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Scattering of a Dirac particle with charge Ze by a fixed magnetic monopole*

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The helicity-flip and helicity-nonflip scattering amplitudes of a Dirac particle with spin 1/2 and charge Ze by a fixed magnetic monopole field are calculated. To make the Hamiltonian meaningful an infinitesimal extra magnetic moment is added to the charged particle. The sign of this extra term has measurable consequences. The differential cross section, which is independent of the sign of Ze, is tabulated. The helicity-flip amplitude vanishes at all angles for incoming beam helicity $=+1$ if $Zeg>0$, and for incoming beam helicity $=-1$ if $Zeg<0$.

The scattering cross section of a nonrelativistic charged particle by an infinitely heavy magnetic monopole has been studied in the literature.[1-3] In the present paper we calculate the helicity-flip and helicity-nonflip amplitudes of the scattering of a Dirac particle with charge Ze at relativistic or nonrelativistic velocities by an infinitely heavy magnetic monopole. This problem has not, to our knowledge, been studied before, although there have been discussions of the radial wave functions for a Dirac particle in a monopole field.[1-3]

The method of calculation uses the monopole harmonics of Ref. 4. The string singularity is completely absent in this formulation, and the steps are parallel to the usual treatment of the scattering of a Dirac particle by a central potential.

It will become apparent, however, that the Hamiltonian that one naturally writes down for the problem is not adequate. An infinitesimal "extra magnetic moment is therefore added to the charged particle. This addition makes the Hamiltonian well defined.

The helicity-flip and -nonflip amplitudes are given in Sec. VI. The corresponding intensities are tabulated in Tables I and II and Fig. 2.

I. TOTAL ANGULAR MOMENTUM $\vec{J}$ OF A CHARGED DIRAC PARTICLE IN THE FIELD OF A FIXED MAGNETIC MONOPOLE

Consider the wave equation of a Dirac particle of charge Ze in the field of a fixed magnetic monopole of strength g. It is clear that the wave equation[5] is dependent only on the quantity

$$q=Zeg. \tag{1}$$

It is known[6] that q is an integral multiple of $\frac{1}{2}$. The wave equation is

$$H\psi=E\psi,\qquad H=\vec{\alpha}\cdot(-i\vec{\nabla}-Ze\vec{A})+\beta M. \tag{2}$$

We choose $\vec{\alpha}$ and β so that

$$\vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3}$$

where $\vec{\sigma}$ are the Pauli matrices and β is diagonal with two diagonal elements equal to $+1$ and two equal to -1. We use the convention

$$\sigma_x\sigma_y=i\sigma_z.$$

The vector potential A is defined[4] as two functions $(A)_a$ and $(A)_b$ in two overlapping regions R_a and R_b. The wave function is a *section* as discussed in detail in Ref. 4. It was further shown there that the components of

$$\vec{L}=\vec{r}\times(\vec{p}-Ze\vec{A})-q\vec{r}/r, \tag{4}$$

are Hermitian operators in the Hilbert space of sections and satisfy the commutation rules with $\vec{r}$, $(\vec{p}-Ze\vec{A})$, and $\vec{L}$ that characterize them as orbital angular momenta. However, they do not commute with the matrices $\vec{\alpha}$ as components of *total* angular momenta should. If one defines, as usual,

$$\vec{J}=\vec{L}+\tfrac{1}{2}\vec{\sigma}, \tag{5}$$

then

$$[\vec{J},\beta]=0,\quad [J_x,\alpha_x]=0,\qquad [J_x,\alpha_y]=i\alpha_z,\quad [J_x,\alpha_z]=-i\alpha_y,\ \text{etc.} \tag{6}$$

These commutation rules imply that $\vec{J}$ is the total angular momentum.

To find the wave functions we endeavor to diagonalize simultaneously the operators H, J^2, and J_z, as usual. Before doing that, we first study the eigensections of J^2 and J_z.

II. PROPERTIES OF TWO-COMPONENT EIGENSECTIONS OF J^2 AND J_z

According to (5), $\vec{J}$ is the sum of two angular momenta. Thus the eigensections of J^2 and J_z with eigenvalues $j(j+1)$ and m are

$$\phi^{(1)}_{j,m} = \begin{bmatrix} \left(\frac{j+m}{2j}\right)^{1/2} Y_{j-1/2,m-1/2} \\ \left(\frac{j-m}{2j}\right)^{1/2} Y_{j-1/2,m+1/2} \end{bmatrix},$$

$$\phi^{(2)}_{j,m} = \begin{bmatrix} -\left(\frac{j-m+1}{2j+2}\right)^{1/2} Y_{j+1/2,m-1/2} \\ \left(\frac{j+m+1}{2j+2}\right)^{1/2} Y_{j+1/2,m+1/2} \end{bmatrix}, \tag{7}$$

where we have omitted the index q from $Y_{q,l,m}$. The range of j is such that, assuming $q \neq 0$,

$$\text{for } \phi^{(1)}_{jm},\ j-\tfrac{1}{2} \equiv l \geq |q|, \qquad \text{for } \phi^{(2)}_{jm},\ j+\tfrac{1}{2} \equiv l \geq |q|. \tag{8}$$

The square-root factors in (7) are Clebsch-Gordan coefficients. It is obvious that the collection of all $\phi^{(1)}$'s and $\phi^{(2)}$'s form a *complete orthonormal set* of two-component sections.

In the case that $q=0$, the sections become ordinary functions, and $\phi^{(1)}_{jm}$ are the $s_{1/2}$, $p_{3/2}$, $d_{5/2}$, etc. states, while $\phi^{(2)}_{jm}$ are the $p_{1/2}$, $d_{3/2}$, etc. states. In such a case we have the simple relations

$$\frac{\vec{\sigma}\cdot\vec{r}}{r}|s_{1/2}\rangle = -|p_{1/2}\rangle, \quad \frac{\vec{\sigma}\cdot\vec{r}}{r}|p_{1/2}\rangle = -|s_{1/2}\rangle, \text{ etc.} \tag{9}$$

When $q\neq 0$, since $\vec{\sigma}\cdot\vec{r}/r$ commutes with $\vec{J}$, $\vec{\sigma}\cdot\vec{r}/r$ operating on $\phi^{(1)}_{jm}$ gives a section that remains an eigensection of J^2 and J_z, with unchanged eigenvalues. In other words,

$$\frac{\vec{\sigma}\cdot\vec{r}}{r}\phi^{(1)}_{jm} = Z_{11}\phi^{(1)}_{jm} + Z_{21}\phi^{(2)}_{jm}, \qquad \frac{\vec{\sigma}\cdot\vec{r}}{r}\phi^{(2)}_{jm} = Z_{12}\phi^{(1)}_{jm} + Z_{22}\phi^{(2)}_{jm}. \tag{10}$$

The coefficients Z will be computed in Appendix A, where it will also be shown that it is convenient to form the following orthonormal combinations of $\phi^{(1)}$ and $\phi^{(2)}$, for $j \geq |q|+\frac{1}{2}$:

$$\xi^{(1)}_{jm} = c\phi^{(1)}_{jm} - s\phi^{(2)}_{jm}, \qquad \xi^{(2)}_{jm} = s\phi^{(1)}_{jm} + c\phi^{(2)}_{jm}, \tag{11}$$

where

$$c = q((2j+1+2q)^{1/2} + (2j+1-2q)^{1/2}) \times |q|^{-1}[2(4j+2)]^{-1/2},$$
$$s = q((2j+1+2q)^{1/2} - (2j+1-2q)^{1/2}) \times |q|^{-1}[2(4j+2)]^{-1/2}. \tag{12}$$

It will be shown in Appendix A that one has the following lemmas.

Lemma 1. If $j \geq |q|+\frac{1}{2}$,

$$(\vec{\sigma}\cdot\vec{r})\xi^{(1)}_{jm} = -r\xi^{(2)}_{jm}, \qquad (\vec{\sigma}\cdot\vec{r})\xi^{(2)}_{jm} = -r\xi^{(1)}_{jm}, \tag{13}$$

$$\vec{\sigma}\cdot(-i\vec{\nabla} - Ze\vec{A})f(r)\xi^{(1)}_{jm} = (i\partial_r + ir^{-1} - i\mu r^{-1})f\xi^{(2)}_{jm},$$
$$\vec{\sigma}\cdot(-i\vec{\nabla} - Ze\vec{A})g(r)\xi^{(2)}_{jm} = (i\partial_r + ir^{-1} + i\mu r^{-1})g\xi^{(1)}_{jm}, \tag{14}$$

where $f(r)$ and $g(r)$ are arbitrary functions of r and

$$\mu = [(j+\tfrac{1}{2})^2 - q^2]^{1/2} > 0. \tag{15}$$

Lemma 2. For

$$j = |q| - \tfrac{1}{2} \geq 0, \tag{16}$$

$$(\vec{\sigma}\cdot\vec{r})\eta_m = rq|q|^{-1}\eta_m \tag{17}$$

and

$$\vec{\sigma}\cdot(-i\vec{\nabla} - Ze\vec{A})f(r)\eta_m = -iq|q|^{-1}(\partial_r + r^{-1})f\eta_m, \tag{18}$$

where $f(r)$ is an arbitrary function of r, and

$$\eta_m \equiv \phi^{(2)}_{jm} \text{ for } j = |q| - \tfrac{1}{2} \geq 0. \tag{19}$$

η_m is not defined if (16) is not satisfied. The ξ's and the η's together form a complete orthonormal set of two-component sections.

In the case $q=0$, the states $s_{1/2}$ and $p_{1/2}$ have opposite parities P. P commutes with the Hamiltonian. For the case $q\neq 0$, P no longer commutes with the Hamiltonian. One can define a pseudoparity P_p that has $\xi^{(1)}_{jm}$ and $\xi^{(2)}_{jm}$ as eigensections with eigenvalues (+1) and (−1), respectively. One can then show that $[H, \beta P_p] = 0$. We shall, however, not develop this idea further in the present paper.

III. RADIAL WAVE FUNCTION

The Hamiltonian in (2) is

$$H = \begin{bmatrix} M & \vec{\sigma}\cdot(-i\vec{\nabla} - Ze\vec{A}) \\ \vec{\sigma}\cdot(-i\vec{\nabla} - Ze\vec{A}) & -M \end{bmatrix}. \tag{20}$$

For fixed j and m, the wave function is a linear combination of $\xi^{(1)}_{jm}$, $\xi^{(2)}_{jm}$, and η_m with coefficients which are functions of r. Using lemmas 1 and 2 we find readily that there are three types of simultaneous eigensections of H, J^2, and J_z:

$$\text{Type (1):}\quad \psi^{(1)}_{jm} = \begin{bmatrix} f(r)\xi^{(1)}_{jm} \\ g(r)\xi^{(2)}_{jm} \end{bmatrix} \quad (j \geq |q| + \tfrac{1}{2}), \tag{21}$$

where

$$(M-E)f + i(\partial_r + r^{-1} + \mu r^{-1})g = 0, \qquad i(\partial_r + r^{-1} - \mu r^{-1})f - (M+E)g = 0. \tag{22}$$

It follows that

$$g=\frac{1}{\sqrt{kr}}J_{\mu+1/2}(kr),\quad f=\frac{ik}{E-M}\frac{1}{\sqrt{kr}}J_{\mu-1/2}(kr). \tag{23}$$

$$\text{Type (2):}\quad \psi^{(2)}_{jm}=\begin{bmatrix} f(r)\xi^{(2)}_{jm} \\ g(r)\xi^{(1)}_{jm}\end{bmatrix}\quad (j\geq |q|+\tfrac{1}{2}), \tag{24}$$

where

$$(M-E)f+i(\partial_r+r^{-1}-\mu r^{-1})g=0,$$
$$i(\partial_r+r^{-1}+\mu r^{-1})f-(M+E)g=0. \tag{25}$$

It follows that

$$f=\frac{1}{\sqrt{kr}}J_{\mu+1/2}(kr),\quad g=\frac{ik}{E+M}\frac{1}{\sqrt{kr}}J_{\mu-1/2}(kr). \tag{26}$$

$$\text{Type (3):}\quad \psi^{(3)}_{m}=\begin{bmatrix} f(r)\eta_m \\ g(r)\eta_m\end{bmatrix}\quad (j=|q|-\tfrac{1}{2}\geq 0), \tag{27}$$

and

$$(M-E)f-iq|q|^{-1}(\partial_r+r^{-1})g=0,$$
$$-iq|q|^{-1}(\partial_r+r^{-1})f-(M+E)g=0. \tag{28}$$

Solutions (23) and (26) for types (1) and (2) both satisfy the boundary condition

$$f=g=0 \text{ at } r=0. \tag{29}$$

In contrast, Eq. (28) does not have any nonvanishing solution that fulfills (29), as one easily verifies by using rf and rg as the dependent variables. This means that Hamiltonian (2) is not a properly defined operator for type (3) of angular dependence. We shall analyze this point further in Sec. IV and resolve the difficulty by adding an *infinitesimally small "extra" magnetic moment* to the charged particle so that its magnetic moment is

$$\frac{Ze}{2M}(1+\kappa). \tag{30}$$

The additional infinitesimal magnetic interaction makes the total Hamiltonian *completely defined.* For type (1) and type (2) angular dependence, this additional interaction does not change solutions (23) and (26). For type (3) angular dependence it changes the boundary condition from (29) to

$$\lim_{r\to 0^+} g/f=i\kappa q/|\kappa q|. \tag{31}$$

The solution of (28) satisfying this boundary condition is unique except for normalization and is given by

$$f=\frac{1}{\sqrt{kr}}\left(\frac{2}{\pi kr}\right)^{1/2}\sin(kr+\delta_3),$$
$$g=\frac{1}{\sqrt{kr}}\left(\frac{2}{\pi kr}\right)^{1/2}\frac{-iq}{|q|}\frac{k}{M+E}\cos(kr+\delta_3), \tag{32}$$

where

$$\tan\delta_3=-k\kappa[|\kappa|(M+E)]^{-1}. \tag{33}$$

With the explicit radial wave functions (23), (26), and (32) we can easily obtain the ingoing and outgoing waves at large r. For this purpose, the asymptotic expression of $J_{\mu+1/2}$ is useful:

$$J_{\mu+1/2}(kr)\sim\left(\frac{2}{\pi kr}\right)^{1/2}\sin\left(kr-\frac{\pi\mu}{2}\right)\quad(r\to\infty), \tag{34}$$

It is to be emphasized that these radial wave functions are valid in *both* regions R_a and R_b. The non-single-valuedness of the wave function due to the fiber-bundle structure resides only in the monopole harmonics Y_{lm} in Eq. (7).

IV. ADDITION OF INFINITESIMAL EXTRA MAGNETIC MOMENT

We have seen in Sec. III that Hamiltonian (2) is not properly defined for wave functions of type (3): The radial wave equation (28) for type (3) has no meaningful solution. The reason for this is contained in the discussion of Lipkin, Weisberger, and Peshkin,[7] who pointed out that the Jacobi identity is not satisfied for $(\vec{p}-Ze\vec{A})$. They showed that

$$[[A,B],C]+[[B,C],A]+[[C,A],B]=-4\pi q\delta^3(r) \tag{35}$$

if we take A, B, and C to be the three components of $\vec{p}-Ze\vec{A}$. They further pointed out that for the nonrelativistic spinless Schrödinger equation, there is no real difficulty, since all radial wave functions vanish at the origin. For the present case of the Dirac particle, wave functions of types (1) and (2) also vanish at the origin. Thus the Lipkin-Weisberger-Peshkin difficulty occurs only for type (3) wave functions.

The difficulty is a manifestation of the ambiguity of a theory of monopole–charged-particle interaction if one allows the charged particle to pass through the monopole. There have been many discussions of this topic in the literature.[8]

To resolve the difficulty we consider, instead of a simple Dirac particle with charge Ze, one with an "extra" magnetic moment so that the total magnetic moment is given by (30). We take κ to be infinitesimal. The Hamiltonian is then

$$H_{\text{new}}=H-\kappa q\beta\vec{\sigma}\cdot\vec{r}(2Mr^3)^{-1}. \tag{36}$$

The effect of the extra magnetic moment, for either sign of κ, does not correspond to the effect of a classical repulsive potential. However, in quantum mechanics it prevents the charged particle from passing through the monopole. To see this we observe that with (36) and for infinitesimal

κ, solutions (23) and (26) are not changed, since they vanish at $r=0$ and are thus little influenced by the infinitesimal extra magnetic moment. For type (3), we have to consider now, instead of (28),

$$[M-E-\kappa|q|(2Mr^2)^{-1}]f-iq|q|^{-1}(\partial_r+r^{-1})g=0\,, \tag{37}$$

$$-iq|q|^{-1}(\partial_r+r^{-1})f-[M+E-\kappa|q|(2Mr^2)^{-1}]g=0\,.$$

Set

$$f=\kappa qF(r)|\kappa qr|^{-1},\quad g=-iG(r)r^{-1}\,. \tag{38}$$

Then

$$\frac{dG}{dr}=\left[-\frac{(E-M)\kappa}{|\kappa|}-\frac{|\kappa q|}{2Mr^2}\right]F\,,$$
$$\frac{dF}{dr}=\left[\frac{(E+M)\kappa}{|\kappa|}-\frac{|\kappa q|}{2Mr^2}\right]G\,. \tag{39}$$

For any $\kappa\neq 0$ these equations can be integrated subject to the initial condition (29), i.e., $G=F=0$ at $r=0$. Thus the introduction of an extra magnetic moment makes the wave function vanish at $r=0$, so that H_{new} is a legitimate Hamiltonian for all wave functions if $\kappa\neq 0$.

Now fix a value of $\kappa\neq 0$, and consider the case of large r. We can neglect the r^{-2} term, so that

$$\frac{d}{dr}[(E+M)G^2+(E-M)F^2]=0\,.$$

That is, in the (F,G) plane, as $r\to\infty$, the solution describes an ellipse. We shall normalize the solution so that this limiting ellipse is

$$(E+M)G^2+(E-M)F^2=\text{constant}$$

and is independent of κ.

As $\kappa\to 0$, the r^{-2} terms in (39) decrease in importance and the limiting ellipse is approached at earlier (i.e., smaller) values of r. On the other hand, for finite κ, as $r\to 0$,

$$G/F\to -1\,,$$
$$F\sim\ \text{const}\times\exp[-|\kappa q|(2Mr)^{-1}]\,. \tag{40}$$

These arguments lead to, in the limit $\kappa\to 0$, the (F,G) curve depicted in Fig. 1. For this limiting curve, the point $r=0$ corresponds to the whole line segment OA. As $r\to 0+$, the point A is approached and we have $G/F\to -1$, which leads to (31). As r increases from $0+$, (F,G) goes around the ellipse, clockwise for $\kappa=0+$ and counterclockwise for $\kappa=0-$, once for every $\Delta r=k^{-1}$. Using (38) one transforms this part of the (F,G) limiting curve into (32).

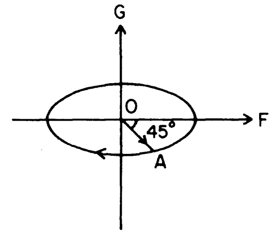

FIG. 1. (F,G) in the limit $\kappa\to 0+$. For the case $\kappa\to 0-$, as r increases from 0 toward $+\infty$, (F,G) moves along OA, then winds around ellipse *counterclockwise*.

V. WAVE FUNCTIONS DESCRIBING SCATTERING

We now try to construct a superposition of $\psi^{(1)}$, $\psi^{(2)}$, and $\psi^{(3)}$:

$$^{\pm}\psi=\sum_{jm}{}^{\pm}K^{(1)}_{jm}\psi^{(1)}_{jm}+\sum_{jm}{}^{\pm}K^{(2)}_{jm}\psi^{(2)}_{jm}+\sum_{m}{}^{\pm}K^{(3)}_{m}\psi^{(3)}_{m}\,, \tag{41}$$

which describes the scattering[5] of a Dirac particle of charge Ze with helicity ± 1 by the fixed monopole at the origin. The coefficients $^{\pm}K$ will be determined by the condition that as $r\to\infty$, the incoming part of the right-hand side of (41) matches that of the incident plane wave in region R_a.

The incoming part at large r of the plane wave e^{-ikrx}, where

$$x=\cos\theta\,, \tag{42}$$

is, according to Banderet,[1]

$$-2\delta(1-x-0)e^{-ikr}(2ikr)^{-1}\,. \tag{43}$$

This will be proved in Appendix B, where it will also be shown that the two-component section in region a,

$$e^{-ikz}\begin{pmatrix}0\\1\end{pmatrix}, \tag{44}$$

has at $r\to\infty$ an incoming part that is

$$\left[e^{-ikz}\begin{pmatrix}0\\1\end{pmatrix}\right]_{\text{inc}}\to-\frac{e^{-ikr}}{2ikr}2\delta(1-x)\begin{pmatrix}0\\1\end{pmatrix}=-\frac{e^{-ikr}\sqrt{\pi}}{ikr}\left[(|q|-q)^{1/2}\eta_m+\sum\frac{q(2j+1)^{1/2}}{\sqrt{2}|q|}(\xi^{(1)}_{jm}+\xi^{(2)}_{jm})\right], \tag{45a}$$

where m stands for $-q-\frac{1}{2}$ and the $\sum$ extends from $j=|q|+\frac{1}{2}$ to ∞. This gives the asymptotic incoming section for a helicity $=+1$ particle. For negative helicity we have

$$\left[e^{-ikz}\begin{pmatrix}1\\0\end{pmatrix}\right]_{\text{inc}}\to-\frac{e^{-ikr}\sqrt{\pi}}{ikr}\left[-(|q|+q)^{1/2}\eta_m+\sum\frac{q(2j+1)^{1/2}}{\sqrt{2}|q|}(\xi^{(1)}_{jm}-\xi^{(2)}_{jm})\right], \tag{45b}$$

where m stands for $-q+\frac{1}{2}$ and the $\sum$ extends from $j=|q|+\frac{1}{2}$ to ∞.

Now for a four-component Dirac particle with helicity $=+1$, the section at $r=\infty$ for the incident beam is given by

$$\frac{ke^{-ikz}}{[2E(E-M)]^{1/2}}\begin{bmatrix}0\\1\\0\\k(E+M)^{-1}\end{bmatrix}. \tag{46}$$

One can apply (45a) separately to the upper two components and the lower two components of this equation to obtain the incoming part at $r=\infty$ of ${}^{+}\psi$:

Upper 2 components of ${}^{+}\psi_{\rm inc}$

$$=k[2E(E-M)]^{-1/2}[\text{RHS of (45a)}], \tag{47}$$

Lower 2 components of ${}^{+}\psi_{\rm inc}$

$$=[(E-M)/2E]^{1/2}[\text{RHS of (45b)}]. \tag{48}$$

These have to be matched with the *incoming part* of the right-hand side of (41), which we easily obtain from the explicit forms of $\psi^{(1)}$, $\psi^{(2)}$, and $\psi^{(3)}$ of Sec. III. Matching first the upper two components, we obtain by examining the coefficients of $\xi_{jm}^{(1)}$, $\xi_{jm}^{(2)}$, and η_{jm}, respectively, the equations

$$\begin{aligned}{}^{+}K_{jm}^{(1)}&=\delta_{m+q+1/2}(q/|q|)\pi e^{-i\pi\mu/2}\\&\quad\times[(2j+1)(E-M)/2E]^{1/2},\end{aligned} \tag{49}$$

$${}^{+}K_{jm}^{(2)}=k(E-M)^{-1}({}^{+}K_{jm}^{(1)}), \tag{50}$$

$$\begin{aligned}{}^{+}K_{m}^{(3)}&=0,\quad \text{if } q>0\\&=\delta_{m+q+1/2}k\pi[-qE^{-1}(E-M)^{-1}]^{1/2}e^{i\delta_3},\quad \text{if } q<0.\end{aligned} \tag{51}$$

Matching the lower two components we obtain these same equations.

In an entirely similar way we obtain for a negative-helicity incoming beam

$${}^{-}K_{jm}^{(1)}={}^{+}K_{j(m-1)}^{(1)}, \tag{52}$$

$${}^{-}K_{jm}^{(2)}=-({}^{+}K_{j(m-1)}^{(2)}), \tag{53}$$

$$\begin{aligned}{}^{-}K_{m}^{(3)}&=-\delta_{m+q-1/2}k\pi[qE^{-1}(E-M)^{-1}]^{1/2}e^{i\delta_3},\ \text{if } q>0\\&=0,\quad \text{if } q<0.\end{aligned} \tag{54}$$

Substitution of (49)–(54) into (41) gives the scattering solutions ${}^{+}\psi$ and ${}^{-}\psi$ for the two helicities of an incident beam.

VI. OUTGOING SECTION AND HELICITY-FLIP AND -NONFLIP AMPLITUDES

With the explicit expression (41) for ${}^{\pm}\psi$ it is straightforward to obtain the outgoing section,

$$({}^{\pm}\psi)_{\rm out}=\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{ikr}}{2ikr}\left\{\sum_{j=|q|+1/2}^{\infty}\left[-\frac{q}{|q|}\left(\frac{2j+1}{2E(E-M)}\right)^{1/2}k\pi e^{-i\pi\mu}\right]\rho_{jm}^{\pm}-({}^{\pm}K_{m}^{(3)})\Omega_{m}^{\pm}\right\}, \tag{55}$$

where μ is given by (15), ${}^{\pm}K_{m}^{(3)}$ by (51) and (54), and

$$m=-q\mp\tfrac{1}{2}. \tag{56}$$

$\rho_{jm}^{\pm}$ and $\Omega_{m}^{\pm}$ are four-component sections defined by

$$\rho_{jm}^{\pm}=\begin{bmatrix}\xi_{jm}^{(1)}\mp\xi_{jm}^{(2)}\\ \pm k^{-1}(E-M)(\xi_{jm}^{(1)}\mp\xi_{jm}^{(2)})\end{bmatrix}, \tag{57}$$

$$\Omega_{m}^{\pm}=\begin{bmatrix}\eta_m\\ \dfrac{q}{|q|}k^{-1}(E-M)\eta_m\end{bmatrix}e^{i\delta_3}. \tag{58}$$

For any given direction θ,ϕ, at large r, we now want to resolve the outgoing section (55) into states of ±1 outgoing helicity. This is very easily done. For given θ,ϕ, let us define $\chi^{\pm}$ as the four-component *normalized* amplitudes for a free plane wave with positive energy E, with helicity ±1, propagating in the direction $\hat{r}=\vec{r}/r$ with momentum k. That is, $\chi^{\pm}$ are defined, except for normalization, by

$$(\vec{\sigma}\cdot\hat{r})\chi^{\pm}=\pm\chi^{\pm}, \tag{59}$$

$$(k\vec{\alpha}\cdot\hat{r}+\beta M)\chi^{\pm}=E\chi^{\pm}. \tag{60}$$

$\chi^{\pm}$ can be explicitly written down as follows. Consider the two-component normalized wave functions

$$\begin{aligned}\zeta^{+}&=\frac{1}{[2(1-\cos\theta)]^{1/2}}\begin{pmatrix}\sin\theta\, e^{-i\phi}\\ 1-\cos\theta\end{pmatrix},\\ \zeta^{-}&=\frac{1}{[2(1-\cos\theta)]^{1/2}}\begin{pmatrix}\sin\theta\\ -(1+\cos\theta)e^{i\phi}\end{pmatrix}.\end{aligned} \tag{61}$$

They satisfy

$$(\vec{\sigma}\cdot\hat{r})\zeta^{\pm}=\pm\zeta^{\pm}. \tag{62}$$

One can then easily verify that

$$\chi^{\pm}=\frac{k}{[2E(E-M)]^{1/2}}\begin{bmatrix}\zeta^{\pm}\\ \pm k^{-1}(E-M)\zeta^{\pm}\end{bmatrix} \tag{63}$$

is normalized and satisfies (59) and (60).

Now we find that $\rho^{\pm}$ and $\Omega^{\pm}$ satisfy equations similar to (59) and (60):

$$(\vec{\sigma}\cdot\hat{r})\rho^{\pm}_{jm}=\pm\rho^{\pm}_{jm}, \tag{64}$$

$$(k\vec{\alpha}\cdot\hat{r}+\beta M)\rho^{\pm}_{jm}=E\rho^{\pm}_{jm}, \tag{65}$$

$$(\vec{\sigma}\cdot\hat{r})\Omega^{\pm}_{m}=q|q|^{-1}\Omega^{\pm}_{m}, \tag{66}$$

$$(k\vec{\alpha}\cdot\hat{r}+\beta M)\Omega^{\pm}_{m}=E\Omega^{\pm}_{m}, \tag{67}$$

which can be proved with the aid of the definition of $\rho^{\pm}$ and $\Omega^{\pm}$ and Eqs. (13) and (17). Thus we conclude that

ρ^{+}_{jm}, $\Omega^{q/|q|}$, and χ^{+} are proportional,

ρ^{-}_{jm}, $\Omega^{-q/|q|}$, and χ^{-} are proportional.

To find the constant of proportionality between ρ^{+}_{jm} and χ^{+}, which according to (56) is the case of interest for $m=-q-\frac{1}{2}$, we compute their second elements. That of ρ^{+}_{jm} is, by (57), (11), (12), and (7),

$$(c-s)\left(\frac{j-m}{2j}\right)^{1/2}Y_{j-1/2,m+1/2}-(s+c)\left(\frac{j+m+1}{2j+2}\right)^{1/2}Y_{j+1/2,m+1/2}$$
$$=\frac{q\mu}{|q|(2j+1)^{1/2}}\left[\frac{1}{\sqrt{j}}Y_{j-1/2,-q}-\frac{1}{(j+1)^{1/2}}Y_{j+1/2,-q}\right].$$

Dividing by the second element of χ^{+} we get

$$\rho^{+}_{j(-q-1/2)}=\frac{2q\mu}{k|q|}\left[\frac{E(E-M)}{2j+1}\right]^{1/2}\frac{1}{(1-\cos\theta)^{1/2}}\left[\frac{1}{\sqrt{j}}Y_{j-1/2,-q}-\frac{1}{(j+1)^{1/2}}Y_{j+1/2,-q}\right]\chi^{+}. \tag{68}$$

In an entirely similar way we can express $\rho^{-}_{j(-q+1/2)}$, $\Omega^{+}_{-q-1/2}$, and $\Omega^{-}_{-q+1/2}$ in terms of χ^{+} and χ^{-}. Substitution into (55) then allows us to read off the helicity-flip and helicity-nonflip amplitudes. They are given by

$$C^{+\to+}=C^{-\to-}=-e^{ikr}(2ikr)^{-1}T_q(\Theta),$$
$$C^{+\to-}=0\quad(q>0),$$
$$=2qe^{ikr}(2ikr)^{-1}(\sin\tfrac{1}{2}\Theta)^{2|q|-1}e^{-i\phi+2i\delta_3}\quad(q<0),$$
$$C^{-\to+}=2qe^{ikr}(2ikr)^{-1}(\sin\tfrac{1}{2}\Theta)^{2|q|-1}e^{i\phi+2i\delta_3}\quad(q>0),$$
$$=0\quad(q<0), \tag{69}$$

where

$$\Theta=\pi-\theta=\text{scattering angle} \tag{70}$$

and

$$T_q(\Theta)=\sqrt{2\pi}\,(\sec\tfrac{1}{2}\Theta)\times\sum_{j=|q|+1/2}^{\infty}\mu e^{-i\pi\mu}\left[\frac{1}{\sqrt{j}}Y_{q,j-1/2,-q}(\theta,\phi)-\frac{1}{(j+1)^{1/2}}Y_{q,j+1/2,-q}(\theta,\phi)\right], \tag{71}$$

and

$$\mu=[(j+\tfrac{1}{2})^2-q^2]^{1/2}. \tag{72}$$

δ_3 is given by (33).

We have not given special attention to whether we should evaluate the monopole harmonics in region a or region b. If one measures the helicity-flip or helicity-nonflip differential cross section, the result is independent of which choice we make since a phase $e^{2iq\phi}$ does not change the angular distribution of intensities. If interference experiments are analyzed so that the phase $e^{2iq\phi}$ becomes important, one should use (55), which is valid for either region a or b.

The phase shift δ_3 plays no role if one only measures helicity-flip or helicity-nonflip intensities. If one does polarization experiments, δ_3 is measurable. It is interesting to observe that according to (33) δ_3 depends on the sign of κ.

VII. EVALUATION OF DIFFERENTIAL CROSS SECTION

By using equation (B7) of Ref. 4 we can easily prove that

$$\frac{1}{\sqrt{j}}Y_{q,(j-1/2),-q} \tag{73}$$

is even in q. Thus

$$T_q(\Theta)=T_{-q}(\Theta). \tag{74}$$

In the rest of this section we take

$$q>0. \tag{75}$$

To evaluate $T_q(\Theta)$ we express[4] (73) in Jacobi polynomials and use the difference equation[9]

$$P_n^{0,2q}-P_{n+1}^{0,2q}=(n+q+1)(n+1)^{-1}(1-x)P_n^{1,2q}, \tag{76}$$

obtaining $(q>0)$

$$T_q(\Theta)=2(\cos\tfrac{1}{2}\Theta)(\sin\tfrac{1}{2}\Theta)^{2q}\times\sum_{n=0}^{\infty}(n+q+1)(n+1)^{-1}\times\mu e^{-i\pi\mu}(-1)^n P_n^{2q,1}(\cos\Theta), \qquad (77)$$

where

$$\mu=[(n+1)(n+1+2q)]^{1/2}. \qquad (78)$$

The sum in (77) is not convergent, but is summable.[10] To see this we notice that[9] for fixed $\Theta\neq 0,\pi$, $P_n^{2q,1}(\cos\Theta)=O(1/\sqrt{n})$ for large n, and oscillates. Furthermore $\epsilon=O(n)$. Thus, if we expand μ in powers of

$$\epsilon=2q(n+1)^{-1},$$

the first two terms contribute divergent but summable sums to (77), while the remainder contributes an absolutely convergent series. To implement this idea we write

$$\begin{aligned}(n+q+1)(n+1)^{-1}\mu e^{-i\pi\mu}&=[1+(\epsilon/2)]\mu e^{-i\pi\mu}\\&=(n+1)e^{-i\pi(n+q+1)}\left[1+\epsilon+\frac{i\pi q}{4}\epsilon+\epsilon^2\left(\frac{1}{8}+\frac{i\pi q}{8}-\frac{\pi^2q^2}{32}\right)\right]+R\\&=(-1)^n e^{-i\pi(q+1)}[n+1+2q+i\pi q^2/2+(n+1)^{-1}(q^2/8)(4+4i\pi q-\pi^2q^2)]+R.\end{aligned} \qquad (79)$$

This equation *defines* R, which is of the order $R=O(n^{-2})$. Substitution into (77) gives, as shown in Appendix C,

$$T_q(\Theta)=-qe^{-i\pi q}\frac{1}{\sin^2(\frac{1}{2}\Theta)}\left\{\frac{\cos\frac{1}{2}\Theta}{1+\sin\frac{1}{2}\Theta}\left[1+\left(1+\frac{i\pi q}{2}\right)\sin\tfrac{1}{2}\Theta\right]+\frac{1}{2}\left(1+i\pi q-\frac{\pi^2q^2}{4}\right)\frac{1-(\sin\frac{1}{2}\Theta)^{2q}}{\cos\frac{1}{2}\Theta}\sin^2(\tfrac{1}{2}\Theta)\right\}+U_q(\Theta), \qquad (80)$$

where

$$U_q(\Theta)=2\cos\tfrac{1}{2}\Theta(\sin\tfrac{1}{2}\Theta)^{2q}\sum_{n=0}^{\infty}(-1)^n R P_n^{2q,1}(\cos\Theta)$$

is an absolutely convergent series with individual terms of the order $O(n^{-2.5})$ for large n.

The differential cross section is dependent on the initial helicity. It is equal to [according to (69)], for an unpolarized incoming beam,

$$\frac{d\sigma}{d\Omega}=(2k)^{-2}[|T_{|q|}|^2+2q^2(\sin\tfrac{1}{2}\Theta)^{4|q|-2}]. \qquad (81)$$

Comparing with Rutherford scattering of charge Ze by Coulomb field $Z'e$,

$$\left(\frac{d\sigma}{d\Omega}\right)_R=(Z'Ze^2/2kv)^2(\sin\tfrac{1}{2}\Theta)^{-4}, \qquad (82)$$

we obtain

$$\left(\frac{d\sigma}{d\Omega}\right)\bigg/\left(\frac{d\sigma}{d\Omega}\right)_R=(gv/Z'e)^2[|T_{|q|}|^2q^{-2}\sin^4(\tfrac{1}{2}\Theta)+2(\sin\tfrac{1}{2}\Theta)^{4|q|+2}]. \qquad (83)$$

This can be compared with Banderet's result[1] for the scattering of a *spinless* nonrelativistic charged particle by a fixed magnetic monopole,

$$\left(\frac{d\sigma}{d\Omega}\right)_B\bigg/\left(\frac{d\sigma}{d\Omega}\right)_R=(gv/Z'e)^2\times[\text{function of }\Theta\text{ and }q,\ \text{independent of }k], \qquad (84)$$

and Mott scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_M\bigg/\left(\frac{d\sigma}{d\Omega}\right)_R=[\text{function of }\Theta,\ Ze,\ \textit{and}\ k]. \qquad (85)$$

The quantities in square brackets in (83)–(85) all approach unity as $\Theta\to 0$. The former two are *independent* of k. Their values have been evaluated with a computer and are tabulated in Tables I and II and plotted in Fig. 2. For comparison we plot similar curves for the ratio of Mott scattering[11] to Rutherford scattering in Figs. 3 and 4.

Notice that in Ze-g scattering the cross section near $\Theta\sim 180°$ represents mostly helicity-flip contributions, while in the forward directions the helicity-nonflip cross section dominates. At $\Theta=180°$ the helicity-nonflip cross sections vanish, because of angular momentum conservation. Furthermore, the scattered beam at $\Theta=180°$ is 100% polarized with helicity $=+1$ for $Zeg<0$ and helicity $=-1$ for $Zeg<0$ [see (69)].

It is also interesting to observe in Figs. 3 and 2 that in the important low-intermediate region of Θ, curves (1) and (2) are lower than curves (3) and

(4) for which $ZZ'<0$, but are higher than curves (5) and (6) for which $ZZ'>0$.

APPENDIX A

To prove lemmas 1 and 2 we observe that since $\vec{\sigma}\cdot\hat{r}$ commutes with $\vec{J}$, the Z_{ij}'s of (10) are independent of m. We therefore evaluate Z_{ij} by taking $m=-j$. In that case (7) gives

$$\phi^{(1)}_{j,-j}=\begin{pmatrix}0\\ Y_{j-1/2,-j+1/2}\end{pmatrix},\qquad \phi^{(2)}_{j,-j}=\begin{pmatrix}-\left(\dfrac{2j+1}{2j+2}\right)^{1/2}Y_{j+1/2,-j-1/2}\\ \left(\dfrac{1}{2j+2}\right)^{1/2}Y_{j+1/2,-j+1/2}\end{pmatrix}. \tag{A1}$$

TABLE I. In the first three columns, k^2 times the helicity-nonflip (HNF), the helicity-flip (HF), and the total (with no superscript) cross sections for the scattering of an unpolarized beam of Dirac particles with spin $\frac{1}{2}$ with charge Ze on a fixed magnetic monopole of strength g are tabulated for the case $q=Zeg=\pm0.5$. k is the momentum of the incoming beam. The entries are independent of k [see (81)]. In the fourth column, we tabulate the ratio to the Rutherford cross section $(d\sigma/d\Omega)_R$ as given by (82). The final column exhibits the ratio involving the cross section $(d\sigma/d\Omega)_B$ obtained by Banderet (Ref. 1) for a spinless nonrelativistic particle of charge Ze by a fixed monopole of strength g. Θ is the scattering angle.

Θ	$k^2\left(\frac{d\sigma}{d\Omega}\right)^{\rm HNF}$	$k^2\left(\frac{d\sigma}{d\Omega}\right)^{\rm HF}$	$k^2\left(\frac{d\sigma}{d\Omega}\right)$	$(Z'e/gv)^2\left(\frac{d\sigma}{d\Omega}\right)\Big/\left(\frac{d\sigma}{d\Omega}\right)_R$	$(Z'e/gv)^2\left(\frac{d\sigma}{d\Omega}\right)_B\Big/\left(\frac{d\sigma}{d\Omega}\right)_R$
5	0.173×10^5	0.125	0.173×10^5	1.00	1.00
10	0.108×10^4		0.108×10^4	1.00	1.00
15	0.215×10^3		0.215×10^3	1.00	1.00
20	0.685×10^2		0.686×10^2	1.00	1.00
25	0.283×10^2		0.284×10^2	1.00	1.00
30	0.138×10^2		0.139×10^2	1.00	1.01
35	0.751×10		0.764×10	1.00	1.01
40	0.444×10		0.457×10	1.00	1.01
45	0.280×10		0.293×10	1.01	1.01
50	0.185×10		0.198×10	1.01	1.02
55	0.127×10		0.140×10	1.02	1.02
60	0.904		0.103×10	1.03	1.03
65	0.658	same	0.783	1.04	1.04
70	0.489		0.614	1.06	1.05
75	0.370	for	0.495	1.09	1.06
80	0.284		0.409	1.12	1.07
85	0.221	all	0.346	1.15	1.08
90	0.173		0.298	1.19	1.09
95	0.136		0.261	1.23	1.10
100	0.108	angles	0.233	1.28	1.11
105	0.857×10^{-1}		0.211	1.34	1.13
110	0.680×10^{-1}		0.193	1.39	1.14
115	0.540×10^{-1}		0.179	1.45	1.15
120	0.426×10^{-1}		0.168	1.51	1.17
125	0.336×10^{-1}		0.159	1.57	1.18
130	0.206×10^{-1}		0.151	1.63	1.19
135	0.200×10^{-1}		0.145	1.69	1.20
140	0.150×10^{-1}		0.140	1.75	1.21
145	0.110×10^{-1}		0.136	1.80	1.22
150	0.775×10^{-2}		0.133	1.85	1.23
155	0.519×10^{-2}		0.130	1.89	1.24
160	0.321×10^{-2}		0.128	1.93	1.25
165	0.174×10^{-2}		0.127	1.96	1.25
170	0.735×10^{-3}		0.126	1.99	1.26
175	0.164×10^{-3}		0.125	1.99	1.26
180	0	0.125	0.125	2.00	1.26

TABLE II. Same as Table I, but with $q=\pm 1$.

Θ	$k^2\left(\frac{d\sigma}{d\Omega}\right)^{HNF}$	$k^2\left(\frac{d\sigma}{d\Omega}\right)^{HF}$	$k^2\left(\frac{d\sigma}{d\Omega}\right)$	$(Z'e/gv)^2\left(\frac{d\sigma}{d\Omega}\right)\Big/\left(\frac{d\sigma}{d\Omega}\right)_R$	$(Z'e/gv)^2\left(\frac{d\sigma}{d\Omega}\right)_B\Big/\left(\frac{d\sigma}{d\Omega}\right)_R$
5	0.691×10^5	0.950×10^{-3}	0.691×10^5	1.00	1.00
10	0.433×10^4	0.380×10^{-2}	0.433×10^4	1.00	1.00
15	0.862×10^3	0.850×10^{-2}	0.862×10^3	1.00	1.01
20	0.275×10^3	0.151×10^{-1}	0.275×10^3	1.00	1.01
25	0.114×10^3	0.234×10^{-1}	0.114×10^3	1.00	1.01
30	0.559×10^2	0.335×10^{-1}	0.559×10^2	1.00	1.01
35	0.307×10^2	0.452×10^{-1}	0.307×10^2	1.00	1.01
40	0.184×10^2	0.585×10^{-1}	0.185×10^2	1.01	1.02
45	0.117×10^2	0.730×10^{-1}	0.118×10^2	1.01	1.02
50	0.788×10	0.895×10^{-1}	0.797×10	1.02	1.02
55	0.552×10	0.107	0.563×10	1.02	1.03
60	0.400×10	0.125	0.413×10	1.03	1.04
65	0.297×10	0.145	0.312×10	1.04	1.05
70	0.226×10	0.165	0.243×10	1.05	1.06
75	0.176×10	0.186	0.195×10	1.07	1.08
80	0.138×10	0.207	0.159×10	1.09	1.10
85	0.110×10	0.228	0.133×10	1.11	1.12
90	0.883	0.250	0.113×10	1.13	1.15
95	0.712	0.272	0.984	1.16	1.18
100	0.577	0.294	0.871	1.20	1.21
105	0.468	0.315	0.783	1.24	1.25
110	0.379	0.336	0.715	1.29	1.29
115	0.306	0.356	0.662	1.34	1.34
120	0.246	0.375	0.621	1.40	1.38
125	0.196	0.394	0.590	1.46	1.43
130	0.154	0.411	0.565	1.52	1.47
135	0.118	0.427	0.545	1.59	1.52
140	0.889×10^{-1}	0.442	0.531	1.66	1.56
145	0.648×10^{-1}	0.455	0.520	1.72	1.60
150	0.456×10^{-1}	0.467	0.513	1.79	1.64
155	0.299×10^{-1}	0.477	0.507	1.84	1.67
160	0.165×10^{-1}	0.485	0.502	1.89	1.70
165	0.804×10^{-2}	0.492	0.500	1.93	1.72
170	0.273×10^{-2}	0.496	0.499	1.97	1.74
175	0.250×10^{-3}	0.499	0.499	1.99	1.75
180	0	0.500	0.500	2.00	1.75

Substitution into (10) gives linear equations for the Z's with coefficients that contain the Y's, $zr^{-1} = \cos\theta$, and $(x \pm iy)r^{-1} = \sin\theta e^{\pm i\phi}$. Using the explicit forms of the Y's of Ref. 4 we can solve for the Z's, obtaining

$$Z_{11} = -Z_{22} = -2q(2j+1)^{-1},$$
$$Z_{12} = Z_{21} = -[(2j+1)^2 - 4q^2]^{1/2}(2j+1)^{-1}. \quad \text{(A2)}$$

We shall write these Z's as a matrix:

$$G = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$
$$= -\{2q\tau_z + [(2j+1)^2 - 4q^2]^{1/2}\tau_x\}(2j+1)^{-1}. \quad \text{(A3)}$$

Then (10) can be written as

$$(\vec{\sigma}\cdot\hat{r})(\phi^{(1)}, \phi^{(2)}) = (\phi^{(1)}, \phi^{(2)})G. \quad \text{(A4)}$$

It follows from $(\vec{\sigma}\cdot\hat{r})^2 = 1$ that

$$G^2 = 1, \quad \text{(A5)}$$

which is evident also from the explicit formula (A3). Now $\vec{\sigma}\cdot(\vec{p} - Ze\vec{A})$ commutes with $\vec{J}$. Thus $\vec{\sigma}\cdot(\vec{p} - Ze\vec{A})\phi_{jm}^{(1)}$ is an eigensection of $\vec{J}^2$ and J_z with the same eigenvalues as $\phi_{jm}^{(1)}$. In other words, there exists a 2×2 matrix B so that

$$\vec{\sigma}\cdot(\vec{p} - Ze\vec{A})(\phi^{(1)}, \phi^{(2)}) = r^{-1}(\phi^{(1)}, \phi^{(2)})B, \quad \text{(A6)}$$

which is entirely similar to (A4). B can be ex-

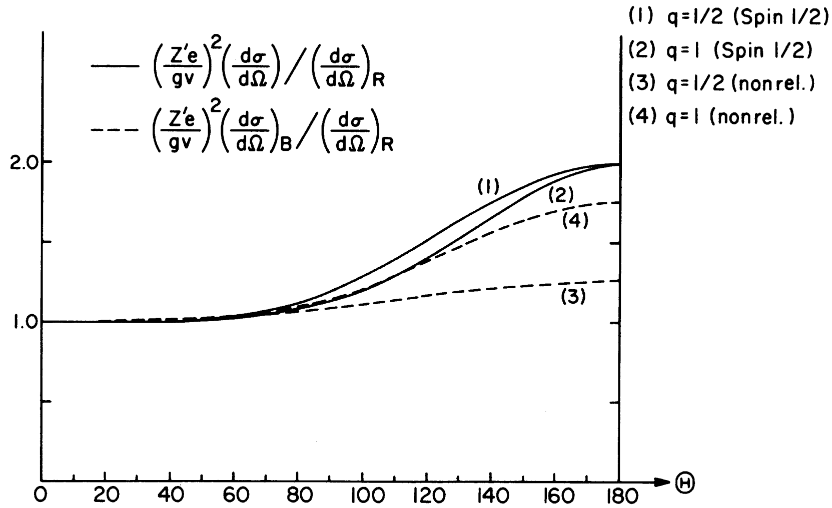

FIG. 2. Ratio of monopole to Rutherford cross sections. $d\sigma/d\Omega$ is the cross section for scattering of a Dirac particle of charge Ze by a fixed monopole g. $(d\sigma/d\Omega)_B$ is the cross section for scattering of a nonrelativistic spinless particle of charge Ze by a fixed monopole g. $(d\sigma/d\Omega)_R$ is the Rutherford cross section (72) for scattering of a particle of charge Ze by a fixed target of charge $Z'e$: $g=Zeg$, $\Theta=$ scattering angle in degrees.

plicitly computed in the same way as G, but we shall here follow a simpler method. It follows from (A6) and (A4) that

$$(\vec{\sigma}\cdot\hat{r})\vec{\sigma}\cdot(\vec{p}-Ze\vec{A})(\phi^{(1)},\phi^{(2)})=r^{-1}(\phi^{(1)},\phi^{(2)})GB. \tag{A7}$$

Now

$$\begin{aligned}(\vec{\sigma}\cdot\hat{r})\vec{\sigma}\cdot(\vec{p}-Ze\vec{A}) &= \hat{r}\cdot(\vec{p}-Ze\vec{A})+i\vec{\sigma}\cdot[\hat{r}\times(\vec{p}-Ze\vec{A})] \\ &= -i\partial_r+ir^{-1}\vec{\sigma}\cdot(\vec{L}+q\hat{r}) \\ &= -i\partial_r+ir^{-1}\vec{\sigma}\cdot\vec{L}+iqr^{-1}\vec{\sigma}\cdot\hat{r}. \end{aligned}\tag{A8}$$

Now

$$\partial_r\phi^{(i)}=0.$$

Also,

$$\vec{\sigma}\cdot\vec{L}\phi^{(1)}=(J^2-L^2-\tfrac{3}{4})\phi^{(1)}=(j-\tfrac{1}{2})\phi^{(1)},$$
$$\vec{\sigma}\cdot\vec{L}\phi^{(2)}=(J^2-L^2-\tfrac{3}{4})\phi^{(2)}=(-j-\tfrac{3}{2})\phi^{(2)}.$$

Using these and substituting (A8) into (A7), we obtain

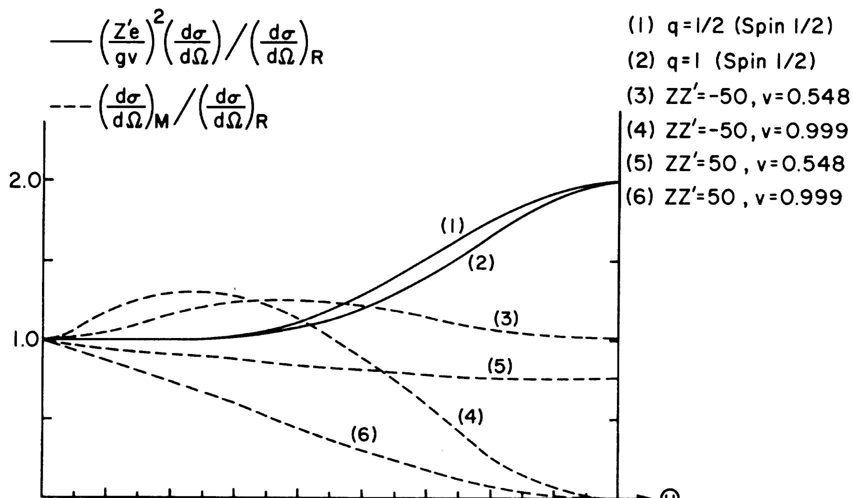

FIG. 3. Ratio of Mott to Rutherford cross sections. $(d\sigma/d\Omega)_M$ is the Mott cross section for scattering of a Dirac particle of charge Ze by a fixed target of charge $Z'e$. $(d\sigma/d\Omega)_R$ is as in Fig. 1. Also plotted for comparison are curves (1) and (2) of Fig. 1. v = velocity of incoming particle.

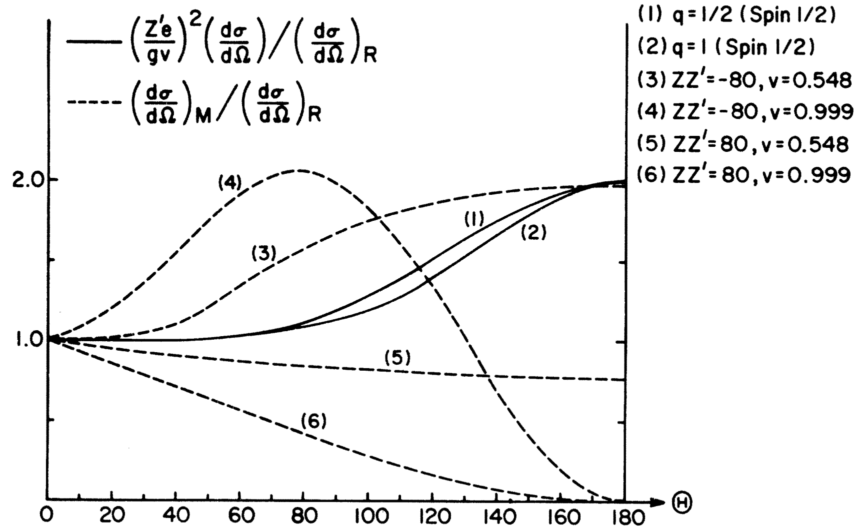

FIG. 4. Ratio of Mott to Rutherford cross sections (continued).

$$i[-1+(j+\tfrac{1}{2})\tau_z]+iqG=GB\,,$$

or (A9)

$$1-iGB=qG+(j+\tfrac{1}{2})\tau_z\,.$$

Now using (12) we can express (A3) and (A9). Thus

$$G=-2cs\tau_z-(c^2-s^2)\tau_x\,, \tag{A10}$$

$$\begin{aligned}1-iGB&=(j+\tfrac{1}{2})(\tau_z+2csG)\\&=(j+\tfrac{1}{2})(c^2-s^2)[(c^2-s^2)\tau_z-2cs\tau_x]\\&=\mu[(c^2-s^2)\tau_z-2cs\tau_x]\,,\end{aligned} \tag{A11}$$

where we have used $c^2+s^2=1$. Now define

$$R=\begin{pmatrix} c & s\\ -s & c\end{pmatrix}.$$

Equations (A10) and (A11) can be written as

$$G=-R\tau_x R^{-1}\,,$$

$$1-iGB=\mu R\tau_z R^{-1}\,.$$

Thus

$$R^{-1}GR=-\tau_x\,, \tag{A12}$$

$$R^{-1}BR=i\tau_x-\mu\tau_y\,. \tag{A13}$$

R is the rotation matrix from ϕ to ξ defined by (11). Operating on the ξ's, $(\sigma\cdot\hat{r})$ generate the matrix $R^{-1}GR$, which by (A12) yields (13). Similarly, (A12) and (A13) yield (14). We have thus proved lemma 1.

To prove lemma 2 we can use (19) and (A1) and the explicit form[4] of the Y's to obtain

$$\eta_{-j}=\text{const}\times\begin{pmatrix}1\\ e^{i\phi}\left(\dfrac{-\cos\theta+q|q|^{-1}}{\sin\theta}\right)\end{pmatrix} \tag{A14}$$

in region R_a. Operating with $\vec{\sigma}\cdot\hat{r}$ one readily obtains (17). A more elegant approach is as follows. Since $\vec{J}$ commutes with $\vec{\sigma}\cdot\hat{r}$ and with $\vec{\sigma}\cdot(\vec{p}-Ze\vec{A})$, one has, similar to (A4) and (A6), the equations

$$\vec{\sigma}\cdot\hat{r}\eta_m=\eta_m G\,, \tag{A15}$$

$$\vec{\sigma}\cdot(\vec{p}-Ze\vec{A})\eta_m=\eta_m Br^{-1}\,, \tag{A16}$$

where G and B are now numbers and not 2×2 matrices. If we now apply (A8) to η_m we obtain

$$i(-j-\tfrac{3}{2})+iqG=GB\,, \tag{A17}$$

which is similar to (A9), but is simpler. Next we compute the anticommutator

$$\begin{aligned}\{(\vec{\sigma}\cdot\hat{r}),\sigma\cdot(\vec{p}-Ze\vec{A})\}_+&=\hat{r}\cdot(\vec{p}-Ze\vec{A})+(\vec{p}-Ze\vec{A})\cdot\hat{r}\\&=\hat{r}\cdot\vec{p}+\vec{p}\cdot\hat{r}\\&=-2i\partial_r-i(\nabla\cdot\hat{r})\\&=-2i\partial_r-2ir^{-1}\,.\end{aligned}$$

Apply this to η_m and we obtain

$$2GB=-2i\,. \tag{A18}$$

Thus

$$G=q|q|^{-1}, \quad B=-iq|q|^{-1}, \tag{A19}$$

which leads to lemma 2.

APPENDIX B

To derive (43), (45a), and (45b) we start with the well-known result

$$e^{-ikrx}=\sum_{l=0}^{\infty}(2l+1)(-i)^l P_l(x)\left(\frac{\pi}{2kr}\right)^{1/2} J_{l+1/2}(kr). \tag{B1}$$

The incoming part at large r is thus, by (34),

$$\sum_{l=0}^{\infty}(2l+1)(-i)^l P_l(x)\left(\frac{\pi}{2kr}\right)^{1/2}\left(\frac{2}{\pi kr}\right)^{1/2}\left(\frac{i}{2}\right)e^{-ikr+i\pi l/2}, \tag{B2}$$

which is summable and gives (43). Another derivation is as follows. For

$$\theta \sim 1/\sqrt{kr},$$
$$e^{-ikz}=\exp[-ikr+ikr(1-\cos\theta)] \tag{B3}$$
$$\cong e^{-ikr}e^{ikr\theta^2/2}$$
$$\cong e^{-ikr}\left(\frac{-2}{ikr}\right)\delta(\theta^2)=-e^{-ikr}\delta(\tfrac{1}{2}\theta^2)(ikr)^{-1},$$

which gives (43).

Equation (43) has the property that it is vanishing in region B. That is, it is a well-defined section with any fixed q. This means that we can expand

$$2\delta(1-x-0)\begin{pmatrix}1\\0\end{pmatrix}=\sum_{ijm}\phi_{jm}^{(i)}A_{jm}^{(i)}. \tag{B4}$$

Multiplying both sides by $\phi_{jm}^{(i)*}$ and integrating over the angles θ and ϕ give the coefficients

$$A_{jm}^{(1)}=\delta_{m+q-1/2}[2\pi(2j+1-2q)]^{1/2}, \quad j\geq |q|+\tfrac{1}{2}$$
$$A_{jm}^{(2)}=-\delta_{m+q-1/2}[2\pi(2j+1+2q)]^{1/2}, \quad j\geq |q|-\tfrac{1}{2}. \tag{B5}$$

Expressing the ϕ's in terms of the ξ's defined in (11) leads from (B4) to (45a). Equation (45b) is derived similarly.

APPENDIX C

To derive (80) the main problem is to evaluate the summable series for $q>0$,

$$W_1=\sum_0^{\infty}P_n^{2q,1}(y)n,$$
$$W_2=\sum_0^{\infty}P_n^{2q,1}(y),$$
$$W_3=\sum_0^{\infty}P_n^{2q,1}(y)(n+1)^{-1},$$

where

$$y=\cos\Theta=-\cos\theta=-x. \tag{C1}$$

We define[9] the generating function

$$H(z)\equiv\sum_0^{\infty}P_n^{2q,1}(y)z^n$$
$$=2^{2q+1}\lambda^{-1}(1-z+\lambda)^{-2q}(1+z+\lambda)^{-1}, \tag{C2}$$

where

$$\lambda=(1-2yz+z^2)^{1/2}. \tag{C3}$$

Then

$$W_1=H'(1-0), \quad W_2=H(1-0), \quad W_3=\int_0^1 H(z)dz. \tag{C4}$$

Thus

$$W_2=H(1-0)=2^{2q+1}(2-2y)^{-q-1/2}[2+(2-2y)^{1/2}].$$

W_1 and W_3 can be evaluated by straightforward differentiation and integration. Collecting terms we obtain (80), where the first term in the curly brackets is a sum of contributions from W_1 and W_2 and the second term in the curly brackets is the contribution from W_3.

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