Pseudoparticles from solitons*

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(Received 29 November 1976)

It is shown that pseudoparticles can be derived from solitons. A transformation is found between the finite-energy solutions of the two-dimensional sine-Gordon theory and the special class of collinear finite-action solutions of an SU(2) gauge theory in Euclidean space.

I. INTRODUCTION

The connection is made between collinear pseudoparticle solutions[1] of pure Yang-Mills theory in four-dimensional Euclidean space and the soliton solutions[2] of the two-dimensional sine-Gordon equation. Finite-action solutions of the Euclidean classical equations of motion—pseudoparticles—have recently been exploited in the computation of the functional integral to describe the corresponding quantum field theory.[3] In earlier work, quantum-mechanical interpretations have been given to solitons—finite-energy solutions of Minkowski classical equations of motion.[4] In this paper, it is shown that the family of classical finite-action Euclidean solutions for an SU(2) gauge theory, corresponding to n pseudoparticles on a line with arbitrary separations and sizes, can be derived from the solitons of a Minkowski equation of motion.[5]

II. COLLINEAR PSEUDOPARTICLES

The gauge Lagrangian is

$$\mathcal{L}(x) = +\frac{1}{2g^2}\,\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu],$$

and

$$A_\mu \equiv \sum_{a=1}^{3} A_\mu^a \frac{\sigma^a}{2i}$$

is an anti-Hermitian matrix, and σ^a are Pauli matrices, generators of SU(2). The equation of motion is

$$D_\mu F^{\mu\nu} = 0, \tag{2.2}$$

where

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}].$$

It has been shown by Witten[1] that a solution to (2.2) with finite action $I = |n|\,8\pi^2/g^2$ for integer n is given by

$$A_j^a(\vec{x}, t) = \frac{\beta^2+1}{x^2}\epsilon_{jak}x_k + \frac{\beta_1}{x^3}(\delta_{ja}x^2 - x_j x_a) + A_1\frac{x_j x_a}{x^2},$$

$$A_0^a(\vec{x}, t) = \frac{A_2 x^a}{x},$$

where

$$\beta_i(x,y) = e^{\psi}\chi_i,$$

$$A_i(x,y) = -\epsilon_{il}\partial_l\psi,$$

$$\psi(x,y) = \ln x - \tfrac{1}{2}\ln ff^* + \tfrac{1}{2}\rho(x,y),$$

$$i, l = 1, 2, \quad j, k, a = 1, 2, 3.$$

In the above

$$f = \chi_1 - i\chi_2 = -\left(\frac{\partial g}{\partial z}\right)\frac{2}{(1+g)^2}$$

is an analytic function of $x+iy$, where g is given below, and

$$x = \left(\sum_{a=1}^{3} x^a x^a\right)^{1/2}, \qquad y = t, \ \text{ the Euclidean time.} \tag{2.3}$$

$\rho(x,y)$ is given by the solutions of

$$\tfrac{1}{2}\nabla^2\rho = e^{\rho}, \tag{2.4}$$

which ensure nonsingular gauge fields A_μ^a. Equation (2.4) is the Euclidean Liouville equation. The solutions we want are

$$\rho = \ln\left[\frac{(\partial g/\partial z)(\partial g/\partial z)^*}{(\mathrm{Re}g)^2}\right],$$

where

$$g = \frac{1 - \prod_{i=1}^{n+1}[(a_i^* - z_i)/(a_i + z_i)]}{1 + \prod_{i=1}^{n+1}[(a_i^* - z_i)/(a_i + z_i)]}, \quad \mathrm{Re}a_i > 0.$$

III. TRANSFORMATION EQUATIONS

The connection between these solutions and the solitons of the sine-Gordon theory is as follows. The scalar curvature of a two-dimensional surface with coordinates (x, y) and metric $g_{\mu\nu} = e^{\rho(x,y)}\delta_{\mu\nu}$ is $R = -\nabla^2\rho e^{-\rho}$. If we consider a coordinate transformation $u(x, y)$, $v(x, y)$, where the metric is now given as a function of u, v:

$$g_{\mu\nu} = \begin{pmatrix} 1 & \cos\alpha(u, v) \\ \cos\alpha(u, v) & 1 \end{pmatrix},$$

the scalar curvature in these coordinates is

$$R = \frac{-2}{\sin\alpha(u, v)} \frac{\partial^2\alpha}{\partial u \partial v}. \tag{3.1}$$

Under a coordinate transformation, R is invariant. Thus

$$-\nabla^2\rho e^{-\rho} = -\frac{2}{\sin\alpha}\frac{\partial^2\alpha}{\partial u \partial v}.$$

For $R = -2$, we have $\frac{1}{2}\nabla^2\rho = e^\rho$ and $\partial^2\alpha/\partial u \partial v = \sin\alpha$. The latter equation is the sine-Gordon equation in light-cone coordinates. It is known to have soliton solutions.[6] These classical solutions will be used to derive the collinear pseudoparticle solutions of the four-dimensional Euclidean SU(2) theory.

Since any two-dimensional surface is conformal to any other, we can find the inverse transformation $x(u, v)$, $y(u, v)$. It satisfies the following differential equations:

$$\begin{aligned} x_u &= \frac{y_v - y_u \cos\alpha}{\sin\alpha}, \\ x_v &= \frac{-y_u + y_v \cos\alpha}{\sin\alpha}, \end{aligned} \tag{3.2}$$

where $x_u \equiv \partial x(u, v)/\partial u$, etc. Such a system of first-order linear partial-differential equations is known as the Beltrami equations.[7] They can be derived in this instance by factoring the line element. Since it is invariant under the coordinate transformation, we have an identity:

$$\begin{aligned} ds^2 &= du^2 + dv^2 + 2\cos\alpha\, du\, dv \\ &= (du + e^{i\alpha}dv)(du + e^{-i\alpha}dv) \\ &= (dx + i\,dy)(dx - i\,dy)(1/\omega\omega^*) \\ &= e^\rho(dx^2 + dy^2). \end{aligned} \tag{3.3}$$

For the moment view ω as an arbitrary complex function of u and v. Let $dx + i\,dy = \omega(du + e^{i\alpha}dv)$ and $dx - i\,dy = \omega^*(du + e^{-i\alpha}dv)$. Equating real and imaginary parts and eliminating ω from the equations, we arrive at (3.2). Now solve for $e^\rho = 1/\omega^*\omega = (x_u^{\,2} + y_u^{\,2})^{-1}$. Had we merely substituted $dx = x_u du + x_v dv$ into (3.3), the equations would be

$$\begin{aligned} x_u^{\,2} + y_u^{\,2} &= x_v^{\,2} + y_v^{\,2}, \\ x_u x_v + y_u y_v &= [\cos\alpha(u, v)](x_u^{\,2} + y_u^{\,2}), \end{aligned} \tag{3.4}$$

$$\rho(x, y) = -\ln(x_u^{\,2} + y_u^{\,2}). \tag{3.5}$$

Equation (3.5) defines $\rho(x, y)$. Equation (3.4) is equivalent to (3.2).

The Beltrami equations can be solved when a particular analytic function $\alpha(u, v)$ is chosen.[7] Define $z(u, v) = x(u, v) + i y(u, v)$. Equation (3.2) implies that $z_u/z_v = e^{-i\alpha}$. Consider the ordinary differential equation for the characteristics in the complex plane.

Let $v(u)$ be a complex analytic function of u where

$$\frac{dv}{du} = -e^{-i\alpha}. \tag{3.6}$$

Therefore

$$\frac{z_u}{z_v} = \frac{-dv}{du}. \tag{3.7}$$

Since ultimately the functions $\rho(x, y)$ are to be solutions of Liouville's equation, $\alpha(u, v)$ must be a (nonzero) solution of the sine-Gordon equation. For simplicity, let

$$\alpha(u, v) = 4\tan^{-1}e^{u+v}. \tag{3.8}$$

This is the one-soliton solution. Solving (3.6) with this $\alpha(u, v)$, we find

$$v(u) = u - i\cosh(u + v) + K$$

or

$$K = v - u + i\cosh(u + v) \equiv i\phi(u, v). \tag{3.9}$$

K is the integration constant. Equation (3.9) is a definition of $\phi(u, v)$. Obviously $dK = 0 = i\phi_u du + i\phi_v dv$. Thus

$$\frac{z_u}{z_v} = \frac{-dv}{du} = \frac{\phi_u}{\phi_v}. \tag{3.10}$$

A particular solution to Eq. (3.10) is

$$z(u, v) = \phi(u, v)$$

or

$$\begin{aligned} x(u, v) &= \cosh(u + v), \\ y(u, v) &= u - v. \end{aligned} \tag{3.11}$$

The solution to Liouville's equation corresponding to this particular solution of the Beltrami equations is from (3.5)

$$\rho = -\ln x^2. \tag{3.12}$$

Inserting this into (2.3), we find the gauge transform of the vacuum.

We have thus established a connection between the one-soliton solution of the sine-Gordon theory and the "0-pseudoparticle" solution of SU(2).

IV. GENERAL SOLUTION

Return now to Eq. (3.10). The general solution to $z_u/z_v = \phi_u/\phi_v$ is $z = f(\phi)$ for any analytic f. The general solution to (3.2) for $\alpha = 4\tan^{-1}e^{u+v}$ is

$$x(u,v) = \mathrm{Re}z(\phi),\qquad y(u,v) = \mathrm{Im}z(\phi), \tag{4.1}$$

where z is an arbitrary analytic function of the complex variable $\phi = \cosh(u+v) + i(u-v)$. The Beltrami equations are two first-order partial differential equations of two functions of two variables. The general solution should depend on two real functions of one variable. This is consistent with (4.1) since any arbitrary analytic function z is also specified by two real functions of one variable. The function $\rho(x,y)$ which corresponds to the general transformation (4.1) is from (3.5)

$$\begin{aligned}\rho(x,y) &= -\ln z_u z_u^* \\ &= -\ln \frac{\partial z}{\partial \phi}\left(\frac{\partial z}{\partial \phi}\right)^* (\mathrm{Re}\phi)^2 \\ &= \ln \frac{\partial \phi}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)^* \left(\frac{1}{\mathrm{Re}\phi}\right)^2 .\end{aligned} \tag{4.2}$$

Any nonzero analytic function $z(\phi)$ has an inverse $\phi(z)$ where

$$\left[\frac{\partial z(\phi)}{\partial \phi}\right]^{-1} = \frac{\partial \phi(z)}{\partial z} .$$

Since the $\rho(x,y)$ are solutions to Liouville's equation, and Liouville's equation is conformally invariant, each $\rho(x,y)$ is related to all the others by a conformal transformation. Indeed this is what (4.2) says. If $\rho = -\ln(\mathrm{Re}z)^2$ is a particular solution to $\frac{1}{2}\nabla^2\rho = e^\rho$, by conformal invariance

$$\rho = -\ln(\mathrm{Re}\phi)^2 + \ln\frac{\partial \phi}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)^*$$

is also a solution as long as ϕ has no zeros.[1] We have thus derived all the solutions to Liouville's equation from the one-soliton solution of the sine-Gordon equation. The special class of $\rho(x,y)$ which yield the n-pseudoparticle solutions are given by $z(\phi)$, to be determined by inverting

$$\phi = \frac{1 - \prod_{i=1}^{n=1} [(a_i - z_i)/(a_i^* + z_i)]}{1 + \prod_{i=1}^{n-1} [(a_i - z_i)/(a_i^* + z_i)]} \tag{4.3}$$

for $\mathrm{Re}a_i > 0$. As an example, for $n=0$, $z=\phi$ and for $n=1$, $z = [1-(1-\phi^2)^{1/2}]/\phi$ for $a_1 = a_2 = 1$. The form of (4.3) has to do with the way in which Liouville's equation is connected to the four-dimensional gauge theory.[1]

If a different $\alpha(u,v)$, say the two-soliton solution, were chosen in (3.6), a different function $\phi^1(u,v)$ would be found. The general solution $z(\phi^1)$ is an arbitrary analytic function of the new $\phi^1(u,v)$. In particular, $z_u(\phi^1)z_u^*(\phi^1)$ would be identical to $z_u(\phi)z_u^*(\phi)$ derived for $\alpha(u,v)$ equal to the one-soliton solution, since $\rho = -\ln z_u z_u^*$ is the general solution of Liouville's equation for any solution of the sine-Gordon equation. That is to say $z_u z_u^*$ is independent of $\alpha(u,v)$ as long as it solves the sine-Gordon equation. However, from (3.4)

$$\frac{\mathrm{Re}(z_u z_v^*)}{z_u z_u^*} = \cos\alpha(u,v).$$

Although $z_u z_u^*$ is independent of $\alpha(u,v)$, the transformation $z(u,v) = x(u,v) + iy(u,v)$ itself is not independent of the choice of α. This transformation cannot be used to write down the general solution of the sine-Gordon theory.

V. CONCLUSION

A transformation has been found between classical finite-energy solutions in Minkowski space and classical finite-action solutions in Euclidean space. Since the phenomena are connected, it might be profitable to pursue the relation between the procedures which are used to quantize these classical entities. A hypothesis inherent in the literature is that theories which have solitonlike solutions, be they Minkowski or Euclidean, are correctly quantized only when the solitons are included. Since we believe solutions do play a role in the quantum sine-Gordon theory, it is consistent that the pseudoparticles should be taken into account in four-dimensional gauge theory as well.

ACKNOWLEDGMENTS

The author thanks R. Jackiw, E. Witten, J. Kiskis, and L. Smarr for discussions.

*Research supported in part by the National Science Foundation under Grant No. MPS75-20427.

†Work supported in part by the Harvard University Society of Fellows.

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