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Novel inconsistency in two-dimensional gauge theories*

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It is shown that the standard infrared-cutoff procedures are inconsistent with the general axial gauge in 't Hooft's two-dimensional model of confinement.

I. INTRODUCTION

The aim of this paper is to investigate the consistency of a non-Abelian gauge theory model.[1-3] The model is quantum chromodynamics in two dimensions with SU(N) as the gauge group, and only the lowest-order diagrams in the $1/N$ expansion are considered. Since the model is in two dimensions color confinement is straightforward; however, the full structure of the resulting color-singlet sector has yet to be studied. In four dimensions this latter task will come after the confinement has been established.

We work in axial gauges in which $n\cdot A=0$, $n^2=-1$. These gauges have the attractive feature that no ghosts are needed for the quantization. This is also true when $n^2=0$, or $n^2=1$; however, in the spacelike or lightlike case all dependent degrees of freedom can be explicitly eliminated, without the need for any operators that are constrained to vanish on the physical states (which is the case for $n^2=1$). The case $n^2=0$ has some singularities in contributions to Feynman integrals of individual terms in the propagator in four-dimensional calculations. Thus the choice $n\cdot A=0$ with $n^2=-1$ seems to be the most advantageous one one.

For the case $n^2=-1$ the most infrared singular terms are of the form $1/(n\cdot k)^2$. Thus in any number of dimensions we will have to encounter integrations of the form

$$\int \frac{d(n\cdot k)}{(n\cdot k)^2} f(n\cdot(k-p)).$$

Therefore, our discussion will be relevant also to four dimensions. (See Ref. 4 where certain terms in the Hamiltonian look exactly like the two-dimensional model considered here.)

In the paper of 't Hooft[1] in which this model was first studied, the calculations were performed in the light-cone gauge (i.e., the $n^2=0$ gauge). The structure of amplitudes in the singlet sector was later investigated by other authors, also in the light-cone gauge.[2,3]

The infrared singularity in the gluon propagator is treated either by the principal-value prescription[5] or by the sharp-cutoff method. Using the principal-value prescription we replace $1/(n\cdot k)^2$ by

$$\frac{1}{2}\left[\frac{1}{(n\cdot k+i\epsilon)^2}+\frac{1}{(n\cdot k-i\epsilon)^2}\right],$$

whereas the sharp cutoff consists of taking out a small region of the $n\cdot k$ integration around the origin. We show that, using the principal-value prescription, there is no solution to the integral equation for the fermion propagator, for sufficiently small but finite m_0/g (m_0 is the bare mass of the fermions and g the gauge field coupling constant). We should mention that the principal-value prescription is an appealing one, since it leads directly to a linear potential and involves no extra dimensional constants; however, we shall show that it is inconsistent in the general axial gauges.

When we employ a sharp-cutoff procedure, a solution to the fermion propagator can be found. We then attempt to solve the integral equations for the bound states at $m_0=0$, and proceed to show that there is no solution with a covariant mass spectrum. This is surprising in view of the fact that in the light-cone gauge the invariant mass spectrum is smooth in the limit $m_0\to 0$.

In Sec. II we introduce our notation and derive the equations of motion and Feynman rules in a general axial gauge, and in Sec. III we derive the integral equation for the fermion propagator. In Sec. IV we demonstrate that this equation has no solution when $m_0=0$, when the principal-value cutoff prescription is used, and we solve the equation with a sharp cutoff. We use this solution in the next section to try and solve the bound-state equation; however, this equation is found to have no covariant solutions. We digress in Sec. VI to show that even in the light-cone gauge some, but not all, components of the wave function have singular behavior as $m\to 0$. In Sec. VII we prove our strongest result, that with the principal-value cutoff prescription the fermion self-energy equation

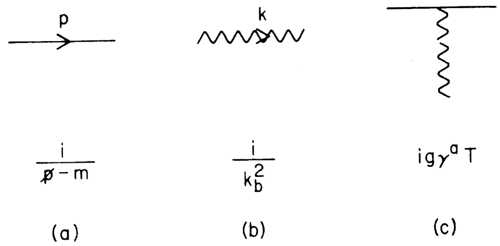

FIG. 1. Feynman rules for a general axial gauge in two dimensions.

has no solutions even for nonzero (although small) masses. Finally in Sec. VIII we present our conclusions.

II. GAUGE CHOICE

For reasons of clarity, we shall not be complete in this paper but shall refer the reader to the original papers[1-3] for the clearest presentation of the model. Since our purpose is to explore the model in a particular family of axial gauges, it is convenient to define rotated coordinates by

$$\begin{aligned} x_a &= \cos\theta\, x_0 + \sin\theta\, x_1 \;, \\ x_b &= -\sin\theta\, x_0 + \cos\theta\, x_1 \;, \end{aligned} \tag{1}$$

so that the invariant length is $x_\mu{}^2 = \cos 2\theta(x_a{}^2 - x_b{}^2) - 2\sin 2\theta(x_a x_b)$, and to work in the class of gauges defined by

$$n\cdot A = A_b = 0 \;. \tag{2}$$

Thus θ is a "gauge parameter" which interpolates between the light-cone gauge ($\theta = \pi/4$) and the axial gauge ($\theta = 0$). In these coordinates, Lorentz-invariant products will be written as $A\cdot B = A^a B_a + A^b B_b$, where $x^{a,b}$ is defined by Eq. (1) but with raised Cartesian indices on the right.

In this family of gauges, only terms linear in the vector field survive in the interaction Lagrangian, and there are no ghosts. When one performs a Lorentz transformation, one also performs a gauge transformation to return to the original gauge. The light-cone gauge is exceptional in the sense that no extra gauge transformation is needed, since the covariant Lorentz change leaves the light-cone gauge invariant. We shall restrict our attention to gauges of the class $-\frac{1}{4}\pi < \theta < \frac{1}{4}\pi$ only, since there are certain complications in dealing with the proper quantization of lightlike and timelike gauges. Of course, while we expect that *nonsinglet* quantities will depend upon the gauge and hence may not be Lorentz covariant, the spectrum of singlet bound states must not depend upon the gauge nor the frame that we choose to work in.

The Feynman rules for the theory under discussion are given in Fig. 1 in the gauge $A_b = 0$. Note that the gluon propagator takes on the simple form $i/k_b{}^2$ (i.e., $i/k_-{}^2$ in the light-cone gauge). A simple way of seeing this is to note that the equations of motion for this vector field achieve the form

$$\partial_b{}^2 A_a = -J^a \;, \tag{3}$$

where indices are raised according to

$$\begin{pmatrix} B^a \\ B^b \end{pmatrix} = \begin{pmatrix} C & -S \\ -S & -C \end{pmatrix} \begin{pmatrix} B_a \\ B_b \end{pmatrix} , \tag{4}$$

and $C = \cos 2\theta$, $S = \sin 2\theta$. In this gauge, $J\cdot A = J^a A_a$. These Feynman rules treat x_a (x_b) as the time (coordinate) variable.

The matrix algebra in terms of a and b components is

$$(\gamma^a)^2 = C, \quad (\gamma^b)^2 = -C, \quad \gamma^a\gamma^b + \gamma^b\gamma^a = -2S \;. \tag{5}$$

As a reminder, note that in the light-cone gauge $C = 0$ and $S = 1$, and the algebra becomes particularly simple.

Before proceeding with any calculations, a prescription must be given to deal with the infrared divergences. In the light-cone gauge, two cutoff procedures have been most popular: the principal-value (PV) prescription and restricting the integration so that $|k_-| > \lambda$. In the former method, the gluon propagator D is written in the form

$$D_{\rm PV} = \tfrac{1}{2} i[(k_- + i\epsilon)^{-2} + (k_- - i\epsilon)^{-2}] \;, \tag{6}$$

whereas in the latter method,

$$D_\lambda = i\theta(k_-{}^2 - \lambda^2)/k_-{}^2 \;. \tag{7}$$

Both procedures have been shown to lead to the same bound-state spectra (at least to leading order in the $1/N$ expansion). The PV prescription has the aesthetic advantage that no new parameters are introduced and, most importantly perhaps, the potential in a quark-antiquark system is automatically linear in their separation distance, whereas using the λ cutoff it is perhaps easier to interpret physically the confining mechanism. A proof of confinement, however, must be carried out in both cases. Below we shall consider both methods and apply them in the more general axial gauges.

III. FERMION PROPAGATOR

First, consider the fermion self-energy which is written in the form

$$\Sigma = A + B\cdot\gamma = i[S_0{}^{-1}(p) - S^{-1}(p)] \;, \tag{8}$$

where S_0 and S are, respectively, the bare and full propagator. Since only rainbow graphs contribute to leading order in $1/N$, the integral equation

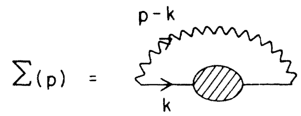

FIG. 2. The fermion self-energy equation.

satisfied by Σ is (in the $C=1$ gauge for simplicity)

$$\Sigma(p_1) = -\frac{ig^2}{4\pi^2}\int d^2k\, D(p_1 - k_1)\gamma_0 S(k)\gamma_0$$

(see Fig. 2) or, transforming to coordinate space,

$$\Sigma(x_1, 0)\delta(x_0) = \tfrac{1}{2}g^2|x_1|\,\delta(x_0)\gamma_0 S(x_1, 0)\gamma_0 \ . \qquad (9)$$

Since $S(x) = \langle 0|T(\psi(x)\bar\psi(0))|0\rangle$, it is easy to show directly that $S^\dagger(x_1, 0) = \gamma_0 S(-x_1, 0)\gamma_0$, where the $x_0 = 0$ limit is the symmetrical limit as $x_0 \to \pm 0$. Equation (9) then implies that $\Sigma^\dagger(x_1, 0) = \gamma_0\Sigma(-x_1, 0)\gamma_0$, and thus

$$\Sigma^\dagger(p) = \gamma_0\Sigma(p)\gamma_0 \ . \qquad (10)$$

This equation implies directly that A and B_μ are real.

Returning to the general gauge, the equation for Σ can be written as

$$\Sigma(p_a, p_b) = \frac{ig^2}{4\pi^2}\int\frac{d^2k}{(p_b - k_b)^2} \times\frac{\gamma^a\{\gamma\cdot[k - B(k)] + m + A(k)\}\gamma^a}{[k - B(k)]^2 - [m + A(k)]^2} \ , \qquad (11)$$

and either the PV or λ cutoff procedures are to be used to regulate the k_b integration. The right-hand side of Eq. (11) is independent of p_a, hence so are $A(p)$ and $B_\mu(p)$. Thus the k_a integration can be done immediately.

Defining the components of B_μ as $B_a = -SB$ and $B_b = -CB$, where B is a scalar function, the integral equations satisfied by A and B are (define $k = k_b$, $p = p_b$)

$$A(p) = \frac{g^2}{4\pi}\sqrt{C}\int\frac{dk}{(p-k)^2}\,a(k) \ ,$$
$$B(p) = \frac{g^2}{4\pi}\int\frac{dk}{(p-k)^2}\,b(k) \ , \qquad (12)$$

where

$$a(k) = \frac{\sqrt{C}\,[m + A(k)]}{\{[k + CB(k)]^2 + C[m + A(k)]^2\}^{1/2}} \ ,$$
$$b(k) = \frac{[k + CB(k)]}{\{[k + CB(k)]^2 + C[m + A(k)]^2\}^{1/2}} \ . \qquad (13)$$

Setting $\theta = \frac{1}{4}\pi$ immediately recovers the familiar light-cone results. Note that $A(p) = A(-p)$, $B(p) = -B(-p)$, and $a^2 + b^2 = 1$.

As one might expect we have been unable to solve the above equations for arbitrary m and θ, nor have we been able to find a meaningful perturbation expansion (in m, $\theta - \frac{1}{4}\pi$, or g for $m = 0$).

IV. ZERO-MASS LIMIT

Let us start by examining the case $m = 0$ for arbitrary C. This should be compared to the familiar discussion of $C = 0$ for arbitrary m. Although the $m = 0$ case has certain problems, its bound-state spectra presents no difficulties if calculated in the light-cone gauge. The case of $m \neq 0$ will be discussed shortly. In this limit, the equations for A and B become (recall $p = p_b$, $k = k_b$)

$$A(p) = 0 \ ,$$
$$B(p) = \frac{g^2}{4\pi}\int\frac{dk}{(p-k)^2}\,\mathrm{sgn}[k + CB(k)] \ . \qquad (14)$$

When the $m \neq 0$ case is discussed, it will be shown that a spontaneous nonzero solution for A cannot develop if $m = 0$.

PV method. By inspection it can be seen that the principal-value definition of the integral will generate poles in $B(p)$ at those values of p where $p + CB(p)$ changes sign. However, since $C > 0$, such a series of poles in B is inconsistent with Eq. (14) because even though the right-hand side of Eq. (14) can reproduce such a series of poles, their residues necessarily have the *wrong sign*. Thus in any gauge with $C > 0$, there is *no solution* to Eq. (14) for B for physical values of g^2.

λ method. Using the λ cutoff, a solution to Eq. (14) can be easily found for all C:

$$B(p) = \frac{g^2}{2\pi}\left(\frac{1}{\lambda} - \frac{1}{|p|}\right)\mathrm{sgn}(p)\theta(p^2 - \lambda^2) \ . \qquad (15)$$

This solution can also be shown to be unique. The θ function is not explicitly denoted in the solution given by 't Hooft, but it is necessary to retain it for all $C \neq 0$ gauges for consistency.

Thus we arrive at the surprising conclusion that there are cutoff procedures which are *inconsistent* with the choice of gauge. In the above simple cases this inconsistency was rather dramatic, namely there was no solution whatsoever in the PV, $C \neq 0$ case. Let us now proceed to check if the λ method is consistent in the singlet sector by examining the bound-state spectrum.

V. BOUND STATES

The eigenvalue condition for the spectrum of bound states in the q-$\bar q$ channel can be discussed by decomposing the bound-state wave function into

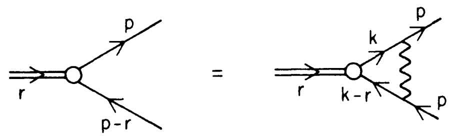

FIG. 3. The bound-state equation.

the form ($\gamma_5 \equiv \gamma_0\gamma_1$)

$$\Gamma = \Gamma_+\gamma^+ + \Gamma_-\gamma^- + \Gamma_1(1+\gamma_5) + \Gamma_2(1-\gamma_5) \ . \tag{16}$$

The reason for using this expansion rather than a series in γ^a, γ^b, for example, is that the integral equations for the above Γ's decouple. In any case, the γ's can be written simply as

$$\gamma^a = \frac{1}{\sqrt{2}}(c+s)\gamma^+ - \frac{1}{\sqrt{2}}(c-s)\gamma^- \ ,$$

$$\gamma^b = \frac{1}{\sqrt{2}}(c-s)\gamma^+ + \frac{1}{\sqrt{2}}(c+s)\gamma^- \ ,$$

where $c = \cos\theta$, $s = \sin\theta$.

The integral equation for the Γ's are depicted diagrammatically in Fig. 3. The equation for Γ can be written as (recall that $m = A = 0$)

$$\Gamma(p,r) = \frac{ig^2}{4\pi^2}\int d^2k\left[\frac{\theta((p_b-k_b)^2-\lambda^2)}{(p_b-k_b)^2} \times \frac{\alpha\cdot\gamma}{\alpha^2+i\epsilon}\gamma^a\Gamma(k,r)\gamma^a\frac{\beta\cdot\gamma}{\beta^2+i\epsilon}\right] , \tag{17}$$

where

$$\alpha_\mu = p_\mu - B_\mu(p_b) \ ,$$

$$\beta_\mu = p_\mu - r_\mu - B_\mu(p_b - r_b) \ .$$

It is convenient to write $\alpha\cdot\gamma = \alpha_-\gamma^- + \alpha_+\gamma^+$, etc, so as to simplify the matrix algebra since $\gamma^{+2} = \gamma^{-2} = 0$. Note the fact that for Γ_-, for example, one can simplify expressions considerably:

$$\frac{\alpha_-}{\alpha^2+i\epsilon} = \frac{c-s}{\sqrt{2}\,C}\left[p_a + D(p_b) - i\epsilon(p_b)\right]^{-1} \ , \tag{18}$$

where

$$D(p_b) = \frac{1-S}{C}p_b + B(p_b) \ .$$

The bound-state equation for Γ_- becomes

$$\Gamma_- = \frac{ig^2}{4\pi^2}\left[p_a + D(p_b) - i\epsilon(p_b)\right]^{-1} \times \left[p_a - r_a + D(p_b - r_b) - i\epsilon(p_b - r_b)\right]^{-1} \times \int \frac{d^2k}{(p_b-k_b)^2}\,\theta((p_b-k_b)^2-\lambda^2)\Gamma_-(k,r) \ . \tag{19}$$

Defining

$$\phi_i(p_b,r) = \int_{-\infty}^{\infty} dp_a \Gamma_i(p,r) \ ,$$

the equation for ϕ_- becomes (for $r_b > 0$)

$$\phi_-(p_b,r) = \frac{g^2}{2\pi}\frac{\theta(p_b)\theta(r_b-p_b)}{r_a + [(1-S)/C]r_b + B(p_b) - B(p_b-r_b)} \times \int_0^{r_b}\frac{dk_b}{(p_b-k_b)^2}\,\phi_-(k_b,r)\theta(p_b-k_b)^2-\lambda^2) \ . \tag{20}$$

The cutoff-dependent terms in this equation cancel identically as can be seen by writing

$$\int_0^{r_b}\frac{dk_b}{(p_b-k_b)^2}\,\phi_-(k_b,r)\theta((p_b-k_b)^2-\lambda^2) = \mathcal{P}\int_0^{r_b}\frac{dk_b}{(p_b-k_b)^2}\,\phi_-(k_b,r) + \phi_-(p_b,r)E(p_b,r) \ , \tag{21}$$

where $\mathcal{P}$ means principal value and

$$E = \frac{1}{p_b}\,\theta(\lambda^2 - p_b^2) + \frac{1}{\lambda}\,\theta(p_b^2-\lambda^2) + \frac{1}{r_b - p_b}\,\theta(\lambda^2-(p_b-r_b)^2) + \frac{1}{\lambda}\,\theta((p_b-r_b)^2-\lambda^2) \ .$$

Inserting this identity into Eq. (20), the λ-dependent terms cancel between E and the B's. The final form of the bound-state equation is achieved by introducing scaled variables according to $x = p_b/r_b$, $y = k_b/r_b$, and one finds

$$\mu_-^2\phi_-(x) = -\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_-(x) - \mathcal{P}\int_0^1 dy\,\frac{\phi_-(y)}{(y-x)^2} \ , \tag{22}$$

where

$$\mu_-^2 = -2\left(r_a r_b + \frac{1-S}{C}r_b^2\right)\frac{\pi}{g^2} \ . \tag{23}$$

If $C \to 0$, the quantity μ_-^2 becomes equal to μ^2, the invariant mass of the bound system, and Eq. (22) is identical to that of 't Hooft. However, the eigenvalue, μ_-^2, is not an invariant for general values of θ, and Eq. (22) is *not a physically meaningful equation in general gauges*.

A similar equation can be derived for the Γ_+ component of the wave function Γ. One finds an equation identical to Eq. (22) but with μ_+^2 in place of μ_-^2, where

$$\mu_+^2 = 2\left(r_a r_b - \frac{1+S}{C}r_b^2\right)\frac{\pi}{g^2} \ , \tag{24}$$

which also suffers from the same difficulties as μ_-^2. It is important to note that the boundary conditions as $x \to 0$ and $x \to 1$ are independent of θ since they are driven by the $1/x$ and $1/(1-x)$ terms on the right-hand side of Eq. (22). The lack of covariance of the eigenvalue cannot be compensated by any corresponding change in the boundary conditions.

The equations for Γ_1 and Γ_2 have even more severe problems than the above; the infrared cutoff λ does not cancel out of the equations.

VI. SCALAR AND PSEUDOSCALAR DENSITIES: AN ASIDE

Before proceeding further with the main issue of this paper let us consider the equations for Γ in the light-cone gauge for $m \neq 0$: The reason for this aside is to remind ourselves as to the type of singularities expected in the $m=0$ limit.[2] One finds only one independent amplitude and

$$\Gamma_1 = \frac{m}{2p_-}\Gamma_- ,$$

$$\Gamma_2 = \frac{-m}{2(r-p)_-}\Gamma_- , \tag{25}$$

$$\Gamma_+ = \frac{m^2}{2p_-(r-p)_-}\Gamma_- ,$$

and

$$\Gamma_-(p,r) = \frac{-ig^2}{\pi^2}\frac{p_-(r-p)_-}{d(p)d(p-r)}\,\mathcal{P}\!\int\frac{d^2k}{(k-p)^2}\Gamma_-(k,r) , \tag{26}$$

where $d(p)=p^2-m^2+g^2/\pi+i\epsilon$.

Since ϕ_- is a function of x only, it follows that

$$\phi_1 = \frac{m}{2r_-}\frac{1}{x}\phi_-(x) ,$$

$$\phi_2 = \frac{-m}{2r_-}\frac{1}{1-x}\phi_-(x) , \tag{27}$$

$$\phi_+ = \frac{m^2}{2r_-^2}\left(\frac{1}{x}+\frac{1}{1-x}\right)\phi_-(x) .$$

Now consider the limit $m\to 0$. Since for $x\to 0$, $\phi_-(x)\sim Ex^h$, and $\phi_-(x)\sim E(1-x)^h$ for $x\to 1$, where $h=(m/g)(3/\pi)^{1/2}$, one sees that $\phi_1(x)$ vanishes as $m\to 0$ for any nonzero x. If $x=0$, however, ϕ_1 blows up in such a way that its integral is finite. Indeed, one finds that as $m\to 0$,

$$\phi_1 = \frac{Eg}{2r_-}\left(\frac{\pi}{3}\right)^{1/2}\delta(x)\sim\delta(p_-) ,$$

$$\phi_2 = \frac{-Eg}{2r_-}\left(\frac{\pi}{3}\right)^{1/2}\delta(1-x)\sim\delta(r_- - p_-) , \tag{28}$$

$$\phi_+ \to 0 .$$

Thus the scalar $(\phi_1+\phi_2)$ and pseudoscalar $(\phi_1 - \phi_2)$ densities become δ functions in momentum space in the light-cone gauge as $m\to 0$.

VII. SMALL, NONZERO, MASS CASE

In this section we prove that the self-energy equations (12), (13) have no solution if the bare mass of the quarks is small and the principal-value cutoff is used to regulate the infrared divergences. In the arguments below we shall frequently use the following two results:

(i) $a(k)$, $b(k)$ [as defined in Eq. (13)] are, respectively, symmetric and antisymmetric functions of k. This follows readily from the requirement that the solution of the self-energy equation (12) be unique.

$$\text{(ii)}\quad \mathcal{P}\int_{-\infty}^{\infty}\frac{dk}{(p-k)^2}=0 . \tag{29}$$

We start by showing that $a(k)$ has no absolute minimum. Assume on the contrary that $a(k)$ has such a minimum at $\bar{k}$.

Now

$$A(\bar{k}) = G\sqrt{c}\,\mathcal{P}\int\frac{dk}{(p-k)^2}[a(k)-a(\bar{k})],\quad G=\frac{g^2}{4\pi} , \tag{30}$$

where we have used (29). But $a'(\bar{k})=0$, so that the integration is now regular and the $\mathcal{P}$ symbol may be omitted. Since by definition $a(k)>a(\bar{k})$ for all k, it follows that $A(\bar{k})>0$ and that the minimum of $a(k)$ be positive. From the integral equation (12) it can be readily shown that the asymptotic behavior of A and B as $k\to\infty$ is

$$A(k)\underset{k\to\infty}{\sim}\frac{GCm}{k^2}\ln\frac{k^2}{m^2} , \tag{31a}$$

$$B(k)\underset{k\to\infty}{\sim}-\frac{2G}{k} . \tag{31b}$$

Now (31) implies that $a(k)\to 0$ as $k\to\infty$ and hence no global minimum of a exists, and a is always positive. Thus for all values of k

$$A(k)+m>0 . \tag{32}$$

Consider now

$$A(0)=G\sqrt{c}\int\frac{dk}{k^2}[a(k)-a(0)] . \tag{33}$$

Again no regulation of this integral is necessary, this time because $a(k)$ is an even function of k. From the antisymmetry of B, and the fact that no singular solutions of B exist at the origin, it follows that B vanishes at the origin and hence that $a(0)=1$, which is the maximal value of $a(k)$. From (33) we now conclude that

$$A(0)<0 . \tag{34}$$

Combining (32) and (34) we see that $A(0)\to 0$ as $m\to 0$. Incidently this result is sufficient to prove

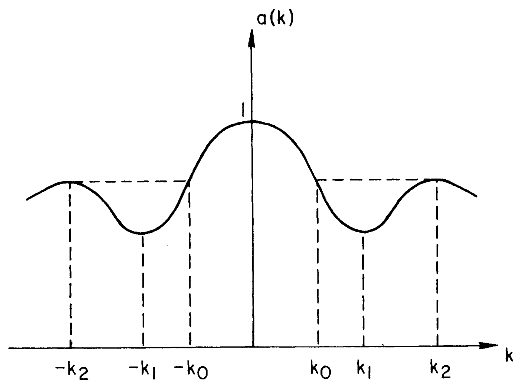

FIG. 4. Hypothetical form for the function $a(k)$, shown to be inconsistent in Sec. VII.

that no spontaneous solution for A is possible if $m=0$.

We now prove that $a(k)$ is a monotonically decreasing function of k in the region of k from zero to infinity, and hence also that $b(k)$ is a monotonically increasing function of k in this region. Assume on the contrary that there is one local minimum at k_1 and one maximum at k_2. The notation is defined in Fig. 4:

$$A(k_1)=G\sqrt{C}\int\frac{dk}{(k-k_1)^2}\,[a(k)-a(k_1)]\;, \tag{35}$$

$$A(k_2)=G\sqrt{C}\int\frac{dk}{(k-k_2)^2}\,[a(k)-a(k_2)]\;. \tag{36}$$

We define the quantity Δ by

$$\Delta(k_1,k_2)\equiv A(k_1)-A(k_2)$$
$$=G\sqrt{C}\int dk\left[\frac{a(k)-a(k_1)}{(k-k_1)^2}-\frac{a(k)-a(k_2)}{(k-k_2)^2}\right]\;. \tag{37}$$

Again since k_1 and k_2 are by definition extrema of a, no regulation of the integrals is necessary. We now show that the integral equation demands that Δ is positive. To this end it is convenient to divide the k integration into three regions:

(i) $-k_0<k<k_0$. In this region

$$[a(k)-a(k_1)]>[a(k)-a(k_2)]$$

and

$$(k-k_1)^2<(k-k_2)^2\;,$$

so that this region contributes positively to Δ.

(ii) $k_0<k<k_3$ and $-k_0>k>-k_3$. In this region $[a(k)-a(k_1)]>0$, but $[a(k)-a(k_2)]<0$ so that this region also contributes positively to Δ.

(iii) $-\infty<k<-k_3$ and $k_3<k<\infty$. The contribution to Δ from this region is

$$2\int_{k_3}^{\infty}dk\left\{[a(k_2)-a(k)]\,\frac{k^2+k_2^{\,2}}{(k^2-k_2^{\,2})^2}-[a(k_1)-a(k)]\,\frac{k^2+k_1^{\,2}}{(k^2-k_1^{\,2})^2}\right\}\;. \tag{38}$$

The first term under the integral in (38) is larger than the second so that the contribution to Δ from this region is also positive. Thus we have shown that

$$A(k_1)>A(k_2)\;. \tag{39}$$

Similarly it can be shown that

$$B(k_1)<B(k_2)\;. \tag{40}$$

Combining (39) and (40) we see that $a(k_1)>a(k_2)$ contrary to the initial assumption. This argument can be readily applied to the situation with an arbitrary number of maxima and minima. Let k_2 be the positive position of the largest maximum of $a(k)$ (except at the origin, of course), and let k_1 be the position of the smallest minimum of $a(k)$ under the condition $0<k_1<k_2$. k_0 is defined by $a(k_0)=a(k_2)$ with $0<k_0<k_1$. Define Δ as in (37), and again it is possible to divide the range of integration into three regions in each of which Δ is positive:

(a) $-k_0<k<k_0$, where argument (i) above applies,

(b) all $k>k_0$, and all $k<-k_0$ such that $a(k)>a(k_1)$, where argument (ii) above applies, and

(c) all $k>k_2$ and all $k<-k_2$ such that $a(k)<a(k_1)$, where argument (iii) above applies.

We have thus shown that $a(k)$ is a monotonically decreasing function for positive k. Let us now choose a k_ϵ such that ($\epsilon\sim 1$)

$$a(k_\epsilon)=\epsilon\;. \tag{41}$$

From (32) we have

$$m\geq -A(0)=2G\sqrt{C}\int_0^{\infty}\frac{dk}{k^2}\,[1-a(k)]$$
$$\geq 2G\sqrt{C}\int_{k_\epsilon}^{\infty}\frac{dk}{k^2}\,[1-\epsilon]=\frac{2G\sqrt{C}\,(1-\epsilon)}{k_\epsilon}\;.$$

Hence

$$k_\epsilon\geq\frac{2G\sqrt{C}(1-\epsilon)}{m}\;. \tag{42}$$

A useful parametrization is now defined by

$$1-\epsilon=\left(\frac{m^2}{G}\right)^a\;, \tag{43}$$

where $m^2<\lambda$ and $0<a<\frac{1}{2}$, so that

$$a(k)\leq\left[1-\left(\frac{m^2}{G}\right)^a\right]\quad\text{for } k\geq k_\epsilon\;, \tag{44a}$$

$$a(k)\geq\left[1-\left(\frac{m^2}{G}\right)^a\right]\quad\text{for } k\leq k_\epsilon\;, \tag{44b}$$

$$k_\epsilon \ge 2\sqrt{C}\, G^{1-a} m^{2a-1} \,, \tag{45}$$

and

$$b(k) = [1 - a^2(k)]^{1/2} \le \sqrt{2}\left(\frac{m^2}{G}\right)^{a/2} \quad \text{for } k \le k_\epsilon \,. \tag{46}$$

Then for a fixed $p < k_\epsilon$

$$\begin{aligned} B(p) &= G\,\mathcal{P}\int \frac{dk}{(p-k)^2}\, b(k) \\ &= G\int_0^{k_\epsilon} dk\, b(k)\left[\frac{1}{(p-k)^2} - \frac{1}{(p+k)^2}\right] \\ &\quad + G\int_{k_\epsilon}^{\infty} dk\, b(k)\left[\frac{1}{(p-k)^2} - \frac{1}{(p+k)^2}\right] . \end{aligned} \tag{47}$$

Both terms on the right-hand side vanish like a positive power of m as $m \to 0$. Similar expressions can be written to show that $A \to 0$ also like a positive power of the mass. Thus $a(p) \to 0$ like a power of the mass in contradiction to (44b), and we have demonstrated that *there is no solution* to the self-energy equations for a sufficiently small bare quark mass.

VIII. CONCLUSIONS

We have shown that the principal-value prescription for regulating the infrared divergences is inconsistent with the general axial gauges, with canonical quantization. The sharp λ cutoff is also inconsistent, but at a more subtle level. While our results have been shown only to leading order in the $1/N$ expansion, it appears to us unlikely that this is the source for the inconsistency.

All our calculations and arguments hold for small bare masses, i.e., $m_0 \ll g$, and it is conceivable that the theory is singular in this strong-coupling regime (perhaps for $m_0 < g/\sqrt{\pi}$). The integral equations that we use should allow us to continue freely in the bare mass and certainly there is no signal for such a difficulty in the light-cone gauge solution. The weak-coupling approximation to the bound-state kernel has been recently studied[6] to see if this inconsistency manifests itself as a lack of invariance of the mass spectrum. It is found that the eigenvalues are invariant; however, the inconsistency is expected to show up only in higher orders.

Our work here is incomplete in that we have been unable to state the general requirements which ensure the consistency of the cutoff procedure and the choice of gauge. We believe that this new phenomenon also occurs in four dimensions, in particular, in axial gauges. The full effects of this new type of inconsistency should be further explored, in view of the interest in confining theories. Even if the above consistency can be guaranteed, one must then prove the uniqueness of the solution. Could it be that two different cutoff procedures lead to different, finite, gauge-invariant solutions. Unfortunately, in the model we have studied we have been unable to find any solution.

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