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Classical description of a particle interacting with a non-Abelian gauge field

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By introducing a new kind of variable we find simple Lagrangian and Hamiltonian descriptions of a classical particle interacting with an external non-Abelian gauge field. Both conventional particles and supersymmetric particles carrying pseudoclassical spin are considered. The physical interpretation of these models is discussed. The models are quantized following Dirac's procedure. Finally, the isospin representations to which the resulting quantized particles belong are investigated.

I. INTRODUCTION

A classical description of particles with non-Abelian charge (we will call it isospin in the following for convenience) has been given by Wong[1] in terms of equations of motion. We show how a Lagrangian description of such particles can be given by using a new set of variables.[2] These new variables can be commuting or anticommuting, giving two different kinds of formalism. We also discuss the associated Hamiltonian formulation and the quantization of the systems using Dirac's well-known method[3] and its generalization to anticommuting variables due to Casalbuoni.[4]

Recently it has been shown[5,6] that a "classical" description of a particle with spin can be given by a supersymmetric Lagrangian. When suitably quantized it describes particles with spin $\frac{1}{2}$. Therefore the description is not a conventional classical description (a conventional classical limit should involve letting the spin quantum number tend to infinity, and the quantization of such a classical description should give particles with arbitrarily large spin), and we will use the term "pseudoclassical" for this description as suggested in Ref. 4. In the following we will show how the spin and isospin descriptions can be combined to give a supersymmetric Lagrangian describing a "classical" particle with spin and isospin. We quantize the models and show how this gives models which can also be described by conventional field theories. The spins of the quantized particles are 0 or $\frac{1}{2}$, depending on whether they are obtained from spinless or pseudospin classical particles, respectively. Starting from commuting isospin variables, the quantized isospin can take arbitrarily large quantum numbers. This means that these variables when quantized give a kind of superfield formalism, which describes an infinite number of different kinds of particles. Starting with anticommuting isospin variables, only a finite number of quantum numbers for the isospin of a particle are obtained. This classical description of isospin should therefore be called pseudoclassical.

In Sec. II we describe the commuting variable description of isospin. In Sec. III we describe the new features encountered when the isospin variables are anticommuting. In Sec. IV we incorporate the pseudoclassical spin in our models. Section V finally deals with the isospin content in the "superfields" mentioned above.

II. CONVENTIONAL CLASSICAL PARTICLE WITH ISOSPIN DESCRIBED BY COMMUTING VARIABLES

We assume that we have a non-Abelian gauge field associated with a compact semisimple gauge group G which we describe by the potential A_μ^α and the field tensor

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha - g f_{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma \quad .$$

We choose an irreducible representation of the Lie algebra of the group. These $n\times n$ matrices T^α then fulfill the commutation relations

$$[T^\alpha, T^\beta] = i f_{\alpha\beta\gamma} T^\gamma \quad .$$

We also introduce n complex dynamical variables $\theta_a(\tau)$, in addition to the position $x^\mu(\tau)$. The dynamics of the particle can now be described by the action

$$\begin{aligned} S &= \int L\, d\tau \ , \\ L &= -m(\dot{x}^\mu \dot{x}_\mu)^{1/2} + i\theta_a^\dagger D_{ab}\theta_b \ , \qquad (2.1)\\ D_{ab}\theta_b &= \dot{\theta}_a + ig\dot{x}^\mu A_\mu^\alpha T_{ab}^\alpha \theta_b \ . \end{aligned}$$

This action fulfills the following consistency requirements:

(i) It is real up to a total time derivative.

(ii) It is invariant under coordinate transformations of the parameter τ.

(iii) It is invariant under gauge transformations in which both θ_a and A_μ^α are transformed. The infinitesimal transformations are

$$A_\mu^\alpha \to (A_\mu^\alpha)^\Lambda = A_\mu^\alpha - g f_{\alpha\beta\gamma}\Lambda^\beta A_\mu^\gamma - \partial_\mu \Lambda^\alpha \ ,$$
$$\theta_a \to (\theta_a)^\Lambda = \theta_a + ig\Lambda^\alpha T_{ab}^\alpha \theta_b \ . \qquad (2.2)$$

(iv) The equations of motion are consistent with Wong's equations as discussed below.

From the action (2.1) follow the equations of motion

$$\dot\theta_a + ig\dot x^\mu A_\mu^\alpha T_{ab}^\alpha \theta_b = 0 \ , \qquad (2.3)$$

$$\frac{d}{d\tau}\left(\frac{m\dot x_\mu}{(\dot x^\nu \dot x_\nu)^{1/2}}\right) - gI^\alpha F_{\mu\nu}^\alpha \dot x^\nu = 0 \ . \qquad (2.4)$$

Here we have introduced the isospin variable

$$I^\alpha = \theta_a^\dagger T_{ab}^\alpha \theta_b \ . \qquad (2.5)$$

The equation of motion for I^α follows from (2.3),

$$\dot I^\alpha - g f_{\alpha\beta\gamma}\dot x^\mu A_\mu^\beta I^\gamma = 0 \ . \qquad (2.6)$$

By adding the action of the kinetic energy of the field

$$-\tfrac{1}{4}\int d^4x\, F_{\mu\nu}^\alpha F^{\alpha\mu\nu}$$

to the action (2.1), and making the field dynamical, one can obtain the field equations of motion

$$D_\mu^{\alpha\beta} F^{\beta\mu\nu} = J^{\alpha\nu} \ , \qquad (2.7)$$

$$D_\mu^{\alpha\beta} = \delta_{\alpha\beta}\partial_\mu - g f_{\alpha\gamma\beta} A_\mu^\gamma \ , \qquad (2.8)$$

where the particle current is

$$J^{\alpha\mu}(x) = g\int d\tau\, \delta^4(x - x(\tau)) I^\alpha(\tau)\dot x^\mu(\tau) \ . \qquad (2.9)$$

The equations (2.4), and (2.6)–(2.9) are those given by Wong.[1] The current (2.9) is the current associated with the gauge symmetry of the particle action.

We now turn to the Hamiltonian formulation of the model, using the method of Dirac.[3] We treat x^μ, θ_a, and $\theta_a^\dagger$ as independent dynamical variables and find their conjugate momenta in the usual way:

$$P_\mu = \frac{\partial L}{\partial \dot x^\mu} = -\frac{m\dot x_\mu}{(\dot x^\nu \dot x_\nu)^{1/2}} - gA_\mu^\alpha I^\alpha \ ,$$

$$P_a = \frac{\partial L}{\partial \dot\theta_a} = i\theta_a^\dagger \ ,$$

$$P_a^\dagger = \frac{\partial L}{\partial \dot\theta_a^\dagger} = 0 \ .$$

This gives the following constraints

$$P_a^\dagger = 0 \ ,$$
$$P_a - i\theta_a^\dagger = 0 \ ,$$
$$\Pi_\mu \Pi^\mu - m^2 = 0 \ ,$$

where

$$\Pi_\mu = P_\mu + gA_\mu^\alpha I^\alpha \ .$$

(Note that P_a is not the Hermitian conjugate of $P_a^\dagger$ since the Lagrangian contains a non-Hermitian total derivative.)

Since $P_\mu \dot x^\mu + P_a \dot\theta_a = L$, the Hamiltonian is just a linear combination of the constraints

$$H = \tfrac{1}{2}C(\Pi_\mu \Pi^\mu - m^2) + C_a(P_a - i\theta_a^\dagger) + \overline{C}_a P_a^\dagger \ .$$

It gives the following time evolution of the constraints:

$$\frac{d}{d\tau}P_a^\dagger = iC_a - gC\Pi^\mu T_{ab}^\alpha \theta_b A_\mu^\alpha \ ,$$

$$\frac{d}{d\tau}(P_a - i\theta_a^\dagger) = -i\overline{C}_a \ ,$$

$$\frac{d}{d\tau}(\Pi_\mu \Pi^\mu - m^2) = 0 \ .$$

These equations determine C_a and $\overline{C}_a$, and give no new constraints. Since the Poisson brackets between our constraints $P_a^\dagger$ and $P_a - i\theta_a^\dagger$ are not zero, we replace them by Dirac brackets. The variables P_a and $P_a^\dagger$ can then be eliminated.

Finally we get the following Hamiltonian description:

$$H = \tfrac{1}{2}C(\Pi_\mu \Pi^\mu - m^2) \ , \qquad (2.10)$$

$$\Pi_\mu \Pi^\mu - m^2 = 0 \ , \qquad (2.11)$$

where

$$\Pi_\mu = P_\mu + gA_\mu^\alpha I^\alpha \ ,$$
$$I^\alpha = \theta_a^\dagger T_{ab}^\alpha \theta_b \ . \qquad (2.12)$$

The nonzero Poisson brackets between the remaining variables are

$$\{x^\mu, P_\nu\} = \delta^\mu{}_\nu \ ,$$
$$\{\theta_a, \theta_b^\dagger\} = -i\delta_{ab} \ . \qquad (2.13)$$

These lead to the correct Dirac brackets for the isospin variables,

$$\{I^\alpha, I^\beta\} = f_{\alpha\beta\gamma} I^\gamma \ .$$

Quantization is straightforward. The nonzero commutators of the fundamental operators P_μ, x^μ, θ_a, and $\theta_a^\dagger$ are

$$[P_\mu, x^\nu] = -i\hbar\delta_\mu{}^\nu \ ,$$
$$[\theta_a, \theta_b^\dagger] = \hbar\delta_{ab} \ .$$

We may diagonalize the x^μ operator and describe the system by an object $\psi(x,\theta^\dagger)$, which is a wave function as far as the variables P_μ and x^μ are concerned, but which is a state vector for the θ_a and $\theta_a^\dagger$ operators. The momentum operator then acts as a differentiation operator. The Hamiltonian is unimportant and the remaining constraint

$$\left[\left(-i\hbar\frac{\partial}{\partial x^\mu}+gA_\mu^\alpha I^\alpha\right)^2-m^2\right]\psi(x,\theta^\dagger)=0 \qquad (2.14)$$

defines the physical states. (Note that for nontrivial representations of semisimple groups there is no ordering problem in the isospin operator since $[\theta_a,\theta_b^\dagger]T_{ab}^\alpha=\hbar\,\mathrm{Tr}T^\alpha=0$.)

Equation (2.14) coincides with the Klein-Gordon equation for a scalar particle interacting with the non-Abelian gauge field except that the isospin dependence of the wave function is represented by a continuous variable rather than by a finite number of components. This has the remarkable consequence that the wave function is able to describe particles belonging to infinitely many different representations.

It is also possible to diagonalize, for example, the real part of θ_a. But it is more appropriate to treat $\theta_a^\dagger$ and θ_a as creation and annihilation operators for harmonic-oscillator modes. An arbitrary state vector can be obtained by applying creation operators on the oscillator vacuum. Thus we may write

$$\psi(x,\theta^\dagger)=\psi(x)+\theta_a^\dagger\psi_a(x)+\tfrac{1}{2}\theta_a^\dagger\theta_b^\dagger\psi_{ab}(x)+\cdots\,. \qquad (2.15)$$

It is obvious that the isospin operator does not mix the terms in this series, and that the nth term describes particles whose representation is the nth symmetrical (reducible) power of the representation T^α. If the gauge group G is SU(2) and if T^α is its fundamental representation, then the wave function can describe each finite-dimensional representation in exactly one way. But this is an exceptional case, usually some representations are omitted and/or some are described in infinitely many ways. If, for example, $G=\mathrm{SU}(3)$ and T^α is the simplest representation whose powers generate all representations, namely $\underline{3}+\underline{\bar{3}}$, then each representation is generated in infinitely many ways.

III. CONVENTIONAL CLASSICAL PARTICLE WITH ISOSPIN DESCRIBED BY ANTICOMMUTING VARIABLES

It is also possible to let the isospin variables θ_a anticommute. In this case it is also possible to use Hermitian variables, provided that the matrices T^α are antisymmetric. Thus, we consider two Lagrangians

$$L_1=-m(\dot{x}^\mu\dot{x}_\mu)^{1/2}+i\theta_a^\dagger D_{ab}\theta_b \qquad (3.1)$$

and

$$L_2=-m(\dot{x}^\mu\dot{x}_\mu)^{1/2}+\frac{i}{2}\theta_a D_{ab}\theta_b\,,\qquad T_{ba}^\alpha=-T_{ab}^\alpha\,. \qquad (3.2)$$

$D_{ab}\theta_b$ is defined as in Eq. (2.1) but the isospin variables are anticommuting in this section

$$[\theta_a,\theta_b]_+=[\theta_a^\dagger,\theta_b]_+=0\,.$$

It is easy to show that these models satisfy the four consistency conditions listed in Sec. II and that the equations (2.3)–(2.9) in the Lagrangian formalism look identical in the above two cases, except that in the case of Hermitian θ_a the expression for the isospin is

$$I^\alpha=\tfrac{1}{2}\theta_a T_{ab}^\alpha\theta_b \qquad (3.3)$$

instead of Eq. (2.5).

In deriving the Hamiltonian formalism, to save space we first bring L_1 to the form L_2 by dividing θ_a into Hermitian and anti-Hermitian parts:

$$\theta_a=\frac{1}{\sqrt{2}}(\theta_{a1}+i\theta_{a2})\,,$$

$$\theta_{a1}=\frac{1}{\sqrt{2}}(\theta_a+\theta_a^\dagger)\,,$$

$$\theta_{a2}=\frac{1}{i\sqrt{2}}(\theta_a-\theta_a^\dagger)\,.$$

Then

$$i\theta_a^\dagger\dot{\theta}_a=\frac{i}{2}\theta_{ai}\dot{\theta}_{ai}-\frac{i}{2}\frac{d}{d\tau}(\theta_{a1}\theta_{a2})$$

and

$$\theta_a^\dagger T_{ab}^\alpha\theta_b=\tfrac{1}{2}\theta_{ai}T_{aibj}^\alpha\theta_{bj}\,,$$

with the following antisymmetric matrix T_{aibj}^α:

$$T_{aibj}^\alpha=\frac{1}{2}\begin{pmatrix}T_{ab}^\alpha-T_{ba}^\alpha & iT_{ab}^\alpha+iT_{ba}^\alpha\\ -iT_{ab}^\alpha-iT_{ba}^\alpha & T_{ab}^\alpha+T_{ba}^\alpha\end{pmatrix}.$$

Omitting a total time derivative we therefore have a Lagrangian of the form of Eq. (3.2).

We now pass to the Hamiltonian formalism from the Lagrangian L_2. The treatment of anticommuting variables has been investigated by Casalbuoni.[4] Following his methods, we introduce the conjugate momenta

$$P_\mu=\frac{\partial L_2}{\partial\dot{x}^\mu}=-\frac{m\dot{x}_\mu}{(\dot{x}^\nu\dot{x}_\nu)^{1/2}}-gA_\mu^\alpha I^\alpha\,,\qquad P_a=\frac{\partial L}{\partial\dot{\theta}_a}=-\frac{i}{2}\theta_a\,. \qquad (3.4)$$

This gives the constraints

$$\chi_1 \equiv P_a + \frac{i}{2}\theta_a = 0 \,,$$

$$\chi_2 \equiv \Pi_\mu \Pi^\mu - m^2 = 0 \,,$$

where Π_μ and I^α are as given by Eqs. (2.12) and (3.3), respectively. The Hamiltonian becomes

$$H = \tfrac{1}{2} C(\Pi_\mu \Pi^\mu - m^2) + C_a\left(P_a + \frac{i}{2}\theta_a\right) ,$$

where C_a is an anticommuting function of the dynamical variables.

We define the following "Poisson" bracket:

$$\{A, B\} = A\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\partial}{\partial P_\mu} - \frac{\overleftarrow{\partial}}{\partial P_\mu}\frac{\partial}{\partial x^\mu} - \frac{\overleftarrow{\partial}}{\partial \theta_a}\frac{\partial}{\partial P_a} - \frac{\overleftarrow{\partial}}{\partial P_a}\frac{\partial}{\partial \theta_a}\right)B \,. \tag{3.5}$$

This Poisson bracket has the following properties:

(i) It gives the time evolution of any function F of the dynamical variables as

$$\dot{F} = \{F, H\} \,. \tag{3.6}$$

(ii) It has the correct symmetry, that is $\{A, B\} = \pm\{B, A\}$, depending on whether A and B both are anticommuting or not.

The Poisson brackets of the constraints are

$$\left\{P_a + \frac{i}{2}\theta_a, P_b + \frac{i}{2}\theta_b\right\} = -i\delta_{ab} \,,$$

$$\left\{\tfrac{1}{2}(\Pi_\mu\Pi^\mu - m^2), P_a + \frac{i}{2}\theta_a\right\} = -i\Pi^\mu g A_\mu^\alpha \theta_b T_{ba}^\alpha \,.$$

This means that there is only one first-class constraint, namely

$$\tfrac{1}{2}(\Pi_\mu\Pi^\mu - m^2) - \Pi^\mu g A_\mu^\alpha \theta_a T_{ab}^\alpha\left(P_b + \frac{i}{2}\theta_b\right) = 0 \,.$$

We then change the Poisson brackets to the "Dirac" brackets,

$$\{A, B\}_D = \{A, B\} - \sum_a \{A, \chi_a\}\{\chi_a, \chi_a\}^{-1}\{\chi_a, B\} ,$$

and eliminate n variables, P_a. After this elimination we have the Dirac bracket

$$\{A, B\}_D = A\left(\frac{\overleftarrow{\partial}}{\partial x^\mu}\frac{\partial}{\partial P_\mu} - \frac{\overleftarrow{\partial}}{\partial P_\mu}\frac{\partial}{\partial x^\mu} - i\frac{\overleftarrow{\partial}}{\partial \theta_a}\frac{\partial}{\partial \theta_b}\right)B . \tag{3.7}$$

When quantizing, these Dirac brackets give the commutation and anticommutation relations

$$\begin{aligned} &[P_\mu, x^\nu]_- = -i\hbar\delta_\mu{}^\nu \,, \\ &[\theta_a, \theta_b]_+ = \hbar\delta_{ab} \,. \end{aligned} \tag{3.8}$$

and the first-class constraint gives the wave equation

$$\left[\left(-i\hbar\frac{\partial}{\partial x^\mu} + g A_\mu^\alpha \tfrac{1}{2}\theta_a T_{ab}^\alpha \theta_b\right)^2 - m^2\right]\psi(x) = 0 \,. \tag{3.9}$$

In this case, since the operators θ_a do not commute, it is impossible to represent them by diagonal matrices. But instead the anticommutation relations (3.8) and the finite number of the variables θ_a permit us to represent them by finite-dimensional matrices. The wave function in (3.9) can therefore be considered to consist of a finite number of components.

In contrast to the case in Sec. II the wave function in Eq. (3.4) can for each choice of matrices T^α only describe particles with a finite number of different isospins. In Sec. V we investigate which representations are generated by an arbitrary matrix representation T^α of a unitary group.

IV. INCLUSION OF PSEUDOCLASSICAL SPIN

Our formalisms for describing a classical particle with classical or pseudoclassical isospin can also be combined with the formalism for pseudoclassical spin[5,6] to describe a particle with both spin and isospin. The Lagrangian in the second-order formalism is

$$\begin{aligned} L &= L_0 + L_m + L_I \,, \\ L_0 &= -\frac{1}{2e}\dot{x}^\nu(\dot{x}_\nu - i\chi\psi_\nu) - \frac{i}{2}\psi^\nu\dot{\psi}_\nu \,, \\ L_m &= -\frac{e}{2}m^2 + \frac{i}{2}(\psi_5\dot{\psi}_5 + m\chi\psi_5) \,, \\ L_I &= \tfrac{1}{2}i\theta_a\dot{\theta}_a - g I^\alpha\left(A_\mu^\alpha\dot{x}^\mu - \frac{i}{2}e\psi^\mu F_{\mu\nu}^\alpha\psi^\nu\right) . \end{aligned} \tag{4.1}$$

Here, for definiteness, we have chosen to describe the isospin by Hermitian anticommuting variables; the other alternatives give only trivial modifications. This Lagrangian differs minimally from the one describing a pseudospinning particle interacting with an external electromagnetic field considered in Ref. 6. The new dynamical variables as compared to the model in Sec. III are e, χ, ψ_5, and ψ_ν. Of these, only the last one is physical, and it is connected with the pseudoclassical spin. The variables χ, ψ_5, ψ_ν, and θ_a are anticommuting, e and x^μ are commuting.

This Lagrangian satisfies three of the consistency requirements listed in Sec. II. Namely, it is Hermitian and manifestly gauge invariant, and the action $\int L d\tau$ is reparametrization invariant if a change of parameter $\tau \to \tau'$ is accompanied by the variable transformations

$$e \to e' = e\,\frac{d\tau}{d\tau'}\ ,$$
$$\chi \to \chi' = \chi\,\frac{d\tau}{d\tau'}\ , \tag{4.2}$$
$$x^\mu,\ \psi^\mu,\ \psi_5,\ \theta_a \text{ unchanged.}$$

The equations of motion are modified by the spin interaction as shown below. In addition, each of the three pieces of the Lagrangian in Eq. (4.1) is invariant up to a total time derivative under the following infinitesimal supersymmetry transformation generated by an anticommuting variable $\alpha(\tau)$:

$$\begin{aligned}
&\delta\chi^\nu = i\alpha\psi^\nu\ ,\\
&\delta\psi^\nu = -\alpha(\dot{x}^\nu - \tfrac{1}{2}i\chi\psi^\nu)/e\ ,\\
&\delta e = -i\alpha\chi\ ,\\
&\delta\chi = 2\dot{\alpha}\ ,\\
&\delta\psi_5 = m\alpha\ ,\\
&\delta\theta_a = g\alpha\psi^\mu A^\alpha_\mu T^\alpha_{ab}\theta_b\ .
\end{aligned} \tag{4.3}$$

This invariance is essential for eliminating the nonphysical degree of freedom ψ_0.

Let us now consider the equations of motion. Varying the variables e, χ, and ψ_5 gives the constraint equations

$$\begin{aligned}
&(\dot{x}^2 - i\chi x^\mu\psi_\mu)/e^2 - m^2 + igI^\alpha\psi^\mu F^\alpha_{\mu\nu}\psi^\nu = 0,\\
&(\dot{x}^\mu\psi_\mu + m\psi_5)/e = 0,\\
&2\dot{\psi}_5 - m\chi = 0.
\end{aligned}$$

We may choose a parametrization in which $e = 1/m$, and make a supersymmetry transformation to get $\chi = 0$. Then by the third constraint, $\dot{\psi}_5 = 0$ so that ψ_5 can be eliminated by a τ independent supersymmetry transformation. The two remaining constraints and the rest of the equations of motion now take the form

$$\begin{aligned}
&m^2\dot{x}^2 - m^2 + igI^\alpha\psi^\mu F^\alpha_{\mu\nu}\psi^\nu = 0,\\
&\dot{x}^\mu\psi_\mu = 0,\\
&\dot{\psi}_\mu - \frac{g}{m}I^\alpha F^\alpha_{\mu\nu}\psi^\nu = 0,\\
&\dot{\theta}_a + ig\left(A^\alpha_\mu\dot{x}^\mu - \frac{i}{2m}\psi^\mu F^\alpha_{\mu\nu}\psi^\nu\right)T^\alpha_{ab}\theta_b = 0,\\
&m\ddot{x}_\mu - gI^\alpha\left(F^\alpha_{\mu\nu}\dot{x}^\nu - \frac{i}{2m}\psi^\rho D^{\alpha\beta}_\mu F^\beta_{\rho\sigma}\psi^\sigma\right) = 0.
\end{aligned} \tag{4.4}$$

This is a manifestly gauge-covariant set of equations. In deriving the last equation, use has been made of the equation of motion for the isospin which follows from the equation of motion for θ_a,

$$\dot{I}^\alpha - gf_{\alpha\beta\gamma}\left(A^\beta_\mu\dot{x}^\mu - \frac{i}{2m}\psi^\mu F^\beta_{\mu\nu}\psi^\nu\right)I^\gamma = 0. \tag{4.5}$$

For interpretation of the equations it is useful to make a nonrelativistic approximation. We introduce the following notations:

$$\begin{aligned}
&\dot{x}^\nu = (\dot{x}_0, \vec{v}),\\
&\psi^\nu = (\psi_0, \vec{\psi}),\\
&-\frac{i}{2}\psi_i\psi_j\epsilon_{ijk} = s_k \quad (i,j,k = 1,2,3),\\
&F^\alpha_{i0} = -E^\alpha_i \quad (i = 1,2,3),\\
&F^\alpha_{ij} = -\epsilon_{ijk}B^\alpha_k \quad (i,j,k = 1,2,3),\\
&\partial_\nu = (\partial_0, \vec{\nabla})
\end{aligned} \tag{4.6}$$

In the limit $\vec{v}/\dot{x}_0 \approx 0$ the second of Eqs. (4.4) tells us that $\psi_0 \approx 0$. Making also a gauge transformation such that A^ν can be neglected locally we get the following approximate equations of motion from (4.4):

$$\begin{aligned}
&\dot{\vec{s}} = \frac{g}{m}I^\alpha\,\vec{s}\times\vec{B}^\alpha,\\
&\dot{I}^\alpha = -gf_{\alpha\beta\gamma}\frac{1}{m}\,\vec{s}\cdot\vec{B}^\beta I^\gamma,\\
&m\dot{\vec{v}} = gI^\alpha(\vec{E}^\alpha + \vec{v}\times\vec{B}^\alpha) + \frac{g}{m}\vec{\nabla}\,I^\alpha\vec{B}^\alpha\cdot\vec{s}.
\end{aligned} \tag{4.7}$$

From the last equation it follows that the particle has a magnetic moment $(g/m)\vec{s}$. Thus if $\vec{s}$ is its intrinsic angular momentum it has gyromagnetic ratio two. (It follows from the quantization below that $\vec{s}$ and not, for example, $2\vec{s}$ is the spin. An alternative way to see this would be to derive the angular momentum from Noether's theorem.)

The isospin current of the particle which enters as the source in the equations of motion for the gauge field,

$$D^{\alpha\beta}_\mu F^{\beta\mu\nu} = J^{\alpha\nu}, \tag{4.8}$$

is

$$J^{\alpha\nu}(x) = g\int d\tau\left(I^\alpha\dot{x}^\nu + i\psi^\mu\psi^\nu I^\alpha\frac{1}{m}\frac{\partial}{\partial x^\mu} - i\psi^\mu\psi^\nu I^\gamma\frac{1}{m}gf_{\alpha\beta\gamma}A^\beta_\mu\right)\delta^4(x - x(\tau)). \tag{4.9}$$

For Eq. (4.8) to be consistent the covariant derivative of $J^{\alpha\nu}$ must vanish. Using Eqs. (4.4) and (4.5) this can indeed be verified. The vanishing of the covariant derivative of $J^{\alpha\nu}$ can also be expressed as the current conservation equation

$$\partial_\nu (J^{\alpha\nu} + g f_{\alpha\beta\gamma} A^\beta_\mu F^{\gamma\mu\nu}) = 0 \,. \qquad (4.10)$$

Concerning the physical interpretation of the different components of the current (4.9) we notice that the first and the second terms correspond to a Gordon decomposition. This corresponds physically to a decomposition of the current into a part associated with the moving charge and a part which describes the intrinsic spin content of the particle. The last term is needed to make the current gauge covariant. At a given point it can always be gauge-transformed away. The current $J^{\alpha\nu}$ can also be shown to coincide with the Noether current corresponding to the gauge symmetry up to the divergence of an antisymmetric Lorentz tensor.

The passage from the Lagrangian (4.1) to the Hamiltonian formalism presents no new problems. The conjugate momenta are

$$\begin{aligned}
&P_{x\mu} = -(\dot{x}_\mu - \tfrac{1}{2} i \chi \psi_\mu)/e - g I^\alpha A^\alpha_\mu, \\
&P_{\psi\mu} = \tfrac{1}{2} i \psi_\mu, \\
&P_e = 0, \\
&P_\chi = 0, \\
&P_5 = -\tfrac{1}{2} i \psi_5, \\
&P_a = -\tfrac{1}{2} i \theta_a \,.
\end{aligned}$$

The Hamiltonian becomes

$$\begin{aligned}
H = &-\tfrac{1}{2} e (\Pi_\mu \Pi^\mu - m^2 + \tfrac{1}{2} i g e \psi^\mu F^\alpha_{\mu\nu} \psi^\nu) \\
&+ \tfrac{1}{2} i \chi (\psi^\mu \Pi_\mu - m\psi_5) + C_e P_e + C_\chi P_\chi + C_5 (P_5 + \tfrac{1}{2} i \psi_5) \\
&+ C_{\psi\mu} (P^\mu_\psi - \tfrac{1}{2} i \psi^\mu) + C_a (P_a + \tfrac{1}{2} i \theta_a).
\end{aligned}$$

As usual, the constraints involving $P_{\psi\mu}$, P_5, and P_a which arise because the Lagrangian is of first order in the anticommuting variables ψ_μ, ψ_5, and θ_a have nonzero Poisson brackets with themselves and are of second class. Changing the Poisson brackets to Dirac brackets and eliminating these momenta gives a Hamiltonian

$$\begin{aligned}
&H = -\tfrac{1}{2} e \phi_{KG} + \tfrac{1}{2} i \chi \phi_D + C_e P_e + C_\chi P_\chi, \\
&\phi_{KG} \equiv \Pi_\mu \Pi^\mu - m^2 + \tfrac{1}{2} i g e \psi^\mu F^\alpha_{\mu\nu} \psi^\nu I^\alpha, \\
&\phi_D \equiv \psi^\mu \Pi_\mu - m \psi_5 .
\end{aligned} \qquad (4.11)$$

The Dirac brackets are now given by

$$\begin{aligned}
\{A, B\}_D = A \Big(&\frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\partial}{\partial P_\mu} - \frac{\overleftarrow{\partial}}{\partial P_\mu} \frac{\partial}{\partial x^\mu} + \frac{\overleftarrow{\partial}}{\partial e} \frac{\partial}{\partial P_e} \\
&- \frac{\overleftarrow{\partial}}{\partial P_e} \frac{\partial}{\partial e} - i \frac{\overleftarrow{\partial}}{\partial \psi_5} \frac{\partial}{\partial \psi_5} - \frac{\overleftarrow{\partial}}{\partial \chi} \frac{\partial}{\partial P_\chi} \\
&- \frac{\overleftarrow{\partial}}{\partial P_\chi} \frac{\partial}{\partial \chi} - i \frac{\overleftarrow{\partial}}{\partial \theta_a} \frac{\partial}{\partial \theta_a} + i \frac{\overleftarrow{\partial}}{\partial \psi_\mu} \frac{\partial}{\partial \psi^\mu} \Big) B .
\end{aligned} \qquad (4.12)$$

The nonzero Dirac brackets of the Hamiltonian and the constraints are

$$\begin{aligned}
&\{P_e, H\}_D = \tfrac{1}{2} \phi_{KG}, \\
&\{P_\chi, H\}_D = -\tfrac{1}{2} \phi_D, \\
&\{\phi_D, \phi_D\}_D = i \phi_{KG}.
\end{aligned} \qquad (4.13)$$

It can now be seen that the spin operator defined in Eq. (4.6) satisfies the angular momentum commutation relations as promised in the discussion of the gyromagnetic ratio. Since it is permissible to add also arbitrary multiples of the secondary first-class constraints to the Hamiltonian,[3] we can choose it to be identically zero so that all variables are constant in τ. The γ matrices afford a representation of the anticommuting spin variables[6]:

$$\begin{aligned}
&\psi_5 = (\tfrac{1}{2}\hbar)^{1/2} \gamma_5, \\
&\psi_\mu = (\tfrac{1}{2}\hbar)^{1/2} \gamma_5 \gamma_\mu .
\end{aligned} \qquad (4.14)$$

In this representation the constraint ϕ_D gives the Dirac equation

$$[\gamma^\mu (i\hbar \partial_\mu - g I^\alpha A^\alpha_\mu) - m] \psi(x) = 0. \qquad (4.15)$$

V. ISOSPIN CONTENT OF THE WAVE FUNCTION

We have constructed the generators of the gauge group G in three different ways. For T^α_{ij} belonging to some irreducible representation R of the Lie algebra of G (we put $\hbar = 1$)

(a) $I^\alpha = \tfrac{1}{2} \theta_i T^\alpha_{ij} \theta_j$, with $[\theta_i, \theta_j]_+ = \delta_{ij}$, $\theta^\dagger_i = \theta_i$

(b) $I^\alpha = \theta^\dagger_i T^\alpha_{ij} \theta_j$, with $[\theta^\dagger_i, \theta_j]_+ = \delta_{ij}$

(c) $I^\alpha = \theta^\dagger_i T^\alpha_{ij} \theta_j$, with $[\theta^\dagger_i, \theta_j] = \delta_{ij}$.

For case (a) the generators T^α_{ij} must be antisymmetric, whereas no symmetry is required in the other two cases. However, in case (c) the method given below works only for antisymmetric generators. In this case it is simpler to note, as briefly mentioned before, that if we represent $\theta^\dagger_i$ by $\delta/\delta\theta^i$ we get

$$\psi = \psi_i \theta_i + \psi_{ij} \theta_i \theta_j + \cdots, \qquad (5.1)$$

with ψ_{ij} symmetric matrices and

$$\begin{aligned}
I^\alpha &= \theta^\dagger_i T^\alpha_{ij} \theta_j \\
&= \frac{\partial}{\partial \theta^i} T^\alpha_{ij} \theta_j, \\
I^\alpha \psi &= \mathrm{Tr}\, T^\alpha \psi + T^\alpha_{ij} \theta_j \frac{\partial}{\partial \theta_i} \psi \\
&= T^\alpha_{ij} \psi_i \theta_j + T^\alpha_{ij} (\psi_{il} \theta_j \theta_l + \psi_{xi} \theta_k \theta_j) + \cdots,
\end{aligned}$$

with

$$\mathrm{Tr}\, T^\alpha = 0. \qquad (5.2)$$

Thus

$$\psi_i' = T^{\alpha}_{ij}\,\psi_i, \quad \psi_{ij}' = T^{\alpha}_{ki}\psi_{kj} + T^{\alpha}_{kj}\psi_{ki} \tag{5.3}$$

and so on, so that the nth-rank symmetric tensor transforms as the symmetric nth tensor product of R.

We have now to consider the other two constructions, (a) and (b). The main point in our method is the following: Let the gauge group be G with Lie algebra L_G. For T^{α}_{ij} belonging to a given representation R of L_G, the generators I^{α} can be considered as an L_G subalgebra of an orthogonal algebra $L_{\mathrm{SO}(n)}$, where n depends on R and on which of the constructions (a) or (b) we have chosen. We can then take the wave function to belong to an irreducible representation of $L_{\mathrm{SO}(n)}$, since different such representations will never be mixed. Knowing the eigenvalues of this latter representation, it is a simple matter to find, by a construction given below, the eigenvalues of I^{α} and thus the reduction to irreducible representations of L_G.

We first show that I^{α} are generators of an orthogonal group. For the two different constructions we have the following:

(a) T^{α}_{ij} are the generators of G in some representation. They are antisymmetric (by assumption) $n\times n$ matrices, where n depends on the representation chosen. I^{α} can then be written as $\sum_{i<j} T^{\alpha}_{ij}\theta_i\theta_j$, $i,j=1,2,\ldots,n$. According to the Clifford algebra theory of the spin representation of an orthogonal group,[7] we have

$$\theta_i\theta_j = -iL_{ij}\ , \tag{5.4}$$

i.e., I^{α} form an L_G subalgebra of $L_{\mathrm{SO}(n)}$. The wave function belongs to the spin representation of $\mathrm{SO}(n)$, i.e., it is a spinor which can be expressed in terms of the θ's.

(b) T^{α}_{ij} are now $n\times n$ matrices with no definite symmetry. We shall assume that they are traceless, i.e., that G is a group of matrices of determinant 1. Then

$$\begin{aligned} T_{ij} &= \tfrac{1}{2}(T_{ij}+T_{ji}) + \tfrac{1}{2}(T_{ij}-T_{ji}) \\ &= T^S_{ij} + T^A_{ij} \end{aligned} \tag{5.5}$$

and we can write I^{α} as

$$I^{\alpha} = \sum_{i<j} T^{\alpha S}_{ij}(\theta_i^*\theta_j + \theta_j^*\theta_i) + \sum_{i<j} T^{\alpha A}_{ij}(\theta_i^*\theta_j - \theta_j^*\theta_i)\,. \tag{5.6}$$

Again from the Clifford algebra theory of spin representations we find

$$\theta_i^*\theta_j + \theta_j^*\theta_i = -L_{2i,2j-1} + L_{2i-1,2j} + \delta_{ij}\ , \tag{5.7}$$

$$\theta_i^*\theta_j - \theta_j^*\theta_i = -iL_{2i-1,2j-1} - iL_{2i,2j}\ , \tag{5.8}$$

and, since T^{α}_{ij} is traceless, I^{α} form an L_G subalgebra of $L_{\mathrm{SO}(2n)}$. The wave function is again a spinor of this group. We note that for anticommuting θ these form a Clifford algebra. The only minimal (irreducible) ideals of this algebra are the spinors or, for even orthogonal groups, the even and odd half-spinors. For commuting θ, on the other hand, we can form from θ_i the full symmetric tensor algebra and thus get all nonspin representations, i.e., all irreducible tensor representations.

Let us now apply this to the cases where the gauge group G is SU(2), SU(3), or SU(4). We first choose T^{α}_{ij} to belong to a given representation of L_G. This we call the generating representation R_g. For SU(2) we will be able to give a general result for any R_g, whereas for the higher groups we study only the fundamental and adjoint representations. The method is in principle simple to apply for other R_g as well, but the calculations become too involved to be motivated by the interest the result could have.

We start with $G=\mathrm{SU}(2)$. For the case where R_g is an integer-spin representation, we can write T^3_{ij} as a block-diagonal antisymmetric matrix

$$T^3_{ij} = \begin{bmatrix} 0 & & & & & & & \\ & 0 & i & & & & & \\ & -i & 0 & & & & & \\ & & & 0 & 2i & & & \\ & & & -2i & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & li \\ & & & & & & -li & 0 \end{bmatrix}. \tag{5.9}$$

T^1_{ij}, T^2_{ij} are also antisymmetric. Therefore, we can consider both constructions and find the following:

$$\text{(a)}\ I^3 = L_{23} + 2L_{45} + \cdots + lL_{2l,2l+1}\,. \tag{5.10}$$

The wave function is an $\mathrm{SO}(2l+1)$ spinor belonging to the representation (see Ref. 8) ${}^{2l+1}D_{\frac{1}{2}\cdots\frac{1}{2}}$ of 2^l components. The eigenvalues of the simultaneously diagonalizable generators $L_{2i,2i+1}$ are $\pm\frac{1}{2}$. Thus the eigenvalues of I^3 can be readily calculated and assembled in irreducible representations of SU(2). As an example we take $l=2$. Then

$$I^3 = L_{23} + 2L_{45} \tag{5.11}$$

TABLE I. For a given representation R_g of the Lie algebra of SU(2) generated by matrices T^{α}_{ab}, and a spinor representation of an orthogonal group generated by Grassmann variables θ_a, the reduction to irreducible components of the SU(2) Lie algebra representation given by (a) $I^{\alpha}=\frac{1}{2}\theta_a T^{\alpha}_{ab}\theta_b$ and (b), (c) $I^{\alpha}=\theta^{\dagger}_a T^{\alpha}_{ab}\theta_b$ is shown. The isospin l of the representation R_g of SU(2) is related to the dimension of the orthogonal group as (a) $2l+1$, and (b), (c) $2(2l+1)$.

(a)

$G=\mathrm{SU}(2)$ R_g: integer spin $[\theta_i,\theta_j]_+=\delta_{ij}$

l	Orthogonal group	Spinor components	Reduction
1	SO(3)	2	$\underline{\tfrac{1}{2}}$
2	SO(5)	4	$\underline{\tfrac{3}{2}}$
3	SO(7)	8	$\underline{3}+\underline{0}$
4	SO(9)	16	$\underline{5}+\underline{2}$
5	SO(11)	32	$\underline{\tfrac{15}{2}}+\underline{\tfrac{9}{2}}+\underline{\tfrac{5}{2}}$

(b)

$G=\mathrm{SU}(2)$ R_g: integer spin $[\theta^{\dagger}_i,\theta_j]_+=\delta_{ij}$

l	Orthogonal group	Spinor components	Reduction
1	SO(6)	4	$\underline{1}+\underline{0}$
2	SO(10)	16	$\underline{3}+\underline{2}+\underline{1}+\underline{0}$
3	SO(14)	64	$\underline{6}+\underline{5}+\underline{4}+3\times\underline{3}+\underline{2}+\underline{1}+2\times\underline{0}$

(c)

$G=\mathrm{SU}(2)$ R_g: half-integer spin $[\theta^{\dagger}_i,\theta_j]_+=\delta_{ij}$

l	Orthogonal group	Spinor components	Spinor	Reduction
$\frac{1}{2}$	SO(4)	2	Even	$\underline{\tfrac{1}{2}}$
			Odd	$2\times\underline{0}$
$\frac{3}{2}$	SO(8)	8	Even	$\underline{2}+3\times\underline{0}$
			Odd	$2\times\underline{\tfrac{3}{2}}$
$\frac{5}{2}$	SO(12)	32	Even	$\underline{\tfrac{9}{2}}+3\times\underline{\tfrac{5}{2}}+\underline{\tfrac{3}{2}}$
			Odd	$2\times\underline{4}+2\times\underline{2}+4\times\underline{0}$

and the eigenvalues of ${}^5D_{\frac{1}{2}\frac{1}{2}}$ are for (L_{23},L_{45}) the four combinations $(\frac{1}{2},\frac{1}{2})$, $(-\frac{1}{2},+\frac{1}{2})$, $(+\frac{1}{2},-\frac{1}{2})$, and $(-\frac{1}{2},-\frac{1}{2})$. Thus the eigenvalues of I^3 are $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, and $-\frac{3}{2}$ and we get the only irreducible representation $l=\frac{3}{2}$. In (a) of Table I we give the reduction for other R_g.

$$\text{(b) } I^3=L_{35}+L_{46}+2L_{79}+2L_{8,10}+\cdots+lL_{4l-1,4l+1}+lL_{4l,4l+2}\,. \tag{5.12}$$

L_{12}, which is diagonalizable together with the generators that enter in I^3, does not occur here. It does, however, occur in I^1 or I^2 and thus the wave function is an SO($4l+2$) spinor belonging to the even representation ${}^{4l+2}D_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}\frac{1}{2}}$ or the odd representation ${}^{4l+2}D_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}-\frac{1}{2}}$. The eigenvalues of the generators $L_{12}, L_{35}, L_{46},\ldots,L_{4l,4l+2}$ are $\pm\frac{1}{2}$ with an even (odd) number of minus signs in the even (odd) representation. Thus we find for R_g with $l=2$ that

$$I^3=L_{35}+L_{46}+2L_{79}+2L_{8,10}\,, \tag{5.13}$$

and the eigenvalues are, for the even representation, 3, 2, 2, 1, 1, 1, 0, 0, 0, 0, −1, −1, −1, −2, −2, −3. The reduction is $\underline{3}+\underline{2}+\underline{1}+\underline{0}$. For the odd representation we always get the same result, since the absent generator L_{12} absorbs the difference in the number of negative eigenvalues. In (b) of Table I we give the reduction for other R_g.

We next take R_g to be a half-integer-spin representation. Then, all three generators T^{α}_{ij} cannot be chosen antisymmetric, and we have only one possible construction, (b). We can still choose T^3_{ij} block-diagonal, i.e., for R_g of spin s we get

TABLE II. Examples of the same reduction as in Table I, but with SU(2) replaced by SU(3).

R_g	Orthogonal group	Spinor components	Spinor	Reduction
$\underline{3}$	SO(6), $\theta \neq \theta^\dagger$	4	Odd	$\underline{\bar{3}}+\underline{0}$
			Even	$\underline{3}+\underline{0}$
$\underline{\bar{3}}$	SO(6), $\theta \neq \theta^\dagger$	4	Odd	$\underline{3}+\underline{0}$
			Even	$\underline{\bar{\bar{3}}}+\underline{0}$
$\underline{8}$	SO(8), $\theta = \theta^\dagger$	8	Odd/even	$\underline{8}$
$\underline{8}$	SO(16), $\theta \neq \theta^\dagger$	128	Odd/even	$2\times(\underline{27}+\underline{10}+\underline{10}+\underline{8}+\underline{8}+\underline{0})$

$$T^3_{ij}=\begin{bmatrix} 0 & \frac{1}{2}i & & & & & & \\ -\frac{1}{2}i & 0 & & & & & & \\ & & 0 & \frac{3}{2}i & & & & \\ & & -\frac{3}{2}i & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & 0 & si \\ & & & & & & -si & 0 \end{bmatrix}. \tag{5.14}$$

This gives

$$I^3=\tfrac{1}{2}L_{13}+\tfrac{1}{2}L_{24}+\tfrac{3}{2}L_{57}+\tfrac{3}{2}L_{68}+\cdots$$
$$+sL_{4s-1,4+1}+sL_{4s,4s+2}\,. \tag{5.15}$$

The wave function is an SO$(4s+2)$ spinor belonging to the even representation ${}^{4s+2}D_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}}$ or the odd representation ${}^{4s+2}D_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}-\frac{1}{2}}$.

For R_g with $s=\frac{3}{2}$ we get

$$I^3=\tfrac{1}{2}L_{13}+\tfrac{1}{2}L_{24}+\tfrac{3}{2}L_{57}+\tfrac{3}{2}L_{68} \tag{5.16}$$

and the eigenvalues are, for the even representation, $2, 1, 0, 0, 0, 0, -1, -2$, giving the reduction $\underline{2}+3\times\underline{0}$. For the odd representation we get the eigenvalues $\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}$, giving the reduction $2\times\underline{\frac{3}{2}}$. The reductions for other R_g are given in (c) of Table I. This completes the study of SU(2). For SU(3) and SU(4) we will consider only the few lowest representations. With $G=\mathrm{SU}(3)$ we get the following:

If R_g is the fundamental representation $\underline{3}$ or $\underline{\bar{3}}$, only case (b) is possible. Then we can write the diagonal operators T^3 and T^8 as (we forget about normalization)

$$T^3=\begin{pmatrix} \frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 0 \end{pmatrix}, \quad T^8=\begin{pmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & -\frac{2}{3} \end{pmatrix}. \tag{5.17}$$

Then

$$I^3=\tfrac{1}{2}L_{12}-\tfrac{1}{2}L_{34}\,, \tag{5.18}$$

$$I^8=\tfrac{1}{3}L_{12}+\tfrac{1}{3}L_{34}-\tfrac{2}{3}L_{56}\,. \tag{5.19}$$

The wave function is an even or odd SO(6) spinor. We find in the former case the eigenvalues of I^3 to be $\frac{1}{2}, -\frac{1}{2}$ and twice zero, and those of I^8 to be $0, -\frac{2}{3}, \frac{1}{3}$, and $\frac{1}{3}$. Thus, the even spinor decomposes into $\underline{3}+\underline{0}$.

If R_g is the adjoint representation, we take $T^\alpha_{ij}=if_{\alpha ij}$, which are antisymmetric, and we get for case (a)

$$I^3=L_{12}+\tfrac{1}{2}L_{45}-\tfrac{1}{2}L_{67}\,, \tag{5.20}$$

$$I^8=L_{45}+L_{67}\,, \tag{5.21}$$

with the wave function an SO(8) spinor, and for case (b)

$$I^3=L_{13}+L_{24}+\tfrac{1}{2}L_{79}+\tfrac{1}{2}L_{8,10}$$
$$-\tfrac{1}{2}L_{11,13}-\tfrac{1}{2}L_{12,14}\,, \tag{5.22}$$

$$I^8=L_{79}+L_{8,10}+L_{11,13}+L_{12,14}\,, \tag{5.23}$$

with the wave function an SO(16) spinor. The reduction for all these cases is given in Table II.

Finally, for $G=\mathrm{SU}(4)$ we have considered the fundamental and antisymmetric 6 representations in a way that completely follows the treatment of SU(3). The result is given in Table III. There is one property that is particular to SU(4). This stems from the fact that $L_{\mathrm{SU}(4)}\sim L_{\mathrm{SO}(6)}$. Therefore there should be one R_g such that the spinors of SO(6), which are four-component, give the fundamental representations of SU(4). The generating representation is $\underline{6}$, which can be given in terms of antisymmetric 6×6 matrices such that I^α coin-

TABLE III. Examples of the same reduction as in Table I, but with SU(2) replaced by SU(4).

R_g	Orthogonal group	Spinor components	Spinor	Reduction
$\underline{4}$	SO(8), $\theta \neq \theta^\dagger$	8	Even	$\underline{6}+\underline{0}+\underline{0}$
			Odd	$\underline{4}+\underline{\bar{4}}$
$\underline{\bar{4}}$	SO(8), $\theta \neq \theta^\dagger$	8	Even	$\underline{\bar{6}}+\underline{0}+\underline{0}$
			Odd	$\underline{4}+\underline{\bar{4}}$
$\underline{6}$	SO(6), $\theta = \theta^\dagger$	4	Even	$\underline{4}$
			Odd	$\underline{\bar{4}}$
$\underline{15}$	SO(15), $\theta = \theta^\dagger$	128		$2\times\underline{64}$

cides with the generators of SO(6). The even and odd spinors then correspond to $\underline{4}$ and $\underline{\bar{4}}$, respectively.

VI. FINAL REMARKS

Wong[1] arrived at his equations of motion for a spinless particle by applying semiclassical considerations to a quantum Hamiltonian. This is not inconsistent with our result since if one arrives first at a particle with pseudoclassical spin, one may neglect this spin in taking the purely classical limit. In that case it would, however, have been more appropriate to start from the Klein-Gordon equation. In fact, if one considers a boson field in the presence of an external gauge field and carries through the same type of arguments as in Ref. 1 one obtains Wong's equations by keeping terms up to order $\hbar$.

The pseudoclassical spin has length zero since $\vec{S}\cdot\vec{S}=0$. This is understandable since the quantum spin is seldom larger than a few multiples of $\hbar$ and in our case the quantum spin is of length $\vec{S}\cdot\vec{S}=(\frac{3}{4})^{1/2}\hbar$. Nevertheless, in the classical equations of motion one cannot distinguish $\vec{S}$ from a classical vector. We suggest that in practice it may be useful to replace it by a classical vector with some finite length. In this way, one can describe either a composite particle with large rotational angular momentum or a single elementary particle in a strong magnetic field. The same suggestion is made for the pseudoclassical isospin.

It is interesting to notice that the correct gyromagnetic ratio already occurs at the pseudoclassical level.

Finally, we remark that the pseudoclassical isospin operators for a given irreducible generating representation in the quantized theory usually belong to a reducible representation of the Lie algebra of the gauge group.

Added note. After the completion of the present paper we received a report by A. Barducci, R. Casalbuoni, and L. Lusanna[9] where a discussion similar to ours of a Yang-Mills particle has been carried through. The authors wish to thank P. Di Vecchia for drawing our attention to this work.

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