Existence of bound states for a charged spin-1/2 particle with an extra magnetic moment in the field of a fixed magnetic monopole*

Yoichi Kazama and Chen Ning Yang
Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794
(Received 23 December 1976)

Bound-state wave functions are found for a Dirac particle of spin 1/2 with an extra magnetic moment in the field of an infinitely heavy magnetic monopole.

Consider a Dirac particle with spin $\frac{1}{2}$, charge Ze, and magnetic moment

$$\frac{Ze}{2M}(1+\kappa) \quad (\kappa \neq 0), \tag{1}$$

in the field of a fixed magnetic monopole of strength $g \neq 0$. According to Dirac[1]

$$q = Zeg \neq 0 \tag{2}$$

must be equal to $\frac{1}{2}$ times an integer. Except in Sec. IX we shall not inquire into the origin of the extra magnetic moment, but take it as a given constant.

It may appear at first sight that because of the extra magnetic moment such a system has no well-defined Hamiltonian. Upon closer examination this turns out to be not correct. We shall show that in fact the Hamiltonian of the system is well defined and possesses bound states. The method used in this analysis is straightforward: For the angular wave function we use the concept of sections.[2] For the radial wave function we follow generally the classical ideas of the Sturm-Liouville theory.

I. TWO TYPES OF RADIAL WAVE FUNCTIONS

The Hamiltonian of the system is

$$H = \vec{\alpha}\cdot(\vec{p} - Ze\vec{A}) + \beta M - \kappa q \beta \vec{\sigma}\cdot\vec{r}(2Mr^3)^{-1}. \tag{3}$$

To avoid singularities in the vector potential, the wave function ψ should be treated as a section.[2] H commutes with[3] the total angular momentum

$$\vec{J} = \vec{r}\times(\vec{p} - Ze\vec{A}) - q\vec{r}r^{-1} + \tfrac{1}{2}\vec{\sigma}. \tag{4}$$

We shall confine our attention to eigenstates belonging to eigenvalues $j(j+1)$ and m of $\vec{J}$ and J_z. There are[3] two two-component angular functions, $\xi^{(1)}_{jm}$ and $\xi^{(2)}_{jm}$, if $J \geq |q| + \frac{1}{2}$, and only one, η_{jm}, if $j = |q| - \frac{1}{2}$. Thus there are two types of eigenfunctions of J^2, J_z, and H:

Type A.

$$j \geq |q| + \tfrac{1}{2},$$

$$\psi = r^{-1}\begin{bmatrix} h_1(r)\,\xi^{(1)}_{jm} + h_2(r)\,\xi^{(2)}_{jm} \\ -i[h_3(r)\,\xi^{(1)}_{jm} + h_4(r)\,\xi^{(2)}_{jm}]\,\kappa q/|\kappa q| \end{bmatrix}. \tag{5}$$

Type B.

$$j = |q| - \tfrac{1}{2}, \quad \psi = \begin{bmatrix} f\eta_{jm} \\ g\eta_{jm} \end{bmatrix}. \tag{6}$$

II. TYPE-A WAVE FUNCTION

Type A is a generalization of types (1) and (2) of wave functions of Ref. 3 to the present case which includes a finite extra magnetic moment κ. Using lemma 1 of Ref. 3 we find

$$\begin{bmatrix} -\partial_r + \mu r^{-1} & 0 & \kappa q(2Mr^2)^{-1} & M+E \\ 0 & -\partial_r - \mu r^{-1} & M+E & \kappa q(2Mr^2)^{-1} \\ \kappa q(2Mr^2)^{-1} & M-E & -\partial_r + \mu r^{-1} & 0 \\ M-E & \kappa q(2Mr^2)^{-1} & 0 & -\partial_r - \mu r^{-1} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ -h_3\kappa q/|\kappa q| \\ -h_4\kappa q/|\kappa q| \end{bmatrix} = 0, \tag{7}$$

where

$$\mu = [(j+\tfrac{1}{2}) - q^2]^{1/2} > 0.$$

Change scale from r to ρ,

$$r = |\kappa q|\,\rho(2M)^{-1}. \tag{8}$$

We find (7) becomes

$$\Omega^{(0)}h^{(0)} = 0, \tag{9}$$

where

$$h^{(0)}=\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}, \tag{10}$$

and

$$\Omega^{(0)}=\begin{bmatrix} \partial_\rho-\mu\rho^{-1} & 0 & \rho^{-2} & A_0+B_0 \\ 0 & \partial_\rho+\mu\rho^{-1} & A_0+B_0 & \rho^{-2} \\ \rho^{-2} & A_0-B_0 & \partial_\rho-\mu\rho^{-1} & 0 \\ A_0-B_0 & \rho^{-2} & 0 & \partial_r+\mu\rho^{-1} \end{bmatrix}, \tag{11}$$

$$A_0=\kappa q/2\neq 0, \quad B_0=\kappa q E(2M)^{-1}. \tag{12}$$

The asymptotic behavior at large distances ρ is obtained by dropping ρ^{-2} and ρ^{-1} in (11). If $(A_0+B_0)(A_0-B_0)>0$, i.e., $M^2>E^2$, the solution of (9) is a sum of hyperbolic sines and cosines. In such a case, for suitable values of A and B we may have bound states. If $(A_0+B_0)(A_0-B_0)<0$, i.e., $E^2>M^2$, the solution $h^{(0)}$ is oscillatory at large ρ, and we have scattering states.

The boundary condition for $h^{(0)}(\rho)$ for a bound state is

$$\lim_{\rho\to 0} h^{(0)}(\rho)=\lim_{\rho\to\infty} h^{(0)}(\rho)=0. \tag{13}$$

We write

$$\Omega^{(0)}=\partial_\rho-\mu a_3\rho^{-1}+b_1\rho^{-2}+A_0 a_1 b_1+iB_0 a_1 b_2. \tag{14}$$

Here a_1, a_2, a_3 and b_1, b_2, b_3 are two sets of Pauli matrices, so that

$$a_1=\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad a_2=\begin{bmatrix} & -i & & \\ i & & & \\ & & & -i \\ & & i & \end{bmatrix}, \quad a_3=\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

$$b_1=\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}, \quad b_2=\begin{bmatrix} & & i & \\ & & & -i \\ -i & & & \\ & i & & \end{bmatrix}, \quad b_3=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{15}$$

For uniformity of notation with type-B solution, we shall perform an orthogonal transformation on $\Omega^{(0)}$ equivalent to multiplying a_1 by $q|q|^{-1}$:

$$\Omega=a_3^{(1-q|q|^{-1})/2}\,\Omega^{(0)}\,a_3^{(1-q|q|^{-1})/2},$$

$$h=a_3^{(1-q|q|^{-1})/2}\,h^{(0)}.$$

Then

$$\Omega=\partial_\rho-\mu a_3\rho^{-1}+b_1\rho^{-2}+Aa_1b_1+iBa_1b_2, \tag{14$'$}$$

where

$$A=\kappa|q|/2, \quad B=\kappa|q|E(2M)^{-1}. \tag{12$'$}$$

Furthermore

$$\Omega h=0. \tag{9$'$}$$

We shall now show that the system has many bound states. First, assume that A, B gives a bound state, $(b_1\Omega b_1)(b_1 h)=0$. But $b_1\Omega b_1$ is the same as Ω, but with the sign of B changed. $b_1 h$ vanishes at $\rho=0$ and $\rho=\infty$. Thus $A, -B$ also gives a bound state. A similar argument shows that $-A, -B$ also gives a bound state. Thus we have the following:

Theorem 1. If we find a bound state for $j\geq|q|+\frac{1}{2}$, then changing the sign of E, κ, or q would produce another type-A bound state of the same j and m.

Next we shall prove the following:

Theorem 2. $E=0$ is always a bound state for every j, m, where $j\geq|q|+\frac{1}{2}$.

Proof. $E=0$ implies $B=0$. Ω then commutes with b_1. Choose $b_1=-1$. Then

$$\Omega=\partial_\rho-\rho^{-2}-\mu a_3\rho^{-1}-Aa_1$$
$$=\begin{bmatrix} \partial_\rho-\rho^{-2}-\mu\rho^{-1} & -A \\ -A & \partial_\rho-\rho^{-2}+\mu\rho^{-1} \end{bmatrix}. \tag{16}$$

The equation $\Omega h=0$ thus has a bound-state solution. Each component of h is of the form

$$\sqrt{\rho}\,e^{-1/\rho}\times(\text{Hankel function of } \rho). \tag{17}$$

This proof also shows that for a given j, m, the type-A bound state $E=0$ is nondegenerate, because $b_1=+1$ does not lead to a bound state.

III. TYPE-B WAVE FUNCTION

Wave function (6) was already discussed in Ref. 3, Eqs. (27), (37), (38), and (39). Thus

$$\frac{dG}{dr}=\left[-\frac{(E-M)\kappa}{|\kappa|}-\frac{|\kappa q|}{2Mr^2}\right]F\,,\qquad \frac{dF}{dr}=\left[\frac{(E+M)\kappa}{|\kappa|}-\frac{|\kappa q|}{2Mr^2}\right]G\,, \tag{18}$$

where

$$f=\kappa q F(r)\,|\kappa q r|^{-1},\quad g=-iG(r)r^{-1}. \tag{19}$$

We shall now change scale of r by using (8) above, and obtain

$$\frac{dG}{d\rho}=\left(A-B-\frac{1}{\rho^2}\right)F\,,\qquad \frac{dF}{d\rho}=\left(A+B-\frac{1}{\rho^2}\right)G\,, \tag{20}$$

where A and B were defined above by (12′). As in Sec. II, $E^2>M^2$ implies $(A-B)(A+B)<0$, which implies oscillating behavior at large ρ. Thus $E^2>M^2$ describes scattering states. We shall concentrate on $E^2\leqslant M^2$, or $A^2\geqslant B^2$. Putting

$$F=R\cos\left(-\frac{\pi}{4}+\frac{\phi}{2}\right),\quad G=R\sin\left(-\frac{\pi}{4}+\frac{\phi}{2}\right), \tag{21}$$

we obtain

$$\frac{d\phi}{d\rho}=-2B+\left(2A-\frac{2}{\rho^2}\right)\sin\phi\,. \tag{22}$$

As $\rho\to 0$, the ρ^{-2} terms dominate in (20), and a possibly meaningful solution is $F\sim -G\sim\exp(-\rho^{-1})$. Other solutions of (20) cannot be meaningful because for them $F\sim\exp(+\rho^{-1})$. Thus

$$G/F\to -1,\ \text{ i.e., }\ \phi\to 0 \text{ as } \rho\to 0+\,. \tag{23}$$

Near $\rho=0+$, ϕ can be expanded in an asymptotic series:

$$\phi=-B\rho^2+B\rho^3-B(A+\tfrac{3}{2})\rho^4+\cdots\,. \tag{24}$$

Changing the sign of B merely switches F and G, and changes the sign of ϕ. Thus we have the following:

Theorem 3. If we find a bound state for $j=|q|-\frac{1}{2}$ then changing the sign of E would produce another bound state.

IV. TYPE-B WAVE FUNCTION FOR $\kappa<0$ (i.e., $A<0$)

Define

$$T(\rho,\phi)\equiv -2B+(2A-2\rho^{-2})\sin\phi\,. \tag{25}$$

Regions in the (ρ,ϕ) plane where $T>0$ and $T<0$ are shown in Fig. 1 for the case

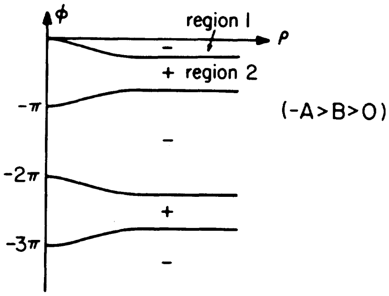

FIG. 1. Regions of definite sign for $T(\rho,\phi)$, for $-A>B>0$.

$$A<0,\quad -A>B>0\,. \tag{26}$$

Integrals of (22) give nonintersecting curves in the (r,ϕ) plane. Maxima of these curves must lie on the boundary lines $T=0$. Therefore the curve with the desired boundary condition (23) and (24) lies entirely in region 1 for $\rho>0$ since it cannot cross into region 2. Thus ϕ is monotonically decreasing. It thus approaches a limit as $\rho\to\infty$:

$$\phi\to\phi_\infty\,. \tag{27}$$

From Fig. 1 and (22) we obtain

$$\phi_\infty=-\sin^{-1}|B/A|,\quad -\pi/2<\phi_\infty<0\,. \tag{28}$$

Thus, for all $\rho>0$,

$$-\pi/4<\phi<0\,,$$

and

$$-G>F>0\,.$$

Thus

$$\frac{dF}{d\rho}=\left(A+B-\frac{1}{\rho^2}\right)G>\left(-A-B+\frac{1}{\rho^2}\right)F>(-A-B)F\,. \tag{29}$$

Hence F increases at least exponentially as $\rho\to\infty$. We have thus shown that there are no meaningful solutions for case (26). It follows that there are no bound states for this case. Using theorem 3 we conclude that there are also no bound states if $-A>-B>0$.

Case $-A>B=0$ can be discussed by inspecting directly (20). One finds that the two solutions of (20) are

$$F=\pm G=e^{\pm(A\rho+\rho^{-1})}\,.$$

No combination of the two solutions can give meaningful results at both $\rho\to 0$ and $\rho\to\infty$. Case $-A=B>0$ yields

$$-\frac{d^2F}{d\xi^2}+(-2A/\xi^2)F=-F,\quad \xi=1/\rho\,. \tag{30}$$

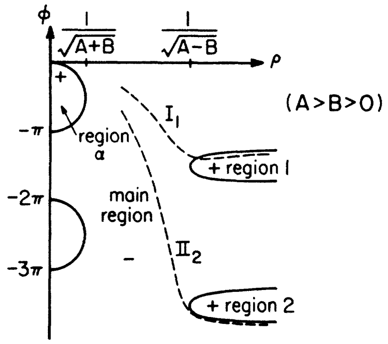

FIG. 2. Regions of definite sign for $T(\rho, \phi)$, for $A>B>0$. An integral of class I_1 and one of class II_2 are schematically displayed.

This is a Schrödinger equation at energy -1 for repulsive potential $-2A\xi^{-2}$. Integrating from $\xi=\infty$ (i.e., $\rho=0$) inwards, we find that $\lim F=\infty$ as $\xi\to 0$. We thus arrive at the following:

Theorem 4. If $\kappa<0$, there are no bound states of total angular momentum $j=|q|-\frac{1}{2}$.

V. TYPE-B BOUND STATES FOR $\kappa>0$ (i.e., $A>0$)

$E=B=0$ is always a bound state, for which (20) gives

$$F=-G=e^{-(A\rho+\rho^{-1})}\,. \tag{31}$$

For this state $\phi=0$ for all ρ.

Because of theorem 3 we now study the case $A>B>0$. We again define T by (25) and show in Fig. 2 the sign of $T(\rho,\phi)$ in different regions of the (ρ,ϕ) plane. We study first the behavior as $\rho\to\infty$ of *all* integrals of

$$\frac{d\phi}{d\rho}=T(\rho,\phi)\,. \tag{32}$$

Since $\phi=0$ is an integral, other integrals must be either above or below the $\phi=0$ axis (horizontal axis). Because of (24), which says that $\phi<0$ for very small ρ, we are interested only in integrals for which $\phi<0$ for $0<\rho$.

In Fig. 2 regions $1,2,\ldots,$ are traps in the sense that once any integral of (32) is in one such region it will remain in the same region for all larger ρ's, and will be monotonically increasing for all larger ρ's. If an integral of (32) is never trapped in any such regions, it must, for $\rho>(A-B)^{-1/2}$, be in the main region and be monotonically decreasing. Since in either case the integral is monotonic for large-enough ρ, *all integrals approach some limit* ϕ_∞ as $\rho\to\infty$. Equation (32) implies then

$$T(\infty,\phi_\infty)=0,\quad \text{i.e.,}\ \sin\phi_\infty=\sin\gamma=B/A\,, \tag{33}$$

where

$$\gamma=\sin^{-1}(B/A),\quad 0<\gamma<\pi/2\,. \tag{34}$$

Thus, according to the behavior of an integral of (32) as $\rho\to\infty$, we put it into one of the following classes:

Classes I_n. All integrals trapped in region n. For these

$$\phi_\infty=(\pi-\gamma)-2n\pi,\quad n=1,2,3,\ldots\,, \tag{35}$$

Classes II_n. All integrals completely outside of any trapping regions. Furthermore

$$\phi_\infty=\gamma-2n\pi,\quad n=1,2,3,\ldots\,. \tag{36}$$

Classes III_n. All integrals completely outside of any trapping region. Furthermore

$$\phi_\infty=(\pi-\gamma)-2n\pi,\quad n=1,2,3,\ldots\,. \tag{37}$$

We can now expand ϕ for large ρ into an asymptotic expansion:

$$\phi=\phi_\infty+b_1\rho^{-1}+b_2\rho^{-2}+\cdots\,. \tag{38}$$

For classes II_n,

$$b_1=0,\ b_2=B[A(A^2-B^2)^{1/2}]^{-1},\ldots\,,$$

for classes I_n and III_n, (39)

$$b_1=0,\ b_2=-B[A(A^2-B^2)^{1/2}]^{-1},\ldots\,,$$

as can be verified from (22). Equation (39) shows that $d\phi/d\rho>0$ for large ρ, for classes I_n and III_n. That means that the integral must be in a region where $T(\rho,\phi)>0$ for large ρ. Thus it must be trapped. These arguments show that class III_n *does not exist* for any n.

In Fig. 2 we exhibit a class-I_1 integral and a class-II_2 integral. The importance of the classification shceme lies in what follows:

Lemma 1. Class-I_n integrals give exponentially divergent F and G as $\rho\to\infty$ (physically inadmissible). Class-II_n integrals give exponentially damped F and G as $\rho\to\infty$ (physically admissible).

Proof. For class I_n, according to definition (21),

$$G/F=\tan[-(\pi/4)+(\phi/2)]\to\tan[(\pi-2\gamma)/4]>0\,.$$

Equation (20) then shows that $|F|$ and $|G|$ become exponentially large for large ρ. Similarly, we prove the lemma for class II_n.

Now define

$$\phi(\rho,A,B)=\text{solution of (32) satisfying (23) and (24)}\,. \tag{40}$$

It is important to remember that $\phi(\rho,A,B)$ is analytic in ρ,A,B, for $0<\rho$ and all A and B. For small positive ρ, (23) and (24) show that ϕ is decreasing with ρ. The ϕ-vs-ρ curve cannot cross into region α of Fig. 2 because in region α, $d\phi/d\rho>0$. Thus the ϕ-vs-ρ curve remains under the ρ axis and above region α.

We shall now fix $A>0$ and study the behavior of the $\phi(\rho,A,B)$-vs-ρ curve as B changes. We shall say that B belongs to class I_n (or II_n) if $\phi(\rho,A,B)$ belongs to class I_n (or II_n). Figure 2 shows that, for $B>0$,

$$\phi((A-B)^{-1/2},A,B)=(\pi/2)-2n\pi \Rightarrow B \text{ is in } \mathrm{I}_n \tag{41}$$

$$\phi((A-B)^{-1/2},A,B)<(\pi/2)-2n\pi\,,$$
$$\phi((A-B)^{-1/2},A,B)>(\pi/2)-2(n+1)\pi \tag{42}$$
$$\Rightarrow B \text{ is in } \mathrm{I}_n,\ \mathrm{II}_n, \text{ or } \mathrm{I}_{n+1}.$$

We now prove three lemmas.

Lemma 2.

$$\frac{\partial\phi(\rho,A,B)}{\partial B}<0 \quad \text{for all } A,B\,. \tag{43}$$

Proof. Take (20) and consider a set of similar equations at B_1 with solutions F_1 and G_1. It follows from these that

$$\frac{d}{d\rho}(F_1G-G_1F)=(B_1-B)(F_1F+G_1G)\,.$$

Thus

$$F_1G-G_1F=\int_0^\rho (B_1-B)(F_1F+G_1G)\,d\rho\,.$$

Divide by B_1-B and approach the limit $B_1\to B$. We obtain

$$\frac{\partial F}{\partial B}G-\frac{\partial G}{\partial B}F=\int_0^\rho (F^2+G^2)\,d\rho>0\,.$$

By (21) this becomes

$$-R^2\frac{\partial\phi}{\partial B}\frac{1}{2}=\int_0^\rho (F^2+G^2)\,d\rho>0\,, \tag{43'}$$

which proves the lemma.

Lemma 3.

$$\phi(\rho,A,0)=0 \quad \text{for } A>0\,. \tag{44}$$

Proof. This case gives the bound-state wave function (31) for which $\phi=0$.

Lemma 4. $\phi(\rho,A,A)$ for $A>\frac{1}{8}$ is a function of ρ that is 0 at $\rho=0$ and approaches $-\infty$ as $\rho\to\infty$.

Proof. When $A=B>0$, (20) can be transformed into

$$-\frac{d^2G}{d\xi^2}+\left(-\frac{2A}{\xi^2}\right)G=-G\,, \tag{45}$$

where $\xi=\rho^{-1}$ and $F=dG/d\xi$. This is a Schrödinger equation with potential energy $-2A\xi^{-2}$ and total energy -1. Near $\xi=0$ the indicial equation is

$$-\alpha(\alpha-1)-2A=0\,.$$

For $A>\frac{1}{8}$

$$\alpha=+\tfrac{1}{2}\pm i\omega,\quad \omega=(8A-1)^{1/2}/2>0\,.$$

Thus integrating from $\xi=\infty$ towards $\xi=0$ we obtain

$$G\to C\sqrt{\xi}\,\sin(\omega\ln\xi+D)\underset{\rho\to\infty}{\sim}-C\rho^{-1/2}\sin(\omega\ln\rho-D) \tag{46}$$

where C and D are constants. Using $F=dG/d\xi$ one obtains

$$F\sim C\rho^{1/2}2^{-1}[-\sin(\omega\ln\rho-D)+2\omega\cos(\omega\ln\rho-D)]\,.$$

Thus (F,G) spirals outwards *indefinitely* in the (F,G) plane *clockwise* as $\rho\to\infty$. By definition (21), this implies

$$\phi(\rho,A,A)\underset{\rho\to\infty}{\sim}-2\omega\ln\rho \quad \text{as } \rho\to\infty\,. \tag{47}$$

Lemma 4 is thus proved.

We shall now separately discuss cases $A>\frac{1}{8}$ and $0<A<\frac{1}{8}$.

VI. TYPE-B BOUND STATES FOR $4\kappa|q|>1$ (i.e., $A>\frac{1}{8}$)

We concentrate on case $B>0$. Since all integrals for case $A>B>0$ approach some limit, $\phi(\infty,A,B)$ exists if $A>B>0$. Its value is given by (35) or (36). The plot $\phi(\infty,A,B)$ vs B (Fig. 3) thus must lie on the dashed curve describing (35) plus the dashed curve describing (36). Now (43) shows

$$\phi(\rho,A,B)>\phi(\rho,A,B_1) \quad \text{if } B<B_1\,.$$

Thus

$$\phi(\infty,A,B)\geqslant\phi(\infty,A,B_1) \quad \text{if } B<B_1\,. \tag{48}$$

That is, $\phi(\infty,A,B)$ is a nonincreasing function of B. Thus the only possibility for $\phi(\infty,A,B)$ is to follow a steplike descent such as $pqst\cdots$.

Lemma 3 shows that $\phi((A-B)^{-1/2},A,B)$ is small and negative for sufficiently small B. Equation

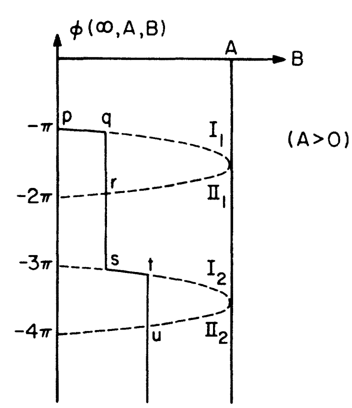

FIG. 3. $\phi(\infty,A,B)$ vs B at fixed $A>0$. The curve is monotonically decreasing. All points on it must be on the broken dashed curve which is $B=A\sin\phi$. Thus it must be stepwise such as $pqrstu\ldots$.

(42) then shows that such B must be in I_0, II_0, or I_1. Since the former two classes give $\phi_\infty > 0$ and are impossible classes for positive B, such small B must be in I_1. Thus the first segment pq must be on I_1 as shown.

Lemma 4 shows that for fixed A there exists ρ_N so that

$$\phi(\rho_N, A, A) < -2N\pi$$

for any large positive number N. $\phi(\rho_N, A, B)$ is continuous in B. Thus there exists a $0<B<A$ so that $\phi(\rho_N, A, B) < -2N\pi + \pi$. Thus $\phi(\infty, A, B) < -2N\pi + 2\pi$. In other words $\phi(\infty, A, B)$ as $B \to A$ is not bounded from below.

We have shown that the $\phi(\infty, A, B)$-vs-B *plot* is a series of steps, with vertical drops starting from pq in I_1, descending towards $-\infty$. We now prove that if the first drop occurs at B_1, then B_1 belongs to II_1 so that r of Fig. 3 is a point on the plot. For $0<B<B_1$, the curve $\phi(\rho, A, B)$ vs ρ is trapped in region 1 (cf. Fig. 2). Its minimum is thus $> -2\pi$. Thus

$$\phi(\rho, A, B) > -2\pi \quad \text{for all } \rho, \quad 0<B<B_1 .$$

Hence

$$\phi(\rho, A, B_1) = \lim_{B \to B_1} \phi(\rho, A, B) \geq -2\pi .$$

Hence

$$\phi(\infty, A, B_1) \geq -2\pi .$$

Thus $\phi(\infty, A, B_1)$ must give a point either at q or r, but not s. If it were at q, then $\phi(\rho, A, B_1)$ would be trapped in region 1, and $\phi(\rho, A, B_1+)$ would still be trapped, showing that B_1 is not the upper end of the I_1 stretch. This is a contradiction. We have thus proved that

$$B_1 \text{ is in } II_1, \text{ giving rise to a bound state,} \qquad (49)$$

according to lemma 1, since the wave function is damped at both $r=0$ and $r=\infty$.

Clearly,

$$\phi((A-B_1)^{-1/2}, A, B_1) > \phi(\infty, A, B_1) > -2\pi .$$

Continuity thus indicates that

$$\phi((A-B_1+)^{-1/2}, A, B_1+) > -2\pi .$$

The $\phi(\rho, A, B_1+)$-vs-ρ curve must therefore be trapped in region 2. That is, B_1+ is in I_2.

Repeating the above arguments we arrive at the conclusion that the $\phi(\infty, A, B)$-vs-B plot (Fig. 3) consists of stretches pq, $st, \ldots$, on all the I_n's, $n=1,2,3,\ldots$. The drops are at $B_1, B_2, B_3, \ldots$, where B_n is in class II_n, $n=1,2,3,\ldots$. Collecting all results together we obtain the following:

Theorem 5. For $4\kappa|q|>1$, $j=|q|-\frac{1}{2}$, there are infinitely many bound states at $B=B_n>0$, and

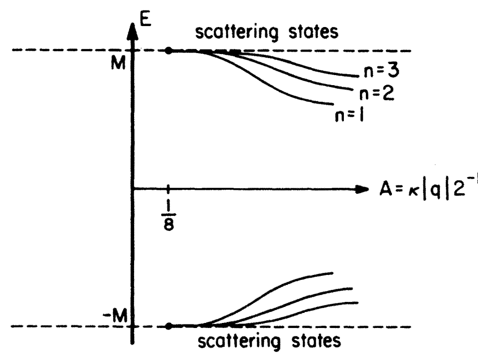

FIG. 4. Bound-state energy for $j=|q|-\frac{1}{2}$. Solid lines show bound-state energies E_n as functions of A. The dashed lines are thresholds outside of which there are only scattering states. $E=0$ is a bound state for all $\kappa|q|>0$. The diagram is symmetrical with respect to a reflection in the A axis. The curves are not numerically accurate.

$B=-B_n$, $n=1,2,\ldots$ (Fig. 4). Besides these and a bound state at $E=0$ there are no others at this j. For the bound state at B_n, $\phi(0, A, B_n)=0$, $\phi(\infty, A, B_n)=\gamma_n - 2n\pi$. ϕ is monotonically decreasing in ρ, and $\gamma_n = \sin^{-1}(B_n/A)$, $0<\gamma_n<\pi/2$. For such a bound state, (F,G) winds monotonically around the origin clockwise in the (F,G) plane. Thus $F(r)$ and $G(r)$ each has n nodes exclusive of $r=0$ and $r=\infty$. The state at $-B_n$ is obtained from that at B_n by the switch $F \leftrightarrow G$. The wave function for the case $E=B=0$ was given in (31) above. All bound-state wave functions vanish in the limit $r \to 0$ as

$$r^{-1}\exp(-|\kappa q/2Mr|), \qquad (50)$$

and vanish in the limit $r \to \infty$ as

$$r^{-1}\exp[-r(M^2-E_n^2)^{1/2}]. \qquad (51)$$

We remark that the damping (50) is provided by the extra magnetic moment.

VII. LACK OF TYPE-B BOUND STATES FOR $E \neq 0$, $0<4\kappa|q|\leq 1$ (i.e., $0<A<\frac{1}{8}$)

For this case, we first consider $B=A$. The differential equation becomes (45), the solution of which is a Hankel function:

$$\xi = \rho^{-1},$$
$$G = -e^{i\pi(p+1)}\sqrt{\xi}\, H_p^{(1)}(i\xi), \quad p = (\tfrac{1}{4} - 2A)^{1/2} \qquad (52)$$
$$F = dG/d\xi .$$

Equation (52) is the only solution, except for a multiplicative constant, that does not exponentially diverge as $\rho \to 0$. The asymptotic forms of the Hankel functions as $\xi \to 0$ (i.e., $\pi \to \infty$) are well known. Using these results we find that as $\rho \to \infty$

$$F \to O(\sqrt{\rho}\ln\rho) \quad \text{if } A=\tfrac{1}{8}, \qquad F \to O(\rho^{p+1/2}) \quad \text{if } 0<A<\tfrac{1}{8}. \tag{53}$$

Thus these wave functions are not square-integrable, and $0<B=A\leq\frac{1}{8}$ does not give a bound state.

We can find ϕ from (52) and (21), obtaining for all ρ,

$$0\geq\phi(\rho,A,A)>-3\pi/2\,,\quad 0<A\leq\tfrac{1}{8}.$$

Now apply lemma 2. We obtain

$$0>\phi(\rho,A,B)>\phi(\rho,A,A)>-3\pi/2\,,\quad 0<B<A\leq\tfrac{1}{8}$$

for all ρ. Referring to Fig. 2 we conclude that $\phi(\rho,A,B)$ is trapped in region 1 and does not give a bound state.

Theorem 6. For $0<4\kappa|q|\leq 1$, $j=|q|-\frac{1}{2}$, there is only one bound state at $E=0$ with wave function given by (31).

VIII. NUMERICAL CALCULATION AND ASYMPTOTIC EXPANSIONS

The bound-state energy $E_n=MB_n/A$ is shown in Fig. 4 schematically. Actual values are tabulated for $n=1,2,3$ for three values of A in Table I. These values are obtained by numerical integration of (22).

Asymptotic values of E_n in various limiting cases can be obtained. We list some of these below:

(a) For fixed n, as $8A-1\to 0+$,

$$\ln(1-E_nM^{-1})\approx-4\pi n(8A-1)^{-1/2}+O(1). \tag{54}$$

(b) For fixed n, as $A\to\infty$,

$$E_nM^{-1}\approx 2\sqrt{n}A^{-1/4}+O(A^{-1/2}). \tag{55}$$

(c) For case $A\to\infty$, $n^2A^{-1}=$fixed,

$$y\left[\left(\frac{1}{1-y}\right)^{-1/2}-\left(\frac{1}{1+y}\right)^{1/2}\right]=\frac{\pi n}{\sqrt{A}}+O\left(\frac{1}{\sqrt{A}}\right),\quad y=E_n/M. \tag{56}$$

(d) For case $n\to\infty$, fixed $A>\frac{1}{8}$,

$$\ln(1-E_nM^{-1})\approx-4\pi n(8A-1)^{-1/2}+O(1). \tag{57}$$

IX. DISCUSSION

(a) Examination of the radial wave function of type A discussed in Sec. II and the radial wave function of type B in the other sections show that Hamiltonian (3) with $\kappa\neq 0$ is *well defined*. Furthermore, for a fixed value of $\kappa\neq 0$, the bound states at $E^2<M^2$ and the scattering states at $E\leq -M$ and $E\geq +M$ together form a complete set. We thus are free of the Lipkin-Weisberger-Peshkin difficulty[4,3] because the wave function at $r=0$ is always damped, for both signs of κ, and all values of E, by the extra magnetic interaction. It is worth mentioning that the other difficulty that has plagued the analysis of the physics of monopoles, the string of singularities, has already been disposed of by the use of the concept of sections.[2]

TABLE I. Lowest three positive-energy levels for some values of $A=\kappa|q|/2$.

A	$(M-E_1)/M$	$(M-E_2)/M$	$(M-E_3)/M$
1.0	5.75×10^{-3}	5.12×10^{-5}	4.47×10^{-7}
1.5	1.41×10^{-2}	3.28×10^{-4}	7.53×10^{-6}
2.0	2.35×10^{-2}	9.45×10^{-4}	3.74×10^{-5}
2.5	3.37×10^{-2}	1.92×10^{-3}	1.08×10^{-4}

(b) For both types A and B states, the addition of an interaction proportional to $\delta^3(r)$ does not change anything since since the damping factor (50) prevents the charged particle from passing through the monopole.

(c) Can the monopole capture more than one electron? This is a very interesting question currently under investigation. Preliminary results indicate that two electrons can be bound simultaneously to one monopole.

We observe that the strong binding of electrons to monopoles applies also to binding of positrons to monopoles. The following possibility then arises. Create a pair of e^+e^-, expending an energy of $2M$. Capture each member into a bound state with $E=0$, of the type described in Theorems 2, 5, and 6. This releases an energy of $2M$. The Coulomb interaction between e^+ and e^- presumably helps to produce a net energy surplus. Thus it is energetically favorable to create pairs and bind them to the monopole. Since there are infinitely many bound states of the type of Theorem 2, this argument suggests that a monopole will be enclothed in a plasma of e^+e^- pairs. In other words, there is a very strong vacuum polarization. The polarization cannot, however, change the magnetic charge on the monopole.

The discussion of the paragraph above is not quite decisive because of the complication to be mentioned below in (e).

(d) Can the analysis of the present paper be used to describe the motion of a proton in the field of a fixed magnetic monopole? The answer is clearly no. The hadronic cloud around the proton (which gives rise to its static extra magnetic moment $\kappa=1.79$) embodies many degrees of freedom which must be taken into consideration in the very strong magnetic field around the monopole.

(e) Can the analysis of the present paper be used to describe the motion of an electron in the field of a fixed magnetic monopole? We are not able to answer this question with certainty since physically the extra magnetic moment is due to radiative corrections, and a complete field theory of electron-monopole interaction remains to be worked out. Lacking a complete theory we argue, however, that the bound state $E=0$ probably does exist for $j=|q|-\frac{1}{2}$. Two points are needed for this argument. (a) For this value of j, the wave function (31) is (for $\kappa>0$)

$$F=-G=\exp(-Mr-|\kappa q/2Mr|)\,, \tag{58}$$

which is independent of κ except for very small values of $r\lesssim|\kappa q/M|$. The wave function is quadratically integrable even if we put $\kappa=0$ in (58). In other words, for $j=|q|-\frac{1}{2}$, $E=0$, the electron distribution is little influenced by the value of κ, provided its sign is positive. (b) For large distances[5] r,

$$\kappa=\frac{1}{(137)2\pi}+\cdots\,. \tag{59}$$

For small distances, the magnetic field becomes very strong. For the distance most relevant for the wave function (58), $r\sim M^{-1}$, and one has the following order of magnitude for the magnetic field:

$$\mathcal{H}=gr^{-2}\sim gM^2\,. \tag{60}$$

The relevant parameter[6] is

$$e\mathcal{H}M^{-2}=q\,.$$

For such a strong magnetic field there are "polarization" effects[6] on the electron which lead to, among other effects, changes in the value of κ. The results of Ref. 6 are not really applicable to the strong nonhomogeneous field we encounter. But they suggest that κ does not change sign.

Combining (a) and (b) we thus suggest that the bound states at $E=0$, $j=|q|-\frac{1}{2}$ exist, with a wave function

$$F\cong-G\cong\exp(-Mr)\,,$$

except at very small distances where F and G are damped in a complicated way.

ACKNOWLEDGMENT

It is a pleasure to thank Professor Tai Tsun Wu for discussions.

[1]P. A. M. Dirac, Proc. R. Soc. London A133, 60 (1931).
[2]Tai Tsun Wu and Chen Ning Yang, Nucl. Phys. B107, 365 (1976).
[3]Yoichi Kazama, Chen Ning Yang, and Alfred S. Goldhaber, preceding paper, Phys. Rev. D 15, 2287 (1977).
[4]H. J. Lipkin, W. I. Weisberger, and M. Peshkin, Ann. Phys. (N.Y.) 53, 203 (1969).
[5]P. Kusch and H. M. Foley, Phys. Rev. 72, 1256 (1947); J. Schwinger, *ibid.* 73, 416 (1948).
[6]R. G. Newton, Phys. Rev. 96, 523 (1954); S. N. Gupta, Nature (London) 163, 686 (1949); M. Demeur, Acad. R. Belg., Mem. Cl. Sci., 28, 1643 (1953); J. Schwinger, *Particles, Sources and Fields* (Addison-Wesley, Reading, Mass.), Vol. III, Chaps. 5, 6; UCLA lectures, 1972 (unpublished); Wu-yang Tsai and Asim Yildiz, Phys. Rev. D 8, 3446 (1973).