Dynamical equations for a Regge theory with crossing symmetry and unitarity. I. Introduction, and the case of weak coupling*

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(Received 26 July 1976)

A program for construction of a crossing-symmetric unitary Regge theory of meson-meson scattering is proposed. The construction proceeds through solution of a nonlinear functional equation, $\psi = G(\psi)$, for certain partial-wave scattering functions ψ. The functional equation is analogous to a conventional dynamical equation, in that the scattering amplitude is generated from input functions which describe the primary forces between mesons and possible inelastic effects. A solution of the equation provides a scattering amplitude having Mandelstam analyticity, exact crossing symmetry, exact unitarity below the production threshold, and meromorphy of partial waves in the right-half l plane, with the consequent Regge asymptotics. Inelastic unitarity $[0 \leq \eta(l,s) \leq 1]$ is not guaranteed, but may perhaps be achieved through constraints on inputs. In any case, the partial waves are bounded throughout the physical region; such a bound was not ensured in earlier schemes based on the Mandelstam iteration. In this first paper of a series, the equations are formulated for the case of weak coupling, in which no Regge poles enter the right-half l plane. Inelastic effects are described by crossed two-particle processes and assigned input functions. Later papers will treat the case of strong coupling, in which Regge trajectories are generated dynamically, and the extension of the formalism to include many coupled channels.

I. INTRODUCTION

We are interested in the theory of the scattering matrix as a possible setting for hadron dynamics.[1-8] The advantages of S-matrix theory as a phenomenology are well recognized, but the value of the theory as a dynamical or quasidynamical scheme has not been fully assessed. During the 1960's there were, to be sure, earnest attempts to construct an S-matrix theory from first principles. Following ideas of Mandelstam,[2] Chew,[3] Regge,[6,7] and others, one attempted to calculate the S-matrix by solving integral equations that were derived from analyticity, crossing symmetry, unitarity, and certain simplifying physical assumptions. The most ambitious schemes were calculations of π-π scattering using one form or another of the "strip model" (Refs. 3, 7, 9, 10, 11, and 12, and other papers quoted therein). In such schemes, the difficulties of the mathematics and the numerical calculations greatly obscured the subject, so that it was hard to separate any shortcomings of the physical models from mere failures of mathematical method.[13] Another disturbing point is that so much work in S-matrix theory has followed the bootstrap point of view,[3] according to which one seeks nontrivial solutions of homogeneous crossing-unitarity equations. There is no real evidence that such solutions exist, and besides, there are compelling reasons to investigate S-matrix theory from a broader perspective in which inhomogeneous equations with driving terms would be allowed. Especially, one should try to incorporate the quark model, which seems to be contrary in spirit to bootstrap theory, and which is now more convincing as a phenomenological scheme than it was in the heyday of the bootstrap philosophy.

Recent studies[14-22] carried out with the help of nonlinear functional analysis[14] have clarified the mathematical structure of S-matrix equations. It has become evident that Mandelstam's original formulation, based on a unitarity equation for the double-spectral function, is not tractable for the case of physical interest in which Regge poles lie to the right of the line Re $l=1$. We are led instead to a pure partial-wave scheme, which will be described in this series of papers. This new scheme seems to avoid the principal mathematical difficulties of earlier approaches. It can be analyzed rigorously to a great extent, and is suitable in principle for controlled numerical calculations. It should allow us to separate the purely technical questions from questions of physical content, and thereby allow a more orderly evaluation of S-matrix theory as a realistic dynamical scheme. We do not adopt the bootstrap viewpoint, but our techniques could be used as well in pursuit of a bootstrap theory.

The extraction of specific predictions of the theory will require lengthy numerical calculations, which are now in a preliminary stage. In view of the difficulty of the calculations, it is fair to ask what may be gained by carrying them out. We do not expect to arrive very soon at a realistic theory of π-π scattering, although that is one of our long-range goals. Rather, we first hope to give an instructive example of a dynamical calculation of scattering, valid at strong coupling, and meeting general requirements of crossing symmetry, analyticity, and unitarity. Since we solve an integral equation exactly, there is no dependence on perturbation theory. Regge trajectories would be generated dynamically, through the action of a sufficiently strong meson-meson force. The proposed calculation provides an interesting alternative to field theory in approaching the strong-coupling problem. The connection to field theory may eventually prove to be rather close. Indeed, in part II of this series we shall describe a scheme which looks as though it could be an approximation to $\lambda\phi^4$ field theory.

Most recent discussions of hadron dynamics have been in terms of field theory, with emphasis on quarks and their confinement, and with little attention to the scattering matrix. It is likely that our approach, with its emphasis on mesons and the scattering matrix, is not incompatible with a field theory of quarks, even if a theory such as ours, which works at the level of ordinary hadrons, would have to be regarded as semiphenomenological from the viewpoint of theories working at the quark level. At important question, which is under investigation but far from being answered, is: How does an underlying quark scheme makes itself felt in an S-matrix theory of hadrons? This is a question which both quark theorists and S-matrix theorists will eventually have to face, since contemporary quark theories usually do not predict an S matrix, while the S-matrix remains the principal observable of interest. One possibility is that the quark structure of hadrons influences the S matrix through Castillejo-Dalitz-Dyson (CDD) poles.[22–25, 18] With analyticity in l the CDD phenomenon is not ruled out, but it becomes more complicated; instead of poles in the D function, one has a more involved analytic structure. The CDD singularities are not yet fully understood, but our scheme is probably able to accommodate them. Another possibility is that quarks, being unobservable, play no overt role in equations for the observable S matrix. The equations incorporate the quantum numbers and symmetries of the quark model, without making direct reference to quarks. The realization of such a scheme would require the treatment of many coupled scattering channels. That would be a difficult task, but not necessarily an impossible one at some level of approximation. A revival of the bootstrap viewpoint, if possible at all, would require a multichannel scheme.

In the remainder of this section we shall indicate how our scheme relates to earlier work. The reader not familiar with earlier literature may find the systematic discussion beginning in Sec. II to be more readable.

First, consider the scattering of unit-mass pseudoscalar mesons without isospin in the simplest case where the Mandelstam representation has no subtractions or single-spectral terms [see Eq. (2.1) of the following section.] By taking the discontinuity of either side of the unitarity equation, Mandelstam[2] found that the double-spectral function $\rho(s,t)$ is represented as follows, for s below the production threshold:

$$\rho(s,t)=\int_4 dt_1 \int_4 dt_2 K(s,t;t_1,t_2)A_t(s_+,t)A_t(s_-,t), \qquad 4\leqslant s\leqslant 16 \tag{1.1}$$

$$A_t(s,t)=\frac{1}{\pi}\int_{\sigma(t)}^{\infty} dx\,\rho(x,t)\left(\frac{1}{x-s}+\frac{1}{x-u}\right), \qquad \sigma(t)=\frac{4t}{t-16}\,. \tag{1.2}$$

The kernel K is well known,[2, 4] and the upper limit of the double integral in (1.1) is defined by the first-quadrant zero of K^{-2}. We define $\rho^{el}(s,t)$, the "s-channel elastic part" of $\rho(s,t)$, as the right-hand side of (1.1), for *all* $s\geqslant 4$. Following Zimmermann[9] and Atkinson,[15] we write the total spectral function as

$$\rho(s,t)=\rho^{el}(s,t)+\rho^{el}(t,s)+v(s,t), \tag{1.3}$$

where $v(s,t)$ is a symmetric function which is nonzero only in a certain subset of the region with $s>16$, $t>16$. The basic premise of the strip model is that $v(s,t)$ can be neglected entirely. For want of a better name, we shall call $v(s,t)$ the "central spectral function." It corresponds to states above the production threshold in both the s and t channels. Without considering many coupled channels, one cannot determine $v(s,t)$ dynamically. In the first papers of this series we suppose provisionally that a model of $v(s,t)$ is given; $v(s,t)$ is regarded as a prescribed "driving term."

For a formulation consistent with the results of π-π phase-shift analysis, a dynamical treatment of the channels that contribute to $v(s,t)$, consistent with that of the π-π channels, will probably be necessary. Indeed, the observed high-energy behavior of phase shifts in π-π scattering suggests that other channels have an important influence on the π-π system.[26] Large positive asymptotic values of the phase shift are seen through Levinson's theorem to be associated with CDD poles,[25] and the latter presumably arise from resonant interactions in channels coupled to the π-π system (for similar remarks on the π-N system, see Ref. 31). It is possible to extend the equations we propose so as to couple and unitarize additional two-body channels, for instance, other meson-meson channels such as $K\bar{K}$ and $\pi\eta$, or baryon-antibaryon channels like $N\bar{N}$. Such an extension, to be described in a later paper, may be carried out so as to preserve crossing symmetry. In either the single-channel or the multichannel formulation, crossing symmetry demands the presence of crossed two-body processes such as are represented in the term $\rho^{el}(t, s)$ in (1.3). Thus, we have inelastic effects somewhat like those of the multiperipheral model, in both the single- and many-channel formulations. A highly realistic theory would probably require explicit treatment of many-body channels, at least in an isobar approximation. Nevertheless, it seems likely that much can be learned from a well-controlled scheme based on direct and crossed two-body processes alone. In particular, it will be interesting to see if the rising of Regge trajectories can be caused by successive openings of channels of higher and higher mass. Our system is such that trajectories $\alpha(s)$ eventually turn back into the left l plane at high s, but with enough channels included, rather large values of $\mathrm{Re}\alpha(s)$ might be attained. (We remind the reader that experimental evidence for infinitely rising trajectories is impossible to acquire, and that theories based on such trajectories meet severe problems of principle). Another interesting question concerns the diffractive effects produced by the crossed two-particle processes. Does one get a Pomeron trajectory, and if so, what are its properties?

Since $v(s, t)$ is henceforth regarded as given, the definition of $\rho^{el}(s, t)$ as the right-hand side of (1.1) amounts to a dynamical equation for determination of ρ^{el}. This nonlinear integral equation was analyzed by Atkinson.[15] As Atkinson proved by means of a fixed-point theorem, there is a unique small solution in a certain function space, which may be computed by iteration, when $v(s, t)$ is sufficiently small and smooth. A solution of (1.1) gives a crossing-symmetric amplitude which is unitary in the elastic region, and has the correct support of double-spectral functions. Atkinson was able to show that the requirement of inelastic unitarity, $0 \leq \eta(l, s) \leq 1$, can be satisfied through constraints on $v(s, t)$. The physical meaning of the particular constraints required is not clear, however. Atkinson's solution of the crossing-unitarity equation is neither a strip-model solution nor a bootstrap solution. His v is larger than ρ^{el}, since v is small and ρ^{el} is second order in v. If one puts v equal to zero, the fixed-point method only suffices to show that the trivial solution, $\rho^{el} = 0$, exists. The question of existence of a bootstrap solution (i.e., a nontrivial solution of the homogeneous equation with $v = 0$) is completely open.

An unsubtracted Mandelstam representation implies that there are no Regge poles in the right half l plane, since the Froissart-Gribov partial wave[7] obtained from such a representation is analytic in the right plane. We wish to allow Regge trajectories $\alpha(s)$ with an arbitrarily large upper bound on $\mathrm{Re}\alpha$, and that requires the Mandelstam representation to have an arbitrary finite number of subtractions. Atkinson extended his analysis of the Mandelstam equation to the case of one subtraction,[16] but when a greater number of subtractions is needed there is an essential difficulty. The problem is that (1.1), with a subtracted version of A_t, is no longer equivalent to s-channel elastic unitarity, as far as the first few partial waves are concerned. Equation (1.1) was obtained by taking the t discontinuity of the unitarity equation for the s-channel absorptive part $A_s(s, t)$. The latter contains subtraction terms, which are polynomials in t, and have zero discontinuity. One would have to know the subtraction terms, which contribute only to s-channel partial waves with $l \leq L$, to reconstitute unitarity from (1.1). One can attempt to deal with the problem by taking the absorptive parts of low partial waves as functions to be determined along with ρ^{el}. The t-channel absorptive part A_t is written as the first L terms of its Legendre series plus a remainder given by a subtracted integral over the double spectral function. The absorptive parts of low partial waves are determined by a set of N/D equations, which are coupled to the Mandelstam equation. A cutoff $h(s)$ is introduced as a factor of the integral in (1.1), as in the work of Chew *et al.*[9,10] The set of equations was studied by Atkinson,[17] who obtained an existence theorem for a crossing-symmetric solution, analogous to that of the unsubtracted case. The solution is unitary in the elastic region, but it was not possible to rule out a truly serious violation of unitarity at high energy. Atkinson could prove only that the partial waves are bounded by a power of s, the power being proportional to the maximum value of $\mathrm{Re}\alpha$.

The trouble arises because the Mandelstam equation is an incomplete statement of elastic unitarity. It is then natural to try a pure partial-wave approach, in which a complete statement of elastic unitarity is the simple equation

$$\frac{1}{2i}[a(l, s_+) - a(l, s_-)] = q(s)a(l, s_+)a(l, s_-). \quad (1.4)$$

Furthermore, a strict partial-wave approach should bring out the true power of Regge theory. The essence of Regge theory is partial-wave structure, which is naturally obscured in the plane-wave Mandelstam equation. The traditional objection to a partial-wave approach has been that it would seem to make the imposition of exact crossing symmetry an awkward matter. We have known for some time, however, that exact crossing symmetry could be built into a partial-wave scheme without Regge poles in the right half l plane.[19,20] Recent work[27] has shown that crossing symmetry in a partial-wave scheme with Regge poles is also possible. We have explored several options in the choice of a partial-wave equation to serve as a dynamical system. In the case with Regge poles in the right plane, the only satisfactory scheme that we could find is one with the N/D method as a principal element. This is not surprising, since the N/D method (i.e., the Jost function method) is indispensable in nonrelativistic Regge theory.[6] The N/D representation inevitably gives the deepest view of the partial-wave amplitude, since it allows one to discuss the genesis of Regge trajectories and the CDD ambiguity as well.

Since full crossing symmetry demands inelasticity, we apply the inelastic N/D equation of Refs. 24 and 25. This equation was first derived, in fact, with the intention of applying it in a crossing-symmetric amplitude construction. The inputs to the equation are $\hat{\eta}(l, s)$ and $B(l, s)$, where $\hat{\eta}$ is related to the usual elasticity function η [in the manner described in Eq. (2.38)] and

$$B(l, s) = c_L(l, s) + \frac{\mathrm{P}}{\pi}\int_{16}^{\infty} \frac{1-\hat{\eta}(l, s')}{2r(l, s')(s'-s)}ds'. \tag{1.5}$$

Here c_L is the left-cut part of an appropriate partial-wave amplitude, and r a phase-space factor. In the simplest case without CDD poles or subtraction constant, the N/D equation is

$$\hat{\eta}(l, s)n(l, s) = B(l, s) + \frac{1}{\pi}\int_{4}^{\infty} \frac{B(l, s)-B(l, s')}{s-s'} r(l, s')n(l, s')ds'. \tag{1.6}$$

The partial-wave amplitide $a(l, s)$ may be constructed from $\hat{\eta}$, B, and the solution n of (1.6). Now crossing and unitarity imply that $\hat{\eta}$ and B are nonlinear functionals of $a(l, s)$ and the central spectral function $v(s, t)$; let us denote these functionals by

$$\hat{\eta} = A[a, v], \quad B = B[a, v]. \tag{1.7}$$

We see that (1.6) amounts to a functional equation for the unknown function pair

$$\psi = (\hat{\eta}, B). \tag{1.8}$$

The equation may be written as

$$\psi = G(\psi; v), \tag{1.9}$$

where the nonlinear operator G is defined in three steps:

(i) Given ψ, solve the N/D equation for n.

(ii) From ψ and n, construct a.

(iii) From a, construct

$$G(\psi) = (A[a, v], B[a, v]). \tag{1.10}$$

There is nothing new or problematical about steps (i) and (ii); the difficult part of the method has to do with the proper definition of the nonlinear functionals (1.7). An important role for the N/D method in relativistic Regge theory has been suggested before, for instance, by Mandelstam[28] and Bali, Chew, and Chu,[10] but serious problems connected with these functionals were not solved, and perhaps not completely recognized. The expressions for $A[a, v]$ and $B[a, v]$ as obtained directly from the Froissart-Gribov representation of $a(l, s)$, contain integrals which on first sight appear to diverge, and other integrals which appear to have bad behavior at large s or large $|l|$. We have to make extensive rearrangements of these expressions, and place restrictions on v, in order to show that A and B are actually well defined and have good asymptotic behavior. The subtlety of Regge theory is evident in some rather intricate cancellations of dangerous terms.

Our study of the dynamical equation (1.9) depends on the methods of functional analysis.[14] We shall not emphasize such methods in the initial papers of this series, but it may be worthwhile to indicate briefly how the study goes. Full details will be published elsewhere. First, we find a function space (a Banach space) on which G is defined as a bounded operator when $v(s, t)$ is suitably restricted (initially, we suppose that v has good asymptotic behavior, and is sufficiently small and smooth). The specification of the space involves analyticity in l, smoothness in s, and asymptotic bounds in l and s. The task of finding an appropriate space involves experimentation with asymptotic behaviors, and the above-mentioned rearrangements of integrals. Second, we use the contraction mapping principle (essentially an iterative method) to show that (1.9) has a unique small solution when v is sufficiently small and smooth. This solution may be computed by iteration. Third, we show that G is still defined as a bounded operator in our space when v is not small, provided that certain conditions are fulfilled. The set of conditions includes the requirement that any Regge trajectory $\alpha(s)$, generated as a zero of the D function, be analytic in a certain finite region of the s plane. Unfortunately, the verification of the conditions depends on quantitative considerations which seem to demand numerical calculations. The conditions required are all reasonable, we think, and are such that they can be checked numerically. If G is in fact defined and bounded at large v, we can take the fourth step, which is to invoke the implicit function theorem. According to that theorem, the small-v solution, $\psi(v)$, has a locally unique continuation in v which extends at least to the first singularity of the Fréchet derivative (i.e., the generalized Jacobian) of the operator $1-G$. In a favorable case we could expect to increase the strength of v (say, by a variable numerical factor), and not reach a singularity of the Fréchet derivative before the strength parameter attained the desired physical value. A singularity does not necessarily stop the continuation, as was explained in Ref. 14, Sec. IV F.

In potential scattering, a continuation from weak to strong potential strength certainly exists. That simple fact provides a motivation for the strength parameter continuation in the relativistic case. For a certain class of potentials such that the scattering amplitude has a Mandelstam representation,[21] potential scattering may be treated by the method of this paper. The functional equation that results has a very close resemblance to (1.9); in fact, the terms in (1.9) which are most bothersome to analyze occur already in potential scattering. We see nothing in the relativistic equation to suggest that it will be different from the nonrelativistic one in allowing a parameter continuation.

As an introduction, this first paper is devoted to the weak-coupling case, in which there are no Regge poles in the right half of the l plane, and the Mandelstam representation may be written without subtractions. For the time being, the only driving term in our equations comes from the central spectral function $v(s, t)$, and "coupling strength" is understood as being some appropriate measure of the size and smoothness of v. In the weak-coupling situation the definition of the functionals A and B of (1.7) is a relatively simple matter. The only complication arises from the necessity of incorporating the correct exponential decrease of partial waves at large Rel. Exponential decrease is required for correct support of the double-spectral function, and the failure to account for it has been responsible for well-known difficulties in earlier applications of the N/D method (for instance, the difficulty[24] of achieving correct threshold behavior of partial waves). Our method to incorporate exponential decrease may be described very simply, but the justification for it is somewhat subtle. We merely write the N/D representation for a reduced amplitude $c(l, s)$ in which both the exponential decrease and the threshold behavior of $a(l, s)$ are taken out by a multiplicative factor. In a linear N/D system such a trick is worthless, since it introduces the singularities of the reciprocal of the reducing factor in the amplitude $a(l, s)$ as finally calculated. In the correct nonlinear system in which the N/D inputs (1.7) are functionals of $a(l, s)$ itself, the inputs automatically adjust so as to prevent appearance of singularities of the reciprocal reducing factor. Our procedure is somewhat similar to Mandelstam's method of strips in the l plane,[28] which was found to be ineffective for our purposes.

In part II of this series we shall treat the definition of the functionals A and B in the strong-coupling case, where Regge poles enter the right half l plane, and the Mandelstam representation involves subtractions. We shall also propose a method for avoiding Regge ghosts, i.e., for preventing bound-state poles at energies below threshold where trajectories pass through integers. In this connection we find it necessary to include a new kind of driving term associated with an s-wave subtraction constant. One may in fact put $v(s, t)=0$ and take the subtraction constant as the sole input. In that way one obtains the scheme mentioned above, which seems to resemble a $\lambda\phi^4$ theory.

In part III we give the proof that a solution of (1.9) in fact yields a crossing-symmetric amplitude with Mandelstam analyticity, and discuss some other technical questions. Part IV will be devoted to the many-channel formalism. Later papers will contain mathematical details, existence proofs and the like, which are not emphasized in parts I–IV. In parts I–IV we hope to explain all of the main issues without giving the tedious estimates which are necessary for a full mathematical discussion.

II. DYNAMICAL EQUATION FOR THE WEAK-COUPLING CASE

For simplicity we discuss pseudoscalar, isoscalar meson-meson scattering, with the meson mass set equal to 1; the inclusion of isospin to account for π-π scattering would be an easy extension of the formalism. Our purpose is to construct an amplitude $A(s, t)$ having an unsubtracted Mandelstam representation without single-spectral functions:

$$A(s,t)=\frac{1}{\pi^2}\int_4^\infty\int_4^\infty dx\,dy\,\rho(x,y)\times\left[\frac{1}{(x-s)(y-t)}+\frac{1}{(x-u)(y-s)}+\frac{1}{(x-t)(y-u)}\right]. \qquad (2.1)$$

Crossing symmetry is ensured by symmetry of the double-spectral function,

$$\rho(s, t) = \rho(t, s) . \quad (2.2)$$

Because of symmetry under interchange of t and u (Bose symmetry) the odd partial waves of A are zero. The even partial waves are equal to the Froissart-Gribov amplitude $a(l, s)$ evaluated at even l. The latter is

$$a(l, s) = \frac{4}{\pi(s-4)} \int_4^\infty dt\; Q_l(z_{st}) A_t(s, t) , \quad (2.3)$$

where Q_l is the Legendre function of the second kind, A_t is the t-channel absorptive part,

$$A_t(s, t) = \frac{1}{\pi} \int_4^\infty ds' \rho(s', t) \left(\frac{1}{s'-s} + \frac{1}{s'-u} \right) , \quad (2.4)$$

and z_{st} is the cosine of the scattering angle for the s channel:

$$z_{st} = 1 + \frac{2t}{s-4} \quad (2.5)$$

$$= \cos\Theta_s . \quad (2.6)$$

The idea of our program, stated in the simplest terms, is to express A_t in terms of the partial waves $a(l, s)$ and the central spectral function $v(s, t)$. Equation (2.3) may then be construed as an integral equation for determination of the partial-wave amplitudes. With the goal of expressing A_t in terms of partial waves, we first note that in the elastic region of the s channel, $4 \leq s \leq 16$, the unitarity equation gives

$$A_s(t, s) = \sum_{l=0}^{\infty} (2l+1) q(s) a(l, s_+) a(l, s_-) P_l^{(e)}(z_{st}), \quad (2.7)$$

where A_s is the s-channel absorptive part and

$$a(l, s_\pm) = \lim_{\epsilon \to 0+} a(l, s \pm i\epsilon) , \quad (2.8)$$

$$P_l^{(e)}(z) = \tfrac{1}{2}[P_l(z) + P_l(-z)], \quad (2.9)$$

$$q(s) = \left(\frac{s-4}{s} \right)^{1/2} . \quad (2.10)$$

The Legendre series (2.7) converges only in the large Lehmann ellipse.[8] If the amplitudes $a(l, s_\pm)$ are analytic in l for $\mathrm{Re}\, l > -\epsilon$, and decrease suitably at infinity in the l plane, the sum (2.7) may be replaced by a Watson-Sommerfeld integral which converges in the entire cut plane, namely

$$A_s(t, s) = \frac{i}{2} \int_{-\epsilon} dl \frac{(2l+1)}{\sin\pi l} q(s) a(l, s_+) a(l, s_-) \times P_l^{(e)}(z_{st}). \quad (2.11)$$

The subscript $-\epsilon$ indicates that the integral follows the path $\mathrm{Re}\, l = -\epsilon - i\infty$ to $-\epsilon + i\infty$. We require that

$$0 < \epsilon < \tfrac{1}{2} . \quad (2.12)$$

The "s-channel elastic" part of the double-spectral function is defined as

$$\rho^{\mathrm{el}}(s, t) = \frac{1}{2i} [A_s(t_+, s) - A_s(t_-, s)], \quad 4 \leq s \leq 16 . \quad (2.13)$$

To compute this function from (2.11), we recall that $P_l(z)$ is analytic in the z plane with cut $(-\infty, -1)$, and that its discontinuity over the cut is given by

$$\frac{1}{2i} [P_l(-z_+) - P_l(-z_-)] = -\sin\pi l\, P_l(z) , \quad z > 1 . \quad (2.14)$$

Hence

$$\rho^{\mathrm{el}}(s, t) = \frac{1}{4i} \int_{-\epsilon} dl (2l+1) q(s) a(l, s_+) a(l, s_-) P_l(z_{st}), \quad 4 \leq s \leq 16. \quad (2.15)$$

Clearly, $\rho^{\mathrm{el}}(s, t)$ is equal to the complete double-spectral function $\rho(s, t)$ only for $s < 16$. For larger s one has to include inelastic contributions to A_s which modify the discontinuity of the latter. It is, nevertheless, convenient to define a function ρ^{el} at all $s, t \geq 4$ as follows:

$$\rho^{\mathrm{el}}(s, t) = \frac{1}{4i} \int_{-\epsilon} dl\; \zeta(l, s) P_l(z_{st}) , \quad (2.16)$$

$$\zeta(l, s) = (2l+1) h(s) q(s) a(l, s_+) a(l, s_-) , \quad (2.17)$$

where $h(s)$ is a prescribed smooth function which is identically equal to 1 for $4 \leq s \leq 16$, and which tends to zero at least as rapidly as an inverse power of s at infinity. The exact rate of decrease that $h(s)$ must have will be specified in part II; it is related to the maximum value of $\mathrm{Re}\,\alpha$, where α is the leading Regge trajectory. In the present case without Regge poles one can actually take h to be 1 identically, but it will be more convenient not to do so. The cutoff function h is similar to that of Chew *et al.*,[9,10] except that we do not require it to be identically zero at large s.

The central spectral function $v(s, t)$ is now defined in terms of the total and elastic double-spectral functions as

$$\rho(s, t) = \rho^{\mathrm{el}}(s, t) + \rho^{\mathrm{el}}(t, s) + v(s, t) . \quad (2.18)$$

We shall see presently that $\rho^{\mathrm{el}}(s, t) = 0$ for $t < 16$. Since

$$\rho(s, t) = \begin{cases} \rho^{\mathrm{el}}(s, t), & s < 16 \\ \rho^{\mathrm{el}}(t, s), & t < 16 \end{cases} \quad (2.19)$$

it follows that $v(s,t)=0$ if either s or t is less than 16. If $v(s,t)$ is regarded as given, then Eqs. (1.2), (2.18), and (2.16) furnish the promised expression of A_t in terms of a and v, and Eq. (2.3) indeed is an integral equation for the partial waves if v is given.

The correct support of $\rho(s,t)$ will be obtained automatically in this scheme. The partial waves that we construct obey a bound at large l and fixed t of the form[29]

$$|a(l,s_\pm)| \leqslant \kappa l_+^{-3/2}[z_{s4}+(z_{s4}^2-1)^{1/2}]^{-\mathrm{Re}\,l}, \quad (2.20)$$

$$l_+=|l|+1 . \quad (2.21)$$

This follows from (2.3) if A_t has appropriate smoothness and asymptotic behavior as a function of t. Also, P_l has the bound

$$|P_l(z)| \leqslant \kappa[z+(z^2-1)^{1/2}]^{\mathrm{Re}\,l} . \quad (2.22)$$

We may close the contour of integration in (2.16) by an infinite semicircle in the right half plane, provided that

$$[z_{s4}+(z_{s4}^2-1)^{1/2}]^2 \leqslant z_{st}+(z_{st}^2-1)^{1/2} . \quad (2.23)$$

Since the integrand is analytic inside the closed contour, $\rho^{\mathrm{el}}(s,t)$ will be zero whenever (2.23) holds; that is, whenever

$$t \leqslant \frac{16s}{s-4} . \quad (2.24)$$

It is convenient to define the boundary functions

$$\tau(x)=\frac{16x}{x-4}, \quad \sigma(x)=\frac{4x}{x-16} ; \quad (2.25)$$

σ is the inverse of τ, and

$$\rho^{\mathrm{el}}(s,t)=\theta(t-\tau(s))\,\rho^{\mathrm{el}}(s,t) = \theta(s-\sigma(t))\,\rho^{\mathrm{el}}(s,t), \quad s,t \geqslant 4 . \quad (2.26)$$

Although (2.3) gives an equation for $a(l,s)$, it is not the equation we want for further work. The principal objection to the equation as it stands is that $a(l,s)$ is a very awkward choice for the unknown function as soon as Regge poles enter the right half l plane. Another objection is that even in the case without Regge poles, there is a difficulty in taking the limit $\mathrm{Re}\,l \to -\epsilon$ in (2.3). Although the limit exists, it cannot be taken inside the t integral, and that causes analytical complications.[20] Both problems are taken care of nicely if we replace Eq. (2.3) by an equivalent equation based on the N/D representation.

We shall now derive the N/D system. We work with the reduced amplitude

$$c(l,s)=a(l,s)\left(\frac{p(s)}{s-4}\right)^l , \quad (2.27)$$

$$p(s)=(s^{1/2}+2)^2 . \quad (2.28)$$

The function $[p(s)/(s-4)]^l$ is defined with a cut $(-\infty, 4]$ in the s plane, in such a way that it is positive for l real, $s>4$. This reducing factor is chosen to remove the exponential decrease of $a(l,s)$ at large l, as well as the threshold behavior $(s-4)^l$. We can then write N/D equations for $c(l,s)$ without the constraints of exponential decrease and threshold behavior. It is evident from (2.3) that $c(l,s)$ is analytic in the s plane with cuts $(-\infty,0]$, $[4,\infty)$. The right cut arises from the cut of $A_t(s,t)$, whereas the left cut comes from that of $p(s)$, and from the s discontinuity of

$$(s-4)^{-l-1}Q_l(z_{st}). \quad (2.29)$$

This follows because $(z-1)^l\,Q_l(z)$ at complex l is analytic in z aside from a cut $[-1,1]$. The discontinuity of $c(l,s)$ over the right cut is obtained from (2.3) and (1.2) as

$$\frac{1}{2i}[c(l,s_+)-c(l,s_-)]=\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st})[\rho^{\mathrm{el}}(s,t)+\rho^{\mathrm{el}}(t,s)+v(s,t)] . \quad (2.30)$$

To recognize the elastic unitarity term in (2.30), we substitute the Watson-Sommerfeld representation (2.16) of $\rho^{\mathrm{el}}(s,t)$, take $\mathrm{Re}\,l>-\epsilon$, and reverse the order of integrations to obtain the first term in (2.30) as

$$\left[\frac{p(s)}{s-4}\right]^l \frac{1}{2\pi i}\int_{-\epsilon} dl'\,\zeta(l',s)\,\frac{W(l,l',z_{s4})}{(l'-l)(l'+l+1)} , \quad (2.31)$$

where with $\mathrm{Re}\,l>\mathrm{Re}\,l'$,[30]

$$(l'-l)(l'+l+1)\int_\xi^\infty dz\, Q_l(z)P_{l'}(z)=W(l,l',\xi)=(1-\xi^2)[Q_l(\xi)P_{l'}'(\xi)-P_{l'}(\xi)Q_l'(\xi)] . \quad (2.32)$$

Note the standard Wronskian identity,[30]

$$W(l, l, \xi) = -1. \tag{2.33}$$

Because of (2.20), (2.22), and a similar bound for $P_l'(z)$, we may close the contour in (2.31) to the right and find with the help of (2.33) that

$$\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st})\rho^{\rm el}(s,t) = \left[\frac{p(s)}{s-4}\right]^l q(s)h(s)a(l,s_+)a(l,s_-) = r(l,s)c(l,s_+)c(l,s_-)\,, \tag{2.34}$$

where

$$r(l,s) = h(s)\left(\frac{s-4}{s}\right)^{1/2}\left[\frac{s-4}{p(s)}\right]^l . \tag{2.35}$$

In the region where $h(s)=1$, which includes at least the interval $[4, 16]$, the expression (2.34) is the elastic contribution to the discontinuity (2.30) of $c(l, s)$. The remainder of the discontinuity is denoted as

$$\frac{1-\hat\eta^2(l,s)}{4r(l,s)} = \frac{4p(s)^l}{\pi(s-4)^{l+1}} \times\int_4^\infty dt\, Q_l(z_{st})[\rho^{\rm el}(t,s)+v(s,t)]. \tag{2.36}$$

The full unitarity relation for $c(l, s)$ is then

$$\frac{1}{2i}[c(l,s_+)-c(l,s_-)] = r(l,s)c(l,s_+)c(l,s_-) + \frac{1-\hat\eta^2(l,s)}{4r(l,s)} . \tag{2.37}$$

The function $\hat\eta(l, s)$ is equal to the usual elasticity $\eta(l, s)$ only at values of s for which $h(s)=1$. The actual overlap function [i.e., the contribution of inelastic states to the absorptive part of $a(l,s)$] is

$$\frac{1-\eta^2(l,s)}{4q(s)} = \frac{1-\hat\eta^2(l,s)}{4q(s)h(s)} + [h(s)-1]\,q(s)a(l,s_+)a(l,s_-)\,. \tag{2.38}$$

Nevertheless, (2.37) has the usual formal structure of a unitarity relation, so that $c(l, s_\pm)$ may be written in terms of the pseudo elasticity $\hat\eta$ and a pseudo-phase shift $\hat\delta$ as

$$c(l,s_\pm) = \frac{\hat\eta(l,s)e^{\pm 2i\hat\delta(l,s)}-1}{2ir(l,s)} . \tag{2.39}$$

We may then apply the N/D method in the standard way,[24, 25] choosing a denominator function D such that (at real l)

$$D(l,s_\pm) = |D(l,s_+)|\,e^{\mp i\hat\delta(l,s)} . \tag{2.40}$$

Because the support of $\rho^{\rm el}(t,s)+v(s,t)$ is confined to the region with $s>16$, both η and $\hat\eta$ are equal to 1 for $s<16$.

The s-plane dispersion relation for $c(l, s)$ has the form

$$c(l,s) = c_L(l,s) + \frac{1}{\pi}\int_4^\infty \frac{ds'r(l,s')c(l,s_+')c(l,s_-')}{s'-s} + \frac{1}{\pi}\int_{16}^\infty \frac{ds'[1-\hat\eta^2(l,s')]}{4r(l,s')(s'-s)} . \tag{2.41}$$

We get the expression for the left-cut term c_L simply by subtracting the right-cut term from (2.3). By (2.3), (1.2), and (2.30) we see that

$$c_L = c_1 + c_2\,, \tag{2.42}$$

$$c_1 = \frac{4}{\pi^2}\int_4^\infty \frac{ds'}{s'-s} \times\int_4^\infty dt\,\rho(s',t)\left[\frac{p(s)^l}{(s-4)^{l+1}}Q_l(z_{st}) - \frac{p(s')^l}{(s'-4)^{l+1}}Q_l(z_{s't})\right], \tag{2.43}$$

$$c_2 = \frac{4}{\pi^2}\frac{p(s)^l}{(s-4)^{l+1}} \times\int_4^\infty dt\,Q_l(z_{st})\int_4^\infty \frac{ds'\rho(s',t)}{s'+t+s-4}\,, \tag{2.44}$$

The term c_2 comes directly from the second term in (1.2). For the N/D method we need the functions C and B, where

$$C = c_L + c_3\,, \tag{2.45}$$

$$B = c_L + c_4\,, \tag{2.46}$$

$$c_3(l,s) = \frac{1}{\pi}\int_{16}^\infty \frac{[1-\hat\eta(l,s')]\,ds'}{2r(l,s')(s'-s)}\,, \tag{2.47}$$

$$c_4(l,s) = \frac{\rm P}{\pi}\int_{16}^\infty \frac{[1-\hat\eta(l,s')]\,ds'}{2r(l,s')(s'-s)} . \tag{2.48}$$

Here P denotes Cauchy's principal value, so that

$$C(l,s_\pm) = B(l,s) \pm i\,\frac{1-\hat\eta(l,s)}{2r(l,s)} . \tag{2.49}$$

The N/D integral equation is[24, 25]

$$\hat\eta(l,s)n(l,s) = B(l,s) + \frac{1}{\pi}\int_4^\infty \frac{B(l,s)-B(l,s')}{s-s'} \times r(l,s')n(l,s')ds'. \tag{2.50}$$

The amplitude c is constructed from C, $\hat\eta$, and the solution n of the N/D equation according to the formula

$$c(l,s)=C(l,s)+\frac{1}{\pi D(l,s)}\int_4^\infty \frac{B(l,s')r(l,s')n(l,s')ds'}{s'-s}, \tag{2.51}$$

where

$$D(l,s)=1-\frac{1}{\pi}\int_4^\infty \frac{r(l,s')n(l,s')ds'}{s'-s}. \tag{2.52}$$

We now have in hand the necessary equations for presentation of the dynamical scheme described in Eqs. (1.9) and (1.10). The functional A is obtained from (2.36) and the functional B from (2.46), with the double-spectral functions that enter being expressed in term of partial waves through (2.16). At the risk of redundancy, we collect here the formulas for A and B:

$$A[a,v]=\left(1-\frac{16h(s)}{\pi(s-4)}\int_4^\infty dt\, Q_l(z_{st})[\rho^{\text{el}}(t,s)+v(s,t)]\right)^{1/2}, \tag{2.53}$$

$$B[a,v]=\frac{\text{P}}{\pi}\int_{16}^\infty \frac{[1-A(a,v;l,s')]\,ds'}{2r(l,s')(s'-s)}+\frac{4}{\pi^2}\int_4^\infty \frac{ds'}{s'-s}\int_4^\infty dt\,\rho(s',t)\left[\frac{p(s)^l}{(s-4)^{l+1}}Q_l(z_{st})-\frac{p(s')^l}{(s'-4)^{l+1}}Q(z_{s't})\right]$$
$$+\frac{4}{\pi^2}\frac{p(s)^l}{(s-4)^{l+1}}\int_4^\infty dt\, Q_l(z_{st})\int_4^\infty \frac{ds'\rho(s',t)}{s'+t+s-4}, \tag{2.54}$$

where

$$\rho(s,t)=\rho^{\text{el}}(s,t)+\rho^{\text{el}}(t,s)+v(s,t), \tag{2.55}$$

$$\rho^{\text{el}}(s,t)=\frac{1}{4i}\int_{-\epsilon} dl'(2l'+1)q(s)h(s)\times a(l',s_+)a(l',s_-)P_{l'}(z_{st}). \tag{2.56}$$

The dynamical equation is written as

$$(\hat\eta,B)=G(\hat\eta,B), \tag{2.57}$$

where the operator G is defined in steps as follows:

(i) Given $\hat\eta, B$ solve the N/D equation (2.50) for n.
(ii) From $\hat\eta, B, n$ compute $a(l,s_\pm)$ by means of (2.49), (2.51), (2.52), and (2.27).
(iii) From $a(l,s_\pm)$ compute

$$G(\hat\eta,B)=(A[a,v],B[a,v])$$

by means of (2.53) and (2.54).

It is understood that the various functions of l and s are calculated for $\text{Re}\,l=-\epsilon$ and $4\le s<\infty$. One can see, however, that the amplitude $c(l,s)$ as calculated from (2.51) is in fact analytic in l for $\text{Re}\,l>-\epsilon$, and also analytic in s in the plane with cuts $(-\infty,0],[4,\infty)$. The function C is analytic in s, but neither B nor $\hat\eta$ need have such analyticity.

III. EXISTENCE OF SOLUTIONS, AND CONSTRUCTION OF THE FULL AMPLITUDE

We have obtained the partial-wave equation (2.57) by formal arguments. To be sure that the scheme makes sense, we should investigate existence and uniqueness of solutions of the equation. We must also show that a solution leads to a full amplitude $A(s,t)$ having unitarity, crossing symmetry, and Mandelstam analyticity.

In order that the transformation G be well defined it is of course necessary that the functions $\hat\eta$ and B satisfy certain conditions of analyticity, smoothness, and asymptotic behavior. We can show that if the pair $(\hat\eta,B)$ satisfies appropriate conditions, then so does the pair $G(\hat\eta,B)$. That is, the transformation is a mapping of a certain function space into itself. The precise description of this space, and the proof that the operator takes the space into itself, will be published subsequently. At present we shall mention only the most essential features of the space. Namely, the functions $\hat\eta(l,s)$ and $B(l,s)$ are to be analytic in l for $\text{Re}\,l>-\epsilon$, continuous in l for $\text{Re}\,l=-\epsilon$, defined for $s\ge 4$, and continuously differentiable in s in that region. Furthermore, both functions should have the reality property

$$f(l,s)=f(l^*,s)^*. \tag{3.1}$$

The functions and their derivatives are to satisfy bounds as follows:

$$l_+|B(l,s)|,\ s|\partial_s B(l,s)|\le\frac{\kappa}{l_+^{1/2}\ln s}, \tag{3.2}$$

$$l_+|1-\hat\eta(l,s)|,\ s|\partial_s\hat\eta(l,s)|\le\frac{\kappa|h(s)|}{l_+^{1/2}\ln s}\left[\frac{s-4}{p_1(s)}\right]^{\text{Re}\,l}, \tag{3.3}$$

with

$$p_1(s)=\frac{1}{s-16}(s+2s^{1/2}-8)^2. \tag{3.4}$$

Further, $\hat\eta(l,s)\equiv 1$ for $4\le s\le 16$. In addition, there are requirements of threshold behavior, and of Hölder continuity of s derivatives, which we shall not relate here. We define a Banach space S of function pairs $(1-\hat\eta,B)$ having the properties mentioned, and show that the operator G takes a subset K of the Banach space into itself, provided that the central spectral function $v(s,t)$ is suitably restricted and sufficiently small. Furthermore, we can use a fixed-point theorem to prove that there is a solution to (2.57), unique in K, for each such $v(s,t)$. The solution, which varies continuously with v, may be constructed by interation.

The scattering amplitude $c(l, s_\pm)$ constructed from the solution $(1-\hat{\eta}, B)$ in the set K will obey bounds similar to (3.2). When stated in terms of the amplitudes $a(l, s_\pm)$, the bounds are

$$|a(l,s_\pm)| \le \kappa \left[\frac{s-4}{p(s)}\right]^{\mathrm{Re}\,l} \frac{1}{l_+^{3/2}\ln s}, \qquad (3.5)$$

$$|\partial_s a(l,s_\pm)| \le \kappa \left[\frac{s-4}{p(s)}\right]^{\mathrm{Re}\,l} \frac{1}{(s-4)l_+^{1/2}\ln s}. \qquad (3.6)$$

A restriction on $v(s,t)$ sufficient for our purpose is that the Froissart-Gribov projection of its contribution to $A(s,t)$ be a member of the space S. If $V(l,s)$ is defined by

$$V(l,s)=\frac{4p(s)^l}{\pi(s-4)^{l+1}}\int_{16}^{\infty} dt\, Q_l(z_{st})V_t(s,t), \qquad (3.7)$$

$$V_t(s,t)=\frac{1}{\pi}\int_{16}^{\infty} ds' v(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right), \qquad (3.8)$$

then we require in the first instance that

$$|V(l,s)| \le \frac{\kappa}{l_+^{3/2}\ln s}, \quad |\partial_s V(l,s)| < \frac{\kappa}{l_+^{1/2} s\ln s}. \qquad (3.9)$$

As we shall show in part IV, it is possible to make a simple model of $v(s,t)$ which has some physical appeal and also yields a $V(l,s)$ that lies in the space S.

For $v(s,t)$ sufficiently small, there will certainly be no Regge pole for $\mathrm{Re}\,l > -\epsilon$. Since the function $n(l,s)$ is uniformly small for small $v(s,t)$, the function $D(l,s)$ is uniformly close to one, and never vanishes [see Eq. (2.52)]. Consequently, $c(l,s)$ as defined in (2.51) is analytic for $\mathrm{Re}\,l > -\epsilon$ since its various ingredients have such analyticity. The function $\hat{\eta}(l,s)$ will never vanish at small v; it is uniformly close to 1, according to its definition (2.53). The linear integral equation (2.51), solved in step (i) of the evaluation of the mapping G, has a unique solution at weak coupling. It is a regular Fredholm equation in this case, and the norm of its kernel is less than 1. If the coupling strength is increased, it may be possible for $D(l,s)$ or $\hat{\eta}(l,s)$ to develop zeros for $\mathrm{Re}\,l > -\epsilon$ or for the linear equation (2.51) to reach a condition such that the corresponding homogeneous equation has a nonzero solution (i.e., the kernel acquires a unit eigenvalue). Zeros of D correspond to the Regge poles, which will be discussed in part II. If $\hat{\eta}$ were to have a zero, the Fredholm character of Eq. (2.51) would be lost, and the problem would be much more complicated. Since $\hat{\eta}(l,16)=1$, one can always choose the cutoff $h(s)$ to be so rapid as to prevent a zero of $\hat{\eta}$. The question of whether a unit eigenvalue of the kernel in (2.51) occurs at strong coupling is a quantitative one, which we cannot answer without numerical calculations. A unit eigenvalue is not necessarily fatal to the scheme, but we shall not discuss the effect that one would have. Earlier calculations with the N/D method have not encountered unit eigenvalues, even at realistic values of coupling strength.[31]

Most of the technique required for proof of the existence theorem mentioned above is given in Ref. 21, but there is one new feature in the present scheme which did not arise in that work. Namely, the t integral in (2.43) does not converge absolutely for $\mathrm{Re}\,l=-\epsilon$, although it does so for $\mathrm{Re}\,l>-\epsilon$. The part of the integral that causes trouble is from $\rho^{\mathrm{el}}(s',t)$. By applying (2.34) we can rewrite that term as

$$\frac{4}{\pi}\int_4^\infty dt\,\rho^{\mathrm{el}}(s',t)\left[\frac{p(s)^l}{(s-4)^{l+1}}Q_l(z_{st})-\frac{p(s')^l}{(s'-4)^{l+1}}Q_l(z_{s't})\right]=\frac{4p(s)^l}{\pi}\int_4^\infty dt\,\rho^{\mathrm{el}}(s',t)\left[\frac{1}{(s-4)^{l+1}}Q_l(z_{st}) - \frac{1}{(s'-4)^{l+1}}Q_l(z_{s't})\right] + \left[\left(\frac{p(s)}{p(s')}\right)^l - 1\right] r(l,s')c(l,s'_+)c(l,s'_-). \qquad (3.10)$$

Now the integral on the right-hand side converges absolutely for $\mathrm{Re}\,l=-\epsilon$ by virtue of a cancellation of its two terms at large t. The other integrated term is well defined on $\mathrm{Re}\,l=-\epsilon$.

After a fixed point of the mapping G is attained, it still remains to construct the full amplitude $A(s,t)$, and to verify that the latter has crossing symmetry, unitarity, and Mandelstam analyticity. Although we have taken the latter properties as ingredients of our equations, it is not immediately obvious that $A(s,t)$, constructed from a solution of the equations, will have these properties. To show that all is in order, we first demonstrate that the amplitude $c(l,s)$, obtained from a solution of (2.57), satisfies the unitarity equation (2.37). We then prove that the corresponding amplitude $a(l, s)$ is actually given by the Froissart-Gribov formula (2.3), with A_t determined through (1.2), (2.55), and (2.56). Since the Froissart-Gribov formula implies the bound (2.20), the double-spectral function (2.55) will have the proper support. Now the solution to our problem is given by the crossing-symmetric formula (2.1), with ρ determined from the N/D partial wave through (2.55) and (2.56). This is true because the Froissart-Gribov amplitude is the partial-wave projection of (2.1). This partial wave is unitary in the elastic region because it satisfies (2.37), with $\hat{\eta}$ being equal to one for $4 \leqslant s \leqslant 16$.

To show that the amplitude (2.51) satisfies the condition (2.37), it suffices to show that

$$c(l, s_\pm) = \frac{1}{2ir(l, s)}\left[\frac{\hat{\eta}(l, s)D(l, s_\mp)}{D(l, s_\pm)} - 1\right] . \quad (3.11)$$

It is readily seen that (3.11) implies (2.37). According to Eqs. (2.50)–(2.52), we have (suppressing the variable l)

$$\begin{aligned} D(s_\pm)c(s_\pm) &= \left[B(s) \pm i\,\frac{1-\hat{\eta}(s)}{2r(s)}\right]D(s_\pm) \\ &\quad + \frac{P}{\pi}\int_4^\infty \frac{B(s')r(s')n(s')ds'}{s'-s} \\ &\quad \pm iB(s)r(s)n(s) \\ &= \hat{\eta}(s)n(s) \pm i\,\frac{1-\hat{\eta}(s)}{2r(s)}\,D(s_\pm) \\ &= \hat{\eta}(s)\frac{D(s_-)-D(s_+)}{2ir(s)} \mp \frac{1-\hat{\eta}(s)}{2ir(s)}D(s_\pm)\ , \end{aligned} \quad (3.12)$$

which is the desired result (3.11).

To verify that $c(l, s)$ of (2.51) is identical to the Froissart-Gribov amplitude (2.3) (multiplied by $[p(s)/(s-4)]^l$), we show that the two amplitudes have the same discontinuities over their cuts in the s plane. Since either amplitude vanishes at infinity in the s plane, uniformly in direction, it will follow that the amplitudes are equal. By (2.34), (2.36), and (2.37) we see that the right-cut discontinuities of both amplitudes are the same. The left-cut part of the N/D amplitude is the left-cut part of $C(l, s)$, which was originally calculated from the Froissart-Gribov formula. Thus, the discontinuities of the two amplitudes agree on both cuts, and the proof of Mandelstam analyticity, crossing symmetry, and elastic unitarity is complete.

For simplicity in notation we have chosen to write the functional $B[a, v]$ in (2.54) in a form which is not recommended for numerical computation or further analysis. One can eliminate some integrations by reordering integrals and employing Watson-Sommerfeld transformations. The recommended form, perhaps not so pretty as (2.54), is obtained by specializing the formula to be given in part II to the case without Regge poles.

The equations with Regge poles, developed in part II, represent a smooth continuation with respect to coupling strength of the equations of this paper.

ACKNOWLEDGMENT

We are grateful for conversations and correspondence with D. Atkinson and M. Kaekebeke. One of us (P. W. J.) enjoyed the hospitality of the Institute for Theoretical Physics, University of Groningen, and received support from the Stichting voor Fundamenteel Onderzoek der Materie of the Netherlands.

*Work supported in part by the National Science Foundation and the Stichting voor Fundamenteel Onderzoek der Materie of the Netherlands.

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model calculations, reported in Refs. 11 and 12, which are in decided disagreement, even though the physical ideas underlying the two calculations are supposed to be identical. We find it hard to understand either calculation, since there was no aspiration to solve a definite equation. Rather, one tried to match "input" and "output" Regge trajectories and residues over a limited energy range to an approximation which degenerates as the energy range increases. The existence of such a match does not convince us that there is a crossing-symmetric, unitary amplitude having the Regge trajectories and residues obtained. The equations employed would not guarantee crossing and unitarity even if solved globally. For instance, Webber uses only the Mandelstam equation, which does not guarantee unitarity of low partial waves. Collins and Johnson do not have a definite equation to be solved, but rather work with a combination of N/D equations and a small number of Mandelstam iterations. This procedure does not ensure crossing symmetry. Furthermore, the definition of certain apparently divergent integrals either has not been justified [as in Webber's definition of his integral (2.6) by analytic continuation] or has not been discussed [as in the case of Eq. (4.19b) of the first Collins and Johnson paper]. As far as we know, the only correct way to handle these integrals is the one we develop in part II of this series.

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