Fermions in a pseudoparticle field*

Joe Kiskis
Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87545
(Received 10 January 1977)

A study is made of the relationship between the triangle anomaly and zero-eigenvalue solutions to the Euclidean Dirac equation in the presence of a pseudoparticle field.

I. INTRODUCTION

It has been evident from the beginning[1,2] that the triangle anomaly[3] is important in determining the properties of a Fermi field in the presence of a pseudoparticle.[1] 't Hooft[2] observed that the massless fermion functional integral vanishes when the Fermi field is coupled to a gauge field with nontrivial topology. This integral can be expressed as a product of the eigenvalues of the Euclidean Dirac operator[4]

$$D=-i\gamma\cdot(\partial-iA)\,. \tag{1.1}$$

When A is a pseudoparticle, the spectrum of D includes a zero-eigenvalue bound state. As a consequence, the functional integral vanishes.

The physical reason behind this result was suggested by 't Hooft[2] and was elaborated upon by Callan, Dashen, and Gross[5] and by Jackiw and Rebbi.[6] They propose that the functional integral over the fermion fields in the presence of the pseudoparticle vanishes because it represents a transition in which a conservation law is violated. The conserved quantity is the gauge-variant chiral charge. The presence of a gauge-variant piece in this charge is a reflection of the existence of the triangle anomaly.

We find ourselves in a remarkable position. The triangle anomaly and the zero-eigenvalue eigenfunction of D appear to be two sides of the same coin. This is surprising because the former arises from subtle ultraviolet renormalization effects in the quantum field theory while the latter is simply the solution to a classical eigenvalue problem.

It is the purpose of this paper to reveal the connection between these two things. Our conclusion will be that if the gauge field to which the fermions are coupled has a nontrivial topology, then the spectrum of D includes either a zero-eigenvalue bound state or a zero-eigenvalue unbound resonance. Either one of these will cause the functional integral to vanish. The connection will be made by studying the equation for the anomalous divergence of the axial-vector current. This approach to the problem was originally suggested by Coleman.[7]

In Sec. II, we will lay out the argument. In Sec. III, we will discuss the fine points more carefully. Section IV will be a detailed calculation in two dimensions. Some interesting features appear there as a consequence of the fact that "infinity" in two dimensions is S^1, which is not simply connected.

II. THE ARGUMENT

Throughout these discussions, the gauge field will appear as an externally applied c-number field. As such, it is not necessarily a solution to the equations of motion. We introduce the Euclidean space propagator for the massive Fermi field in the presence of the gauge field. It satisfies

$$(D-im)S(x,y)=\delta(x-y)\,. \tag{2.1}$$

Using the continuum $\psi_{\lambda K}$ and bound ψ_i solutions to

$$D\psi=\lambda\psi\,, \tag{2.2}$$

we can express S as

$$S=\sum_K\int d\lambda\,\frac{\psi_{\lambda K}(x)\psi^\dagger_{\lambda K}(y)}{\lambda-im}+\sum_j\frac{\psi_j(x)\psi_j^\dagger(y)}{\lambda_j-im}\,. \tag{2.3}$$

K denotes a set of angular-momentum-like quantum numbers.

We will deal with a Pauli-Villars regulated S

$$S_R=\lim_{\mu\to\infty}[S(m)-S(\mu)]\,. \tag{2.4}$$

The existence of the triangle anomaly is the statement that

$$\frac{i}{2}\partial_\mu\operatorname{Tr}[\gamma_\mu\gamma_5S_R(x,x)]=im\operatorname{Tr}[\gamma_5S_R(x,x)]+t_d\,. \tag{2.5}$$

In four dimensions,

$$t_4=\frac{1}{16\pi^2}\operatorname{Tr}[F_{\mu\nu}\tilde F_{\mu\nu}]\,, \tag{2.6}$$

and, in two dimensions,

$$t_2=\frac{1}{2\pi}\,\frac{1}{2}\,\epsilon_{\mu\nu}F_{\mu\nu}\,. \tag{2.7}$$

The first term on the right-hand side (RHS) of

(2.5) is

$$
\begin{aligned}
im\,\mathrm{Tr}[\gamma_5 S_R(x,x)] &= -\sum_K \int d\lambda\, \psi^\dagger_{\lambda K}\gamma_5\psi_{\lambda K}\left(\frac{-im}{\lambda - im} - \frac{-im}{\lambda - i\mu}\right) \\
&\quad - \sum_j \psi^\dagger_j \gamma_5 \psi_j \left(\frac{-im}{\lambda_j - im} - \frac{-im}{\lambda_j - i\mu}\right) \qquad (2.8) \\
&= -\sum_K \int d\lambda \frac{-1}{2i\lambda}\,\partial_\mu(\psi^\dagger_{\lambda K}\gamma_\mu\gamma_5\psi_{\lambda K})\frac{-im}{\lambda - im} \\
&\quad - \sum_j \psi^\dagger_j\gamma_5\psi_j \frac{-im}{\lambda_j - im}\,. \qquad (2.9)
\end{aligned}
$$

This step is justified because it is only the $\lambda = 0$ region that will be contributing in later expresions.

We now integrate (2.5) over all space and assume the the surface term from the left-hand side (LHS) is zero. This gives

$$
\begin{aligned}
0 &= -\int dV \sum_K \int d\lambda \frac{-1}{2i\lambda}\,\partial_\mu(\psi^\dagger_{\lambda K}\gamma_\mu\gamma_5\psi_{\lambda K})\frac{-im}{\lambda - im} \\
&\quad - \int dV \sum_j \psi^\dagger_j\gamma_5\psi_j \frac{-im}{\lambda_j - im} + T_d\,. \qquad (2.10)
\end{aligned}
$$

The topologically interesting quantities

$$T_4 = \frac{1}{16\pi^2}\int d^4x\, \mathrm{Tr}[F_{\mu\nu}\tilde{F}_{\mu\nu}] \qquad (2.11)$$

and

$$T_2 = \frac{1}{2\pi}\int d^2x\, \tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu} \qquad (2.12)$$

have now appeared.

If we recall that γ_5 anticommutes with D, we can observe that

$$\gamma_5\psi_\lambda = \psi_{-\lambda} \quad \text{unless } \lambda = 0\,. \qquad (2.13)$$

By making reference to (2.8), we conclude that all contributions to the first two terms of (2.10) are zero except for those coming from $\lambda = 0$. If the continuum did not extend to $\lambda \to 0$ of if the contribution it made in that region were zero, we would not have to consider the first term on the RHS of (2.10). We would have

$$0 = -\sum_{\substack{j \\ \text{such that} \\ \lambda_j = 0}} \int dV\, \psi^\dagger_j\gamma_5\psi_j + T_d\,. \qquad (2.14)$$

When $\lambda_i = 0$, we can choose things so that

$$\gamma_5\psi_i = \pm\psi_i\,. \qquad (2.15)$$

If n_+ and n_- are numbers of these two kinds of eigenstates, then we obtain

$$n_+ - n_- = T_d\,. \qquad (2.16)$$

The conclusion from (2.16) would be that, if the gauge field is nontrivial so that $T_d \neq 0$, then there must exist at least one zero-eigenvalue eigenfunction of D. This would be the connection that we seek.

III. THE FINE POINTS

The conditional tense which was used in the last part of Sec. II was meant to indicate that there may be some problems associated with the argument which was given.

The first problem is with the surface term which was assumed to be zero in (2.10). The contribution from the bound states to this surface term is certainly zero. The contribution from the continuum is of the same form as the first term on the RHS of (2.10) except that $1/\lambda$ is replaced by $1/m$. We will see that the first term of (2.10) makes a contribution only in special cases, and, in those cases, it is finite. Thus when $1/\lambda \to 1/m$ the surface term will be zero.

This brings us to the second problem which has to do with the continuum contribution on the RHS of (2.10). In the usual case, the continuum *will* extend to $\lambda \to 0$ and, therefore, may contribute in (2.10). Even so, if the interchange of the integrations in (2.10) were valid, we would conclude that the contribution is zero. However, the two-dimensional example worked out in Sec. IV shows that this interchange is not justified in general.

Let us investigate the circumstances which are associated with a nonvanishing contribution from the continuum. We will argue that the continuum contribution is zero unless there are bound states or unbound resonances at $\lambda = 0$ in the spectrum of D. In order to do this, some additional assumptions will be used. We will assume that there exists a region characterized by a radius R outside of which the field strengths $F_{\mu\nu}$ are negligible In the case of the pseudoparticle, $F_{\mu\nu}$ falls rapidly ($\sim r^{-4}$) when r is greater than the pseudoparticle size.

Now, we will cut off the integral over all space at some large radius r which is much greater than R. We can interchange the integrations if we leave the $r \to \infty$ limit outside of the λ integration. The resulting expression involves

$$
\begin{aligned}
C(\lambda, r) &\equiv \int^r dV \sum_K \frac{-1}{2i\lambda}\,\partial_\mu(\psi^\dagger_{\lambda K}\gamma_\mu\gamma_5\psi_{\lambda K}) \\
&= r^3 \frac{-1}{2i\lambda}\sum_K \int d\Omega\, \psi^\dagger_{\lambda K}\gamma\cdot\hat{r}\gamma_5\psi_{\lambda K}\,. \qquad (3.2)
\end{aligned}
$$

We would like to know how C behaves in λ as $r \to \infty$.

For $R < r$, A is approximately a pure gauge. Thus, we can expect that the behavior of the summand in C will be similar to its behavior when A

is zero. That is, we expect that for any $\lambda \neq 0$

$$r^3 \frac{-1}{2i\lambda} \int d\Omega\, \psi^\dagger_{\lambda K} \gamma \cdot \hat{r} \gamma_5 \psi_{\lambda K} \sim \frac{1}{\lambda} \sin(\lambda r + \delta) . \quad (3.3)$$

From (3.3), we can see that in an expression such as

$$\lim_{r \to \infty} \int d\lambda f(\lambda) C(\lambda, r) , \quad (3.4)$$

it is only the point at $\lambda = 0$ that will contribute. Thus,

$$\lim_{r \to \infty} C(\lambda, r) = c\delta(\lambda) . \quad (3.5)$$

We would like to determine c. In the free case, c is zero due to a cancellation between contributions from different K values. So there is no contribution to (2.10) for that case.

In order to determine c, we will consider the expression

$$B(\Delta, r) = \int_{-\Delta}^{\Delta} d\lambda\, C(\lambda, r) . \quad (3.6)$$

c is obtained from B by

$$c = \lim_{\Delta \to 0} \lim_{r \to \infty} B(\Delta, r) . \quad (3.7)$$

Because of the oscillations, the contributions to B from λ's which are much greater than $1/r$ is small. An estimate for B is

$$B \sim \int_{-1/r}^{1/r} d\lambda\, C(\lambda, r) . \quad (3.8)$$

We now need an estimate for C in the region

$$\lambda r \lesssim 1 , \quad (3.9)$$

$$R \ll r . \quad (3.10)$$

In this region, A is approximately a pure gauge so the behavior of C in r will be *similar* to the free case. We are also inside the radius at which the oscillations of C as a function of r begin. Thus, we can expect that the radial-part wave functions will behave like

$$\lambda^{3/2} (\lambda R)^l [a_0(l, \lambda)(r/R)^l + b_0(l, \lambda)(r/R)^{-(l+2)}] \quad (3.11)$$

in four dimensions. As K runs over all its values, l will run over the set $\{0, 1, 2, 3, \ldots\}$. For the free case, $b_0 = 0$. Normalization requires that

$$a_0{}^2 + (\lambda R)^{2(l+1)} b_0{}^2 \sim 1 . \quad (3.12)$$

The constants a_0 and b_0 are determined by the boundary conditions at $r = R$. At small λ, the wave function inside R will be independent of λ except for normalization. Equation (3.11) becomes

$$\lambda^{3/2} (\lambda R)^l [a_0(l, 0)(r/R)^l + b_0(l, 0)(r/R)^{-(l+2)}] . \quad (3.13)$$

We expect both a_0 and b_0 to be of order one in general. Special cases are the following:

(1) Free case.

$$b_0 = 0 , \quad a_0 \sim 1 . \quad (3.14)$$

(2) Bound state or unbound resonance at $\lambda = 0$.

$$b_0 \sim 1 , \quad a_0 \sim (R\lambda)^2 . \quad (3.15)$$

[In this case, (3.12) is altered.]

Thus, the first term in (3.13) will completely dominate the second unless

$$a_0 \to 0 \text{ as } \lambda \to 0 . \quad (3.16)$$

Except when this happens, (3.13) will make *exactly* the same contribution to C that it does in the free case. But, in the free case, c is zero. Thus, we conclude that the continuum can contribute to (2.10) only when there is a bound state or unbound resonance at $\lambda = 0$.

This spoils Eq. (2.16). However, while (2.16) is not true, we can still conclude that in the presence of a topologically nontrivial gauge field, the spectrum of D must include either a bound state or an unbound resonance at zero eigenvalue. In either case, when the system is placed in a box of radius R_0 the lowest eigenvalue of D will approach zero faster than $R_0{}^{-1}$. The argument given by 't Hooft[8] allows us to conclude that the functional integral is zero.

This completes the general discussion. We can summarize our results by saying that through the anomalous divergence equation the triangle anomaly determines the structure of the spectrum of D to a sufficient extent to guarantee that

$$\frac{\int d\psi d\psi^\dagger \exp\{-\int dV\, \psi^\dagger [-i\gamma \cdot (\partial - iA)] \psi\}}{\int d\psi d\psi^\dagger \exp\{-\int dV\, \psi^\dagger [-i\gamma \cdot \partial] \psi\}} = \frac{\det D}{\det D_0} = 0 , \quad (3.16)$$

when A has nontrivial topology.

The calculations of the next section will provide an example of the general arguments which have been presented here.

IV. TWO DIMENSIONS

A. The topology

Two dimensions would present us with a paradox if (2.16) were true. This would arise from the fact that while the LHS of (2.16) is an integer, the RHS need not be an integer for the external-field problem in two dimensions. If we restrict our attention to gauge-field configurations of finite flux, then $F_{\mu\nu}$ must vanish at infinity and A must be a pure gauge at infinity. Thus, in a simply connected region, A can be expressed as

$$A = ig^{-1}\partial g \tag{4.1}$$

with

$$g \in \mathrm{U}(1) \text{ for each } x . \tag{4.2}$$

However, when we try to extend (4.1) to the entire circle at infinity, we will not succeed because this region is not simply connected. In higher dimensions, infinity is simply connected, and this difficulty does not arise. Now, if we are not assured of the existence of a continuous function

$$g:\ S^1 \to \mathrm{U}(1)\,, \tag{4.3}$$

then we cannot carry through the topological arguments which would require that T_2 be an integer. We emphasize that the failure of the usual topological arguments in the external-field problem in two dimensions is a characteristic of that problem. When the gauge field is a part of a coupled-field problem, other physical conditions may require that the flux be quantized.

Although the RHS of (2.16) need not be an integer in two dimensions, we have also seen that there may be contributions to the LHS which come from the continuum and are not included in (2.16). In order to understand this more fully, we have obtained an exact solution for a particular choice of A.

B. The example

For A, we take

$$A_\mu = \epsilon_{\mu\nu}\partial_\nu \Phi \tag{4.4}$$

with

$$\Phi = \begin{cases} -\frac{1}{2}f\dfrac{r^2}{R^2}\,, & r<R \\[2mm] -\frac{1}{2}f\left(1+\ln\dfrac{r^2}{R^2}\right), & R<r \end{cases}$$

$$r^2 = x_1^{\,2} + x_2^{\,2}\,. \tag{4.5}$$

If we define F by

$$F_{\mu\nu} = \epsilon_{\mu\nu}F\,, \tag{4.6}$$

then

$$F = \begin{cases} 2f\dfrac{1}{R^2}\,, & r<R \\[2mm] 0\,, & R<r \end{cases} \tag{4.7}$$

and

$$T_2 = f\,. \tag{4.8}$$

The Euclidean Dirac eigenvalue equation is

$$D\psi = -i\gamma\cdot(\partial - iA)\psi = \lambda\psi\,,$$
$$\gamma_1 = \sigma_1\,,\quad \gamma_2=\sigma_2\,,\quad \gamma_5=\sigma_3\,. \tag{4.9}$$

Out of the continuum $\psi_{\lambda l}$ and bound ψ_i solutions to this equation, we can construct the Green's function

$$S(x,y) = \sum_l \int d\lambda\, \frac{\psi_{\lambda l}(x)\psi^\dagger_{\lambda l}(y)}{\lambda - im} + \sum_j \frac{\psi_j(x)\psi^\dagger_j(y)}{\lambda_j - im}\,, \tag{4.10}$$

which solves

$$(D-im)S(x,y) = \delta^2(x-y)\,. \tag{4.11}$$

Since the angular momentum operator commutes with D, the solutions will be of the form

$$\psi_+ = \begin{pmatrix} \phi_+(r)e^{il\theta} \\ \chi_+(r)e^{i(l+1)\theta} \end{pmatrix}, \tag{4.12}$$

with

$$l = 0, 1, 2, \ldots$$

and

$$j = l + \tfrac{1}{2}\,, \tag{4.13}$$

or of the form

$$\psi_- = \begin{pmatrix} \chi_-(r)e^{-i(l+1)\theta} \\ \phi_-(r)e^{-il\theta} \end{pmatrix}, \tag{4.14}$$

with

$$l = 0, 1, 2, \ldots$$

and

$$j = -l - \tfrac{1}{2}\,. \tag{4.15}$$

Equation (4.9) becomes

$$\left[\frac{\partial}{\partial r} + \frac{1}{r}\left(l+1+r\frac{\partial\Phi}{\partial r}\right)\right]\chi_+ = i\lambda\phi_+\,,$$
$$\left[\frac{\partial}{\partial r} - \frac{1}{r}\left(l+r\frac{\partial\Phi}{\partial r}\right)\right]\phi_+ = i\lambda\chi_+\,, \tag{4.16}$$

or

$$\left[\frac{\partial}{\partial r} + \frac{1}{r}\left(-l+r\frac{\partial\Phi}{\partial r}\right)\right]\phi_- = i\lambda\chi_-\,,$$
$$\left[\frac{\partial}{\partial r} - \frac{1}{r}\left(-l-1+r\frac{\partial\Phi}{\partial r}\right)\right]\chi_- = i\lambda\phi_-\,. \tag{4.17}$$

C. The bound states

We will now find the $\lambda=0$ bound states. It is easy to see from the equations in the next subsection that there are no bound states for $0<\lambda^2$. The bound states that we find here have no nodes. Therefore, there are no bound states with $\lambda^2<0$.

The solutions to (4.16) and (4.17) are not difficult to find when $\lambda = 0$. They can be chosen to be eigenstates of γ_5. For

$$-1 \le f \le 1\,, \tag{4.18}$$

there are no bound states. For

$$1 < f \le 2\,, \tag{4.19}$$

there is one bound state at $j = \frac{1}{2}$. For

$$2 < f \le 3\,, \tag{4.20}$$

an *additional* bound state appears at $j = \frac{3}{2}$. This pattern continues. For

$$-2 \le f < -1\,, \tag{4.21}$$

there is one bound state with $j = -\frac{1}{2}$. For

$$-3 \le f < -2 \tag{4.22}$$

an *additional* bound state appears at $j = -\frac{3}{2}$. This pattern continues. The form of these solutions is

$$\psi_j = \begin{cases} N_j \begin{pmatrix} \exp[-\frac{1}{2}f(r^2/R^2)](r/R)^{j-1/2}\exp[i(j-\frac{1}{2})\theta] \\ 0 \end{pmatrix}, & r < R \\ N_j \begin{pmatrix} \exp(-\frac{1}{2}f)(r/R)^{j-1/2-f}\exp[i(j-\frac{1}{2})\theta] \\ 0 \end{pmatrix}, & R < r \end{cases} \tag{4.23}$$

when j is positive. The solutions for j negative have a similar form except that $\gamma_5 = -1$.

As an example of an unbound resonance at $\lambda = 0$ which appears for

$$0 < f \le 1\,, \tag{4.24}$$

we have

$$\begin{pmatrix} \exp[-\frac{1}{2}f(r^2/R^2)] \\ 0 \end{pmatrix}, \quad r < R$$
$$\begin{pmatrix} \exp(-\frac{1}{2}f)(r/R)^{-f} \\ 0 \end{pmatrix}, \quad R < r \tag{4.25}$$

There is a similar solution with $\gamma_5 = -1$ for $-1 \le f < 0$. These are actually a part of the continuum.

D. The continuum

For nonzero λ, we consider the second-order equations which follow from (4.16) and (4.17):

$$-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\phi_+ + (U_+ - \lambda^2)\phi_+ = 0\,, \tag{4.26}$$

$$-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\phi_- + (U_- - \lambda^2)\phi_- = 0\,, \tag{4.27}$$

with

$$U_+ = \begin{cases} \dfrac{l^2}{r^2} - \dfrac{2lf}{R^2} + r^2\left(\dfrac{f}{R^2}\right)^2 - \dfrac{2f}{R^2}\,, & r < R \\ \dfrac{(l-f)^2}{r^2}\,, & R < r \end{cases} \tag{4.28}$$

and

$$U_- = \begin{cases} \dfrac{l^2}{r^2} + \dfrac{2lf}{R^2} + r^2\left(\dfrac{f}{R^2}\right)^2 + \dfrac{2f}{R^2}\,, & r < R \\ \dfrac{(l+f)^2}{r^2}\,, & R < r\,. \end{cases} \tag{4.29}$$

Our analysis will assume that f is not an integer. A separate analysis verifies that the continuum contribution that we are calculating has a value at integer f which is the limit as f approaches the integer from the side closer to zero.

The solution to these equations for the inside region is

$$\phi_+(r) = c_+ e^{-z/2} z^{l/2} F(d_+ | l+1 | z)\,,$$

with

$$z = |f|\frac{r^2}{R^2}\,,$$
$$d_+ = \tfrac{1}{2}[l + 1 - \epsilon(f)(l+1) - (\lambda R)^2/|f|\,]\,. \tag{4.30}$$

$F(a|b|x)$ is the confluent hypergeometric function. Also

$$\phi_-(r) = c_- e^{-z/2} z^{l/2} F(d_- | l+1 | z)$$

and

$$d_- = \tfrac{1}{2}[l + 1 + \epsilon(f)(l+1) - (\lambda R)^2/|f|\,]\,. \tag{4.31}$$

For the outside region, we have the solutions

$$\phi_+ = a_+ J_{l-f}(|\lambda| r) + b_+ J_{-(l-f)}(|\lambda| r)\,, \tag{4.32}$$

$$\phi_- = a_- J_{l+f}(|\lambda| r) + b_- J_{-(l+f)}(|\lambda| r)\,. \tag{4.33}$$

The constants $a_\pm$, $b_\pm$, and $c_\pm$ are determined by normalization and by continuity at $r = R$.

As we have seen in Sec. III, it is only the region $\lambda \to 0$ and $r \to \infty$ that is important. In this region, we find that for $f < 0$

$$\phi_+ = \left(\frac{|\lambda|}{2}\right)^{1/2} J_{l-f}(|\lambda| r) \tag{4.34}$$

and

$$\phi_- = \left(\frac{|\lambda|}{2}\right)^{1/2} \times \begin{cases} J_{l+f}(|\lambda| r)\,, & -1 < l + f \\ J_{-(l+f)}(|\lambda| r)\,, & l + f < -1\,, \end{cases} \tag{4.35}$$

and for $0 < f$

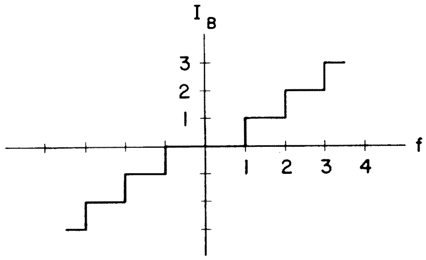

FIG. 1. I_B as a function of f.

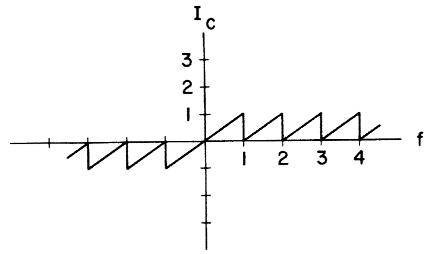

FIG. 2. I_C as a function of f.

$$\phi_+ = \left(\frac{|\lambda|}{2}\right)^{1/2} \times \begin{cases} J_{l-f}(|\lambda| r), & -1 < l-f \\ J_{-(l-f)}(|\lambda| r), & l-f < -1 \end{cases} \tag{4.36}$$

and

$$\phi_- = \left(\frac{|\lambda|}{2}\right)^{1/2} J_{l+f}(|\lambda| r).$$

E. The result

Let us now split

$$I \equiv -im\int d^2x\, \mathrm{Tr}[\gamma_5 S(x,x)] \tag{4.37}$$

into its bound-state and continuum parts

$$I = I_B + I_C . \tag{4.38}$$

I_B is $n_+ - n_-$ and it is shown in Fig. 1 as a function of f. Since we know that

$$0 = -I + T_2 \tag{4.39}$$

or

$$I = f , \tag{4.40}$$

it must be that I_C is as shown in Fig. 2.

F. I_C

Our last section will verify Fig. 2. The expression for I_C is

$$I_C = -im\int d^2x \sum_l \int d\lambda\, \frac{\psi^\dagger_{\lambda l+}\gamma_5\psi_{\lambda l+} + \psi^\dagger_{\lambda l-}\gamma_5\psi_{\lambda l-}}{\lambda - im} . \tag{4.41}$$

Using the expressions for the wave functions that have been derived, we find

$$I_C = \lim_{r\to\infty} -im\int d\lambda \frac{K(\lambda,r)}{\lambda - im} ; \tag{4.42}$$

$$\begin{aligned} K(\lambda,r) = &-\sum_{l=0}^{\infty} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{l-f}(|\lambda| r) - (l-f)J^2_{l-f}(|\lambda| r)\right] \\ &+ \sum_{\substack{l \\ l+f<-1}} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{-(l+f)}(|\lambda| r) - (l+f)J^2_{-(l+f)}(|\lambda| r)\right] \\ &+ \sum_{\substack{l \\ -1<l+f}} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{l+f}(|\lambda| r) - (l+f)J^2_{l+f}(|\lambda| r)\right] \end{aligned} \tag{4.43}$$

for $f < 0$, and, for $0 < f$,

$$\begin{aligned} K(\lambda,r) = &-\sum_{-1<l-f} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{l-f}(|\lambda| r) - (l-f)J^2_{l-f}\right] \\ &- \sum_{l-f<-1} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{-(l-f)}(|\lambda| r) - (l-f)J^2_{-(l-f)}(|\lambda| r)\right] \\ &+ \sum_{l=0}^{\infty} \frac{1}{2|\lambda|}\left[\tfrac{1}{2}r\frac{\partial}{\partial r} J^2_{l+f}(|\lambda| r) - (l+f)J^2_{l+f}(|\lambda| r)\right] . \end{aligned} \tag{4.44}$$

From these expressions, it is not difficult to verify that

$$I_C(-f) = -I_C(f) \tag{4.45}$$

and that

$$I_C(f+1) = I_C(f) \quad \text{for } 0<f \tag{4.46}$$

and that

$$I_C(f-1) = I_C(f) \quad \text{for } f<0 . \tag{4.47}$$

Thus, we need only evaluate I_C for f in the range

$$-1<f<0 . \tag{4.48}$$

If Fig. 2 is correct, we should find

$$I_C = f . \tag{4.49}$$

It is important to carry out the sum on l before the integral on λ is attempted. The sums involving the derivatives of the Bessel functions are not difficult. After the use of a recurrence relation, they telescope. The other sums can be done after relating them to a formula in Watson.[9] The expression for $K(\lambda, r)$ which results is

$$K(\lambda,r) = -\frac{1}{2|\lambda|}\{\tfrac{1}{2}|\lambda| r J_{-f}(|\lambda|r)J_{-f-1}(|\lambda|r) - \tfrac{1}{4}(|\lambda|r)^2[J_{-f}^2(|\lambda|r) - J_{-f-1}(|\lambda|r) + J_{-f+1}(|\lambda|r) + J_{-f-1}^2(|\lambda|r) - J_{-f-2}(|\lambda|r)J_{-f}(|\lambda|r)]\} . \tag{4.50}$$

From this, we can show that

$$\lim_{r\to\infty} K(\lambda, r) = c\,\delta(\lambda) . \tag{4.51}$$

To determine c, we use

$$c = \lim_{\Delta\to 0}\lim_{r\to\infty}\int_{-\Delta}^{\Delta} d\lambda\, K(\lambda, r) . \tag{4.52}$$

A tedious calculation gives the expected result

$$c = f . \tag{4.53}$$

Thus

$$I_C = f \tag{4.54}$$

for

$$-1<f<0 \tag{4.55}$$

and, with (4.45) and (4.46), Fig. 2 is verified.

G. The conclusion of section IV

In this calculation, the ideas that were discussed in Sec. III are seen at work. In the region

$$-1 \leq f \leq 1 , \tag{4.56}$$

there is no bound state that can contribute to (4.39). It is the continuum which makes the proper contribution. However, although there are no bound states when f satisfies (4.56), there is an unbound resonance [Eq. (4.25)].

V. CONCLUSION

We have argued that in the presence of a topologically nontrivial gauge field, the spectrum of D must include either a bound state or an unbound resonance at zero eigenvalue. Either one of these is sufficient to give

$$\frac{\int d\psi d\psi^\dagger \exp(-\int dV\, \psi^\dagger D\psi)}{\int d\psi d\psi^\dagger \exp(-\int dV\, \psi^\dagger D_0\psi)} = 0 . \tag{5.1}$$

Note added. It has come to our attention that this work is related to a result in differential topology known as the index theorem.[10] Indeed, it appears that there is also a local version[11] of this theorem which is closely related to the triangle anomaly. Our thanks to L. Dolan and K. Macrae for bringing this work in mathematics to our attention.

ACKNOWLEDGMENT

It is a pleasure to acknowledge helpful conversations with S. Coleman, R. Jackiw, and C. Zemack.

*Work supported in part by the United States Energy Research and Development Administration.

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[4]The γ matrices satisfy

$$\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} = \delta_{\mu\nu} .$$

γ_5 is $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. The internal symmetry is U(1) in two dimensions and SU(2) in four dimensions. In four dimensions the gauge field A is a Hermitian matrix normalized so that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

Internal-symmetry indices are suppressed.

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