, as we do in Sec. IV that $\alpha(4)<2$, the only possible occurrence of a Regge pole is in $a(0,t)$ at $t=4$. As usual, such a pole is eliminated by the subtraction technique of Sec. IV, or by making $\beta(4)=0$. In either event we have no difficulty from a pole of $a(0,t)$, and we find easily that ϕ_1 obeys the same inequalities (3.48) as ϕ_2 and ϕ_3.

Of all our estimates of terms in A and B, the estimates of ϕ_1, ϕ_2, and ϕ_3 are the largest. The latter then determine the choice of exponent γ in (3.12) and (3.13). From (3.48) it is seen that a suitable choice is

$$\gamma = L_0 - \tfrac{1}{4}. \tag{3.50}$$

(e) One must give attention to the definition of Watson-Sommerfeld integrals at the energy s_1 at which Regge poles cross the path of integration, $\mathrm{Re}\,l' = -\epsilon$. This problem occurs in connection with c_3, as defined in (2.29) and (2.32), and c_5, defined in (2.39) and (2.36). The l' integrals in question have contributions from Regge poles of the form

$$\int_{-\epsilon} \frac{dl' f(l',s')}{l' - \alpha(s'_+)}, \tag{3.51}$$

for $4 \leq s' < s_1$. As s' tends to s_1 from below, this integral is defined by the Plemelj formula,

$$\mathrm{P}\int_{-\epsilon} \frac{dl' f(l',s_1)}{l' - \alpha(s_{1+})} + \pi i f(\alpha(s_{1+}), s_1). \tag{3.52}$$

For $s' > s_1$, there is no Regge pole in the region $\mathrm{Re}\,l' > -\epsilon$, and the Watson-Sommerfeld integrand is analytic in l' for $\mathrm{Re}\,l' > -\epsilon$. In this case, the path of integration may be translated a small distance to the right without changing the integral. The limit of the translated integral (as s' tends to s_1 from above) clearly exists, since no Regge pole touches the integration path. Thus, the limits from both sides of s_1 are defined, but one should keep in mind that the two limits are not equal, in general. It is easy to check that the s' discontinuity does not spoil continuity in s of the s derivatives of c_3 and c_5. The existence of the Plemelj limit (3.52) is ensured if $f(l, \omega(\alpha))$ is Hölder-continuous in l and α. Appropriate continuity can be provided, in that it can be incorporated in the full definition of the space T.

(f) We have now touched on all major problems connected with regularization of the N/D input functionals, A and B. Upon returning to the N/D equation itself, Eq. (I2.50), one sees that there are no further problems of regularization, thanks to the cutoff which appears in the factor r. When the N/D equation is regarded as a linear integral equation for n with given $(\hat{\eta}, B)$, it is a regular Fredholm equation on a Banach space V of functions $n(l,s)$ which obey the same bound as B [viz. the first of the inequalities (3.12)]. Barring unit eigenvalues of the kernel, there will be a unique solution n in V.

IV. EQUATION FOR THE PHYSICAL s-WAVE AND GHOST ELIMINATION

In our model of neutral isoscalar mesons, the physical partial waves of odd l vanish, so that ghost poles at integer l in the even-signature Froissart-Gribov amplitude $a(l,s)$ can cause a difficulty only for $l=0,2,4,\ldots$. [The amplitude $a(l,s)$ is not defined for negative integer l, and is not needed for a construction of the complete amplitude $A(s,t)$ valid at all s,t]. Furthermore, we can rule out ghosts with $l \geq 2$ by the Jin-Martin theorem,[16] which asserts that at most two subtractions are needed in a fixed-t dispersion relation for $0<t<4$. The conditions of the theorem are met in our scheme, if we succeed in enforcing unitarity to the extent of making absorptive parts positive. Of course, we aspire to enforcing the even stronger condition of inelastic unitarity $[0<\eta(l,s) \leq 1]$ by constraints on inputs. Thus, the Jin-Martin theorem and Regge asymptotic behavior of $A(s,t)$ at large s imply $\alpha(t)<2$ for $0<t<4$, and we see that $\alpha(t_*)=2,4,\ldots$ is impossible with $t_*<4$. If our theory were extended to account for pion isospin, we would have to be concerned about ghosts at $l=1$ in the odd-signature amplitude. If the theory is realisitic there will be no ghost at $l=1$, however, since the spectrum of π-π resonances and various Regge-pole fits of data show that odd-signature trajectories are well below $l=1$ at $t=4$.

We propose to treat the remaining problem of a ghost at $l=0$ by making the physical s wave $a_0(s)$ different from $a(0,s)$, the l-analytic amplitude evaluated at $l=0$. The amplitude $a(l,s)$ will still be defined, and will form an essential part of the theory, for $\mathrm{Re}\,l \geq -\epsilon$. We allow a bound-state pole to occur in $a(0,s)$ at $s=s_*<4$, but this pole does not appear in $A(s,t)$. The latter is constructed so that it has $a_0(s)$ as its s wave, but $a(l,s)$ as its partial wave for $l=2,4,\ldots$. We expect that a pole of $a(0,s)$ will in fact enter at some critical value of the coupling as the latter is increased from an initial small value. In order that a trajectory go far enough to the right in the l plane to produce a resonance of nonzero spin ($l \geq 2$ in our model of neutral mesons) it must pass through zero at a point below threshold.

We introduce a separate N/D equation for $a_0(s)$. Since this equation has the same left-cut term as that for $a(0,s)$, it is coupled to the l-analytic system. The s-wave equation will have some new input quantities: for instance, a subtraction term, or CDD poles, or both. It may have either its own inelastic function $\hat{\eta}_0(s)$, which can be chosen freely, or else may have the same inelastic function $\hat{\eta}(0,s)$ that appears in the l-analytic system. To choose the inputs to our scheme in the simplest possible way, we would take only a subtraction term in the s-wave equation, put $\hat{\eta}_0(s) = \hat{\eta}(0,s)$, and set the central spectral function $v(s,t)$ equal to zero. The subtraction term alone provides a meson-meson interaction, reminiscent of that in a $\lambda\phi^4$ field theory, which should lead to nontrivial amplitudes. As we shall see presently, the subtraction term induces an attractive force in the l-analytic N/D system, at least when the subtraction constant is sufficiently small, and this force has nearly the same effect as a superposition of attractive Yukawa potentials. If this force remains attractive at large values of the subtraction constant, one can expect the theory to have interesting Regge trajectories. To extend this simplest model so as to resemble a field theory having both meson-baryon and meson-meson interactions, one would add a central spectral function $v(s,t)$ corresponding to meson-meson scattering through a box diagram, the box being formed by a baryon loop.

A procedure of writing a separate N/D equation for the s wave in a crossing-symmetric scheme was introduced in Ref. I18. Our present method will be slightly different, in that we make one subtraction in the Cauchy representation of the s-wave denominator function $D_0(s)$. The subtraction point is $\hat{s}$, with $0<\hat{s}<4$, and the N/D equations is an integral equation for the function

$$\phi(s)=\frac{n_0(s)}{s-\hat{s}}=\frac{-\mathrm{Im}D_0(s_+)}{r_0(s)(s-\hat{s})}, \tag{4.1}$$

where

$$r_0(s)=\left(\frac{s-4}{s}\right)^{1/2}h_0(s). \tag{4.2}$$

Here $h_0(s)$ is a cutoff function, equal to 1 for $s\le 16$, which need not be the same as our previous cutoff $h(s)$. The denominator function is allowed to have n CDD poles at the points $s_i \ge 4$, with residues $c_i(s_i-s)$, and also a pole at infinity with residue c. It is represented as

$$D_0(s)=1+(s-\hat{s})\left[c-\sum_{i=1}^{n}\frac{c_i}{s_i-s} -\frac{1}{\pi}\int_4^\infty \frac{r_0(s')\phi(s')ds'}{s'-s}\right]. \tag{4.3}$$

We suppose for the moment that the inelastic function and cutoff are the same as in the l-analytic system at $l=0$, since there is a complication in the arbitrary choice of $\hat{\eta}_0(s)$ which is best discussed later. With the s-wave force function denoted by B_0, we then have

$$\hat{\eta}_0(s)=\hat{\eta}(0,s)\,,\quad B_0(s)=B(0,s)\,,\quad h_0(s)=h(s)\,, \tag{4.4}$$

and the N/D equation is

$$\hat{\eta}_0(s)\phi(s)=\frac{\lambda_0+B_0(s)}{s-\hat{s}}+cB_0(s)+\sum_{i=1}^{n}c_i\frac{B_0(s_i)-B_0(s)}{s_i-s} +\frac{1}{\pi}\int_4^\infty\frac{B_0(s)-B_0(s')}{s-s'}r_0(s')\phi(s')ds'\,. \tag{4.5}$$

The real constant λ_0 is the subtraction constant mentioned above. The scattering amplitude $a_0(s_\pm)$ is constructed from $\hat{\eta}_0$, B_0, and the solution ϕ of the N/D equation by means of the formula

$$a_0(s_\pm)=B_0(s)\pm i\,\frac{1-\hat{\eta}_0(s)}{2r_0(s)} +\frac{s-\hat{s}}{D_0(s)}\left[\frac{\lambda_0}{s-\hat{s}}+\sum_{i=1}^{n}c_i\frac{B_0(s_i)}{s_i-s} +\frac{1}{\pi}\int_4^\infty\frac{B_0(s')r_0(s')\phi(s')ds'}{s'-s_\pm}\right]. \tag{4.6}$$

By a calculation analogous to that of (I3.12), one verifies that (4.3)–(4.6) imply

$$\frac{1}{2i}\left[a_0(s_+)-a_0(s_-)\right]=r_0(s)a_0(s_+)a_0(s_-)+\frac{1-\hat{\eta}_0(s)^2}{4r_0(s)}\,. \tag{4.7}$$

The amplitude $a_0(s)$ is influenced by the l-analytic N/D system by having at least the same left-cut part as $a(0,s)$. Correspondingly, the l-analytic system must be influenced by a_0 if we are to have a complete crossing-symmetric scheme. The required modification of the l-analytic system is accomplished by replacing its s-wave absorptive part by the absorptive part of a_0. If A_t is the original complete absorptive part as given in (2.9), then the modified one is

$$\tilde{A}_t(s,t)=A_t(s,t)-\frac{1}{t-4}\int_{4-t}^{0}ds\,A_t(s,t) +r_0(t)\left|a_0(t_+)\right|^2+\frac{1-\hat{\eta}_0(t)^2}{4r_0(t)}\,. \tag{4.8}$$

If one chooses $\hat{\eta}_0(t)=\hat{\eta}(0,t)$, then only the t-channel-elastic part of A_t is affected:

$$\tilde{A}_t^{\mathrm{el}}(s,t)=r_0(t)\left|a_0(t_+)\right|^2 +\left(1-\frac{1}{t-4}\int_{4-t}^{0}ds\right)A_t^{\mathrm{el}}(s,t)\,. \tag{4.9}$$

The change $A_t\to\tilde{A}_t$ amounts to adding a function of t alone. The Froissart-Gribov transform of this function of t has only a left cut in the s plane, and resembles the Born term of a superposition of

Yukawa potentials:

$$\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st})$$
$$\times\left[r_0(t)\,|a_0(t_+)|^2+\frac{1-\hat{\eta}_0(t)^2}{4r_0(t)}-\frac{1}{t-4}\int_{4-t}^0 ds\, A_t(s,t)\right]. \tag{4.10}$$

The addition of (4.10) to $B(l,s)$ is the only change required in the l-analytic system. This term satisfies all the required bounds, if r_0 and $\hat{\eta}_0$ are chosen suitably [for instance, as in (4.4)]. Since the smallest integer l for which $a(l,t)$ occurs in $\bar{A}_t$ is now $l=2$, a bound-state pole in $a(0,t)$ cannot contribute a singularity to the N/D system.

For future reference, we note that the integral in (4.9) may be represented as follows:

$$-\frac{1}{t-4}\int_{4-t}^0 ds\, A_t^{\text{el}}(s,t)=\frac{1}{2\pi i}\int_{-\epsilon}\frac{dl\,\zeta(l,t)}{l(l+1)}$$
$$+\theta(s_1-t)\Delta\left[\frac{(2\alpha+1)\beta(t)}{\alpha(\alpha+1)}\right]_{\alpha=\alpha(t)}. \tag{4.11}$$

This result comes from (2.26) and a known integral of the Legendre function (Ref. 12, formula 7.112).

We can now justify the above remark about the attractive nature of the force produced by the subtraction term. Let us take a system in which the subtraction term, $\lambda_0/(s-\hat{s})$ in (4.5), is the only inhomogeneity (that is, c, c_i, and v are all zero), and $\hat{\eta}_0(s)=\hat{\eta}(0,s)$. Then for small λ_0, the term $|a_0|^2$ will be of order $\lambda_0{}^2$, and will produce a left-cut contribution of similar order in the l-analytic N/D equation. It follows that $a(l,s)$ will be of order $\lambda_0{}^2$, and the second and third terms in (4.10), being quadratic in $a(l,s)$, will be of order $\lambda_0{}^4$. At small λ_0 the positive first term in (4.10) dominates, and it has exactly the effect of a superposition of attractive Yukawa potentials, so far as the l-analytic N/D system is concerned.

It is necessary that $D_0(s)$ have no zero in the cut s plane. Otherwise, the putative ghost-elimination mechanism would merely trade one type of ghost for another. It seems likely that one can in fact avoid zeros of D_0, since one has a free choice of parameters (λ_0, c, c_i, s_i) that have a strong influence on the behavior of D_0. In particular, the signs of λ, c, and c_i are probably crucial for preventing zeros.

The parameters λ_0 and $\hat{s}$ are not independent. Given a solution of the equations for a given λ_0 and $\hat{s}$, one can show that there is an N/D representation of the corresponding a_0 with a denominator function subtracted at a different point $\hat{s}'$. The corresponding ϕ satisifies (4.5), with new parameters λ_0', $\hat{s}'$, but with the same $\hat{\eta}_0$ and B_0. This is an application of standard N/D theory (see Ref. I24 and I25).

The bounds on $B(0,s)$ and its derivative, obtained in Sec. III, are sufficient to show that (4.5) is a regular Fredholm equation in an appropriate Banach space. For instance, if $\alpha(0)=1$ and $c\neq 0$, the appropriate space consists of all continuous real functions ψ such that

$$\sup_{4\leq s<\infty}|\ln^2 s\,\psi(s)|=\|\psi\|<\infty, \tag{4.12}$$

whereas if $\alpha(0)=1$ and $c=0$, the space would be made up of all continuous real functions ψ such that

$$\sup_{4\leq s<\infty}|s\psi(s)|=\|\psi\|<\infty. \tag{4.13}$$

If $c\neq 0$ [and $\alpha(0)>-\epsilon$, say] we see from (4.3) and (4.6) that

$$a_0(s_\pm)\sim B_0(s)\pm i\frac{1-\hat{\eta}_0(s)}{2r_0(s)},\quad s\to\infty, \tag{4.14}$$

provided the cutoff decreases at least as rapidly as s^{-1}. The asymptote (4.14) is the same as that of $a(0,s\pm)$. If $c=0$, the amplitude $a_0(s_\pm)$ is asymptotic, in general, to a nonzero real constant.

There are potential advantages in taking $\hat{\eta}_0(s)$ and $h_0(s)$ to be different from $\hat{\eta}(0,s)$ and $h(s)$. For instance, it might happen that $\hat{\eta}(0,s)$ violated inelastic unitarity by giving a negative overlap function (I2.38). One could then try to restore unitarity with a different choice of $\hat{\eta}_0(s)$. To allow a free choice of $\hat{\eta}_0$, one would take

$$B_0(s)=c_L(0,s)+\frac{\text{P}}{\pi}\int_{16}^\infty\frac{1-\hat{\eta}_0(s')}{2r_0(s')(s'-s)}\,ds', \tag{4.15}$$

where $c_L(0,s)$ is the left-cut part of $c(0,s)$,

$$c_L(0,s)=B(0,s)-\frac{\text{P}}{\pi}\int_{16}^\infty\frac{1-\hat{\eta}(0,s')}{2r(0,s')(s'-s)}\,ds'. \tag{4.16}$$

There is a complication, however, since the prescription makes sense only if $\alpha(0)<1$. If $\alpha(0)=1$, the integral of $1-\hat{\eta}$ in (4.16) diverges, and one would have to match the asymptotic behavior of the input function $\hat{\eta}_0(s)$ to that of the dynamical variable $\hat{\eta}(0,s)$. With an appropriate match, the sum of the s' integrals in (4.15) and (4.16) could be made convergent.

V. SUMMARY OF EQUATIONS IN REGULAR FORM

It seems worthwhile to collect the basic equations of our theory. We choose the forms best suited to calculation and further analysis, and in some cases adopt more convenient notation. Let us first summarize our notation:

$$z_{st}=1+\frac{2t}{s-4}\,,\quad u=4-s-t\ ,$$

$$\sigma(s)=\frac{4s}{s-16}\,,\quad \tau(s)=\frac{16s}{s-4}\,,\quad \theta(x)=\begin{cases}1, & x\geq 0\\ 0, & x<0\end{cases}$$

$$q(s)=(1-4/s)^{1/2},\quad p(s)=[(s+50)^{1/2}+2]^2\ ,$$

$$r(l,s)=q(s)h(s)\left[\frac{s-4}{p(s)}\right]^l\ ,$$

L_0 is arbitrary and such that

$$L\leq\alpha_m<L_0<L+1=\text{integer}\ ,$$

$$\alpha_m=\max_{s\geq 4}\operatorname{Re}\alpha(s_+)\ ,$$

$$\beta(s)=\left[\frac{s-4}{p(s)}\right]^{\alpha(s)}\tilde{\beta}(s)$$

$$=[\text{residue of Regge pole in } a(l,s)]\ ,$$

$$\omega(l)=[\text{inverse of } \alpha(s)]\ ,$$

ϵ and s_0 are arbitrary and such that

$$0<\epsilon<\tfrac{1}{2},\quad -4<s_0<12,\quad \alpha(-s_0)<1-\epsilon\ .$$

The trajectory leaves (enters) the half plane $\operatorname{Re}l\geq-\epsilon$ at $s=s_1$ (s_2):

$$\operatorname{Re}\alpha(s_{1+})=-\epsilon=\alpha(s_2),\quad s_2\geq-50\ .$$

$$\Delta f(s)=\frac{1}{2i}[f(s_+)-f(s_-)]\ ,$$

$$\int_a dl\,f(l)=\left[\text{integral along } \operatorname{Re}l=a,\ \int_{a-i\infty}^{a+i\infty}dl\,f(l)\right]\ ,$$

$$\Gamma=[\text{union of paths } \Gamma_+ \text{ and } \Gamma_-,\ \text{Fig. 4(a)}]\ ,$$

$$\omega(\Gamma)=[\text{image of } \Gamma \text{ under mapping } \omega,\ \text{Fig. 4(b)}],$$

$$W(l,l',z)=(1-z^2)[Q_l(z)P_{l'}'(z)-P_{l'}(z)Q_l'(z)]\ ,$$

$$\Phi(l,l',s,s')=\frac{2}{s'-s}\int_{\tau(s')}^{\infty}dt\,P_{l'}(z_{s't})\left[\frac{p(s)^l}{(s-4)^{l+1}}Q_l(z_{st})-\frac{p(s')^l}{(s'-4)^{l+1}}\left(\frac{p(s)}{p(s')}\right)^{l\theta(L_0-\operatorname{Re}l)}Q_l(z_{s't})\right]\ ,$$

$$\Psi(l,l',s,s')=\frac{2p(s)^l}{(s-4)^{l+1}}\int_{\tau(s')}^{\infty}dt\,P_{l'}(z_{s't})Q_l(z_{st})\frac{1}{s'-u}\ ,$$

$$\zeta(l,s)=q(s)h(s)(2l+1)a(l,s_+)a(l,s_-)\ .$$

The input parameters of the scheme are the following:

$$V(l,s)=\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_{16}^{\infty}dt\,Q_l(z_{st})$$

$$\times\frac{1}{\pi}\int_{16}^{\infty}ds'v(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]\ ,$$

$$h(s)=[\text{cutoff function}]=1,\quad s\leq 16,$$

$$h(s),\ sh'(s)=O(s^{-(2L_0+1/2)}),\quad s\to\infty,$$

$$\lambda_0=[\text{subtraction constant}]\ ,$$

$$\hat{s}=[\text{subtraction point}]\ ,$$

$$[c_i,s_i]=[\text{CDD pole residue, position}]\ ,$$

$$c=[\text{residue of CDD pole at infinity}]\ ,$$

$$h_0(s)=[s\text{-wave cutoff function}]\ ,$$

$$\hat{\eta}_0(s)=[s\text{-wave inelastic function}]\ .$$

The parameters $\lambda_0,\ldots,\hat{\eta}_0$ appear only in the separate s-wave equation (Sec. IV).

We shall state the dynamical equations so as to incorporate the scheme of Sec. IV. The simple change to make the alternative scheme, in which only l-analytic partial waves are involved, will be indicated at the end. For simplicity in writing, we also suppose that $\hat{\eta}_0(s)=\hat{\eta}(0,s)$, although this is not necessary [see (4.15) ff.].

The dynamical equation is an equation to determine the function pair $\psi=(\hat{\eta},B)$:

$$\psi=G(\psi)\ . \tag{5.1}$$

The operator G is defined in four steps as follows:

(1) Given $(\hat{\eta},B)$, solve the linear integral equations for n and ϕ:

$$\hat{\eta}(l,s)n(l,s)=B(l,s)+\frac{1}{\pi}\int_4^{\infty}K(l,s,s')r(l,s')n(l,s')ds'\ , \tag{5.2}$$

$$\hat{\eta}(0,s)\phi(s)=\frac{\lambda_0+B(0,s)}{s-\hat{s}}+cB(0,s)+\sum_{i=1}^{n}c_iK(0,s,s_i)+\frac{1}{\pi}\int_4^{\infty}K(0,s,s')r(0,s')\phi(s')ds'\ , \tag{5.3}$$

where

$$K(l,s,s')=\frac{B(l,s)-B(l,s')}{s-s'}\ . \tag{5.4}$$

The solution of (5.2) is required for $s>4$, for $\mathrm{Re}l=-\epsilon$, and also for l on the trajectory $\alpha(s_+)$, $s_2<s<s_1$ [to be determined in step (3)].

(2) Determine the amplitudes $a(l,s_\pm),a_0(s_\pm)$ from the formulas

$$\left[\frac{p(s)}{s-4}\right]^l a(l,s_\pm)=B(l,s)\pm i\,\frac{1-\hat\eta(l,s)}{2r(l,s)}+\frac{1}{\pi D(l,s)}\int_4^\infty \frac{B(l,s')r(l,s')n(l,s')ds'}{s'-s_\pm}\ , \tag{5.5}$$

$$a_0(s_\pm)=B(0,s)\pm i\,\frac{1-\hat\eta(0,s)}{2r(0,s)}+\frac{s-\hat s}{D_0(s_\pm)}\left[\frac{\lambda_0}{s-\hat s}+\sum_{i=1}^{n}c_i\frac{B(0,s_i)}{s_i-s}+\frac{1}{\pi}\int_4^\infty\frac{B(0,s')r(0,s')\phi(s')ds'}{s'-s_\pm}\right]\ , \tag{5.6}$$

where

$$D(l,s)=1-\frac{1}{\pi}\int_4^\infty\frac{r(l,s')n(l,s')ds'}{s'-s}\ , \tag{5.7}$$

$$D_0(s)=1+(s-\hat s)\left[c-\sum_{i=1}^{n}\frac{c_i}{s_i-s}-\frac{1}{\pi}\int_4^\infty\frac{r(0,s')\phi(s')ds'}{s'-s}\right]\ . \tag{5.8}$$

These amplitudes are needed only for $s>4$, $\mathrm{Re}l=-\epsilon$.

(3) Compute the Regge trajectory α from the equation

$$D(\alpha(s_+),s_+)=0,\quad s_2<s<s_1 \tag{5.9}$$

and its inverse ω from

$$D(l,\omega(l))=0,\quad \mathrm{Re}l=-\epsilon,\ 0<\mathrm{Im}l<\mathrm{Im}\alpha(s_{1+})\ . \tag{5.10}$$

Thus, we find $\alpha(s)$ on the real s axis, and $\omega(l)$ on the curve Γ. Calculate $\tilde\beta(s_+)$, $s_2<s<s_1$, and $\tilde\beta(\omega(l))$, $l\in\Gamma$, from

$$\tilde\beta(s)=\left[\frac{1}{\pi D_l(l,s)}\int_4^\infty\frac{B(l,s')r(l,s')n(l,s')ds'}{s'-s}\right]_{l=\alpha(s)}. \tag{5.11}$$

(4) Determine $G(\psi)=(A,B)$ from $a,a_0,\alpha,\omega,\tilde\beta$ by by the following equations:

$$A(l,s)=[1-4r(l,s)I(l,s)]^{1/2}\ , \tag{5.12}$$

$$I(l,s)=\Delta V(l,s)+\frac{p(s)^l}{(s-4)^{l+1}}\int_4^\infty dt\,Q_l(z_{st})\left(\frac{1}{\pi i}\int_{-\epsilon}dl'\zeta(l',t)P_{l'}(z_{ts})+2\theta(s_1-t)\Delta\{[2\alpha(t)+1]\beta(t)P_{\alpha(t)}(z_{ts})\}\right)\cdot \tag{5.13}$$

$$B(l,s)=\sum_{i=1}^{6}B_i(l,s)\ , \tag{5.14}$$

$$B_1(l,s)=\tfrac{1}{2}[V(l,s_+)+V(l,s_-)]\ , \tag{5.15}$$

$$B_2(l,s)=\frac{\mathrm{P}}{\pi}\int_{16}^\infty\frac{[1-A(l,s')]^2ds'}{4r(l,s')(s'-s)}\ , \tag{5.16}$$

$$B_3(l,s)=\frac{4p(s)^l}{\pi(s-4)^{l+1}}\times\int_4^\infty dt\,Q_l(z_{st})\left\{r(0,t)|a_0(t_+)|^2-\pi\theta(s_1-t)\Delta\left[(2\alpha+1)\beta(t)\left(\frac{1+\cos\pi\alpha}{2\sin\pi\alpha}P_\alpha(z_{ts})-\frac{1}{\pi}Q_\alpha(z_{ts})-\frac{1}{\pi\alpha(\alpha+1)}\right)\right]_{\alpha=\alpha(t)}+\frac{i}{2}\int_{-\epsilon}dl'\zeta(l',t)\left(\frac{1+\cos\pi l'}{2\sin\pi l'}P_{l'}(z_{ts})-\frac{1}{\pi}Q_{l'}(z_{ts})-\frac{1}{\pi l'(l'+1)}\right)\right\}\ , \tag{5.17}$$

$$B_4(l,s)=\frac{1}{\pi}\int_4^\infty ds'\frac{1}{2\pi i}\int_{-\epsilon}dl'\zeta(l',s')[\Phi(l,l',s,s')+\Psi(l,l',s,s')]+\theta(L_0-\mathrm{Re}l)\frac{1}{\pi}\int_4^\infty\frac{ds'}{(s'-4)^l}\left[\frac{p(s)^l-p(s')^l}{s-s'}\right]\frac{\zeta(l,s')}{2l+1}\ , \tag{5.18}$$

$$B_5(l,s)=\frac{p(s)^l}{\pi i(s-4)^{l+1}}$$
$$\times\int_{\tau(s_1)}^{\infty}dt\,Q_l(z_{st})\left\{\int_{\omega(\Gamma)}ds'(2\alpha+1)\left(\frac{1}{s'-s}\frac{\tilde\beta(s')}{p(s')^{\alpha}}[(s'-4)^{\alpha}P_\alpha(z_{s't})-(s-4)^{\alpha}P_\alpha(z_{st})]+\frac{\beta(s')}{s'-u}P_\alpha(z_{s't})\right)\right.$$
$$\left.+\frac{4i}{\pi}\int_{-s_0}^{4}ds'\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)(2\alpha+1)\tilde\beta(s')\left[\frac{4-s'}{p(s')}\right]^{\alpha}\sin^2\pi\alpha\,Q_\alpha(-z_{s't})\right\}_{\alpha=\alpha(s')}, \tag{5.19}$$

$$B_6(l,s)=\frac{1}{\pi}\int_4^{s_1}ds'\Delta\left\{\frac{(2\alpha+1)\beta(s')}{s'-s}\,\frac{1}{\alpha-l}\,\frac{1}{\alpha+l+1}\right.$$
$$\times\left[\left(\frac{p(s)}{s-4}\right)^{l}\left(\frac{s-4}{s'-4}\right)^{\alpha}W(l,\alpha,z_{s\tau(s_1)})-\left(\frac{p(s')}{s'-4}\right)^{l}\left(\frac{p(s)}{p(s')}\right)^{l\theta(L_0-\mathrm{Re}\,l)}W(l,\alpha,z_{s'\tau(s')})\right]$$
$$\left.+\frac{2(2\alpha+1)\beta(s')p(s)^l}{(s-4)^{l+1}}\int_{\tau(s')}^{\tau(s_1)}dt\,Q_l(z_{st})P_\alpha(z_{s't})\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)\right\}_{\alpha=\alpha(s')}. \tag{5.20}$$

In the case of weak coupling, no Regge pole enters the half plane $\mathrm{Re}\,l>-\epsilon$, and B_5, B_6, and the second term of B_3 drop out. Correspondingly, if two or more trajectories enter the half plane, these terms are understood as sums over all trajectories.

To obtain the alternative system with only l-analytic partial waves, we ignore the s-wave equations (5.3), (5.6), and drop those terms in B_3 which involve

$$|a_0|^2,\quad \frac{1}{\alpha(\alpha+1)},\quad \frac{1}{l'(l'+1)}. \tag{5.21}$$

With such a system one must somehow constrain the input $V(l,s)$ so as to guarantee that $\beta(s_*)=0$, if $\alpha(s_*)=0$ for $s_*<4$.

The scheme as described above furnishes the partial waves $a(l,s_\pm)$ for $\mathrm{Re}\,l=-\epsilon$, $s\geq 4$, and the Regge parameters $\alpha(s_\pm),\beta(s_\pm)$ for $4\leq s\leq s_1$. As we show in part III, these quantities are the ingredients of a rather direct construction of the total crossing-symmetric $A(s,t)$, valid at all s and t. Alternately, one may continue $a(l,s_\pm)$ to physical l by means of the equations given above, and then sum the Legendre series to obtain a representation of A valid in the Lehmann ellipse in the z_{st} plane. Also, one may find $a(l,s)$ in the entire cut s plane from the continuation of (5.5), the latter being given by Eqs. (2.51) and (2.46).

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[9]For instance, the equation $\mathrm{Re}\,\alpha(s)=-\epsilon$ might define a continuous but nonsimple curve in the s plane; i.e., a curve that intersected itself. In that case, the boundary $\partial\Omega$ would have kinks at a point of self-intersection of the above-mentioned curve. Since $\mathrm{Im}\,\alpha$

would be discontinuous at the kinks, our later procedure of distorting contours to follow $\partial\Omega$ would be in trouble.

[10]E. Hille, *Analytic Function Theory* (Chelsea, New York, 1973), Vol. II.

[11]C. Lovelace and D. Masson, Nuovo Cimento 26, 472 (1962).

[12]I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965), formula 8.833.

[13]The second term in (3.21) is analytic for $\mathrm{Re}\,l > -\epsilon$, unless $\alpha(-a^2) > -\epsilon$, in which case there is a cut from $\alpha(-a^2)$ to $-\epsilon$. The factor $\zeta(l, s')$ has Regge poles at $l = \alpha(s'_\pm)$. Continuation in l of the integral through the trajectories gives the original integral plus a term which has a factor $p[\omega(l)]^l$. Because of the cut of $p(s)$ beginning at $-a^2$, that factor has an l-plane cut $[-\epsilon, \alpha(-a^2)]$, if $\alpha(-a^2) > -\epsilon$. Since we demand that $c_5(l, s)$ be analytic for $\mathrm{Re}\,l > -\epsilon$, we must require that $s_2 > -a^2$. We shall find presently that there is a limit to the magnitude of a^2, imposed by our technique of handling large-l behavior. Thus, α is restricted: we cannot allow $\alpha(-a_{\max}^2)$ to be larger than $-\epsilon$. This restriction is probably not intrinsic to the theory, but we have been unable to avoid it thus far.

[14]The restriction on a^2 implied by (3.39) and (3.42) leads to the condition $\alpha(-a_{\max}^2) < 0$, which may not be met by all realistic trajectories. One might attempt to move the branch point of $p(s)$ to a point less than -50 by taking a different functional form for $p(s)$. One can show, however, that no $p(s)$ in a rather broad class is acceptable if its branch point is less than $-a_{\max}^2 \approx 89$. The class consists of functions $p(s) = p(s^*)^*$ which are (a) analytic except for the cut $(-\infty, -a^2]$, (b) are bounded uniformly by a polynomial, and (c) have spectral functions $\mathrm{Im}\,p(s_+)$ with at most a finite number of zeros. Any such function is the product of a rational function and a Herglotz function [K. Symanzik, J. Math. Phys. 1, 249 (1960); S. Weinberg, Phys. Rev. 124, 2049 (1961)]. The only rational factor that we can allow is a constant. Simple properties of Herglotz functions and the inequalities (3.39) and (3.42) then lead to the stated bound on a^2. It seems that the prospects for improving the situation by a better choice of $p(s)$ are not very good, since $\alpha(-89) < 0$ is still a stronger condition than one would like.

[15]These bounds may be derived from the Laplace integral representations of Legendre functions. See Appendix A of Ref. I21 for some of the relevant steps.

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