Some restrictions on possible supergroups and flavor groups

Cihan Saclioglu*†
Department of Theoretical Physics, University of Oxford, Oxford, England
(Received 14 October 1976)

It is pointed out that a suggestion of Nambu which yields diquark currents from the usual flavor or color quark currents may be relevant for classifying similar currents which arise in gauge theories unifying strong, weak, and electromagnetic interactions. The requirement that charges of $SU(3)_{color} \times SU(n)_{flavor}$ plus the new diquark currents complete the generators of a simple vectorlike supergroup G can be met only in the cases $n = 3, G = F_4$; $n = 6, G = E_7$; and $n = 7, G = SU(15)$ for $n \leq 30$. Black-hole "no-hair" theorems seem to indicate an interesting but speculative motivation for diquark and leptoquark currents. The exceptional groups G_2 and E_6 result from an analogous generalization of SU(3) and $SU(3) \times SU(3) \times SU(3)$. Explicit generators involving diquark and leptoquark charges are constructed for the groups G_2 and F_4.

I. INTRODUCTION

As first observed by Georgi and Glashow[1] and by Georgi, Quinn, and Weinberg,[2] theories which attempt to unify strong, weak, and electromagnetic interactions lead, in the absence of special precautions, to processes violating baryon and lepton number conservation. If these transitions are assumed to occur through diagrams involving a single superheavy boson propagator and if the quarks are nonintegrally charged, currents changing quarks to leptons must be employed together with diquark currents which can simultaneously create or annihilate two quarks. Naturally, the corresponding charges constitute part of the generators of the supergauge group. Charges of this type have already been mentioned in a number of articles[3,4,5]. It is also an intriguing coincidence that the most direct way of avoiding an unacceptably short proton lifetime, i.e., the use of extraordinarily large masses for the superheavy bosons, leads to values comparable to the Planck mass,[2] suggesting that such a theory might be connected to gravitation as well. This last view is further encouraged by theoretical work on black holes in general relativity; since these are believed to be described completely by their charges, angular momenta, and masses,[6] and since it has been argued that they can explode through blackbody radiation,[7] a process in which a proton is dropped into a black hole and a final state with zero baryon and odd lepton number is obtained becomes a natural theoretical possibility. Obviously, baryon annihilation without the creation of an odd number of leptons is forbidden by angular momentum conservation. If a microscopic local-field-theory description of a baryon → lepton + mesons transition with local currents coupling to quarks is now sought, it is easily seen that diquark and leptoquark currents are again needed so that both nonintegral charge and half-integral spin can be conserved. (As black holes are not expected to conserve fermion number we are not considering models[3] in which the proton can decay into three leptons plus mesons through higher-order diagrams.) After such a consideration of the connection between diquark/leptoquark currents and black-hole physics, the tempting next step is to interpret literally the superheavy propagator appearing in the proton-decay matrix element as a virtual black hole with the Planck mass. In this case, the simultaneous violation of lepton and baryon number conservation can simply be attributed to the presence of a black hole as in the discussion involving macroscopic black holes. Not surprisingly, the possibility that virtual black holes may act as baryon sinks has already been mentioned in articles on general relativity.[8]

Diquark currents have also been introduced in a more general context: Nambu has argued[9] a massless fermion Lagrangian has a built-in internal-symmetry group larger than whatever specific internal symmetry the fermion fields may explicitly possess. This results from the invariance under Pauli-Gürsey[10,11] transformations which mix fermion and antifermion fields, thus also yielding diquark operators when applied to the generators associated with the explicit internal-symmetry group. It is then economical to assume that the diquark currents involved in baryon-lepton transitions are obtained according to Nambu's suggestion from the currents generating the internal-symmetry group $SU(3)_{color} \times SU(n)_{flavor}$. This fixes the combinations of γ matrices entering into the diquark currents and allows us to count and classify them by simple methods developed in Ref. 9.

If the supergroup is assumed to be vectorlike,[12] one can thus obtain a specific quadratic expression in n for the number of its generators involving *quark* color and flavor charges, together with diquark/leptoquark charges by the methods outlined above. A vectorlike supergroup is chosen here because of the attractiveness of the idea that if both color and weak currents are taken as generators of a simple group, it is more symmetric to let them have the same γ-matrix structure and assume parity violation is caused by an additional symmetry breakdown. Otherwise, the following analysis could be repeated allowing an arbitrary number of the supergroup currents to have axial-vector components as well. To find the total number of generators of the supergroup, one must next specify the number of purely *leptonic* charges. Unfortunately, we have no very reliable guiding principles here except one of maximum economy and a generalized "universality" between *observable* hadron and lepton currents; we assume[13] the notion that hadronic and leptonic weak charges *together* make up an SU(2) generalizes to the remaining flavor currents, and that this exhausts all leptonic charges. Thus the leptonic charges are already counted among the n^2-1 color-singlet flavor currents. Then the quadratic expression in n can be set equal to quadratic expressions in k for the number of generators of supergroups of the SU(k), SO(k), or Sp(k) type. We looked for solutions to the resulting Diophantine equations up to thirty flavors, arbitrarily choosing this number as a marking point beyond which the economic appeal of the quark model can be safely considered as lost.[14]

The plan of the paper is as follows: In Sec. II, we introduce, classify, and count the new diquark charges. We then present the Diophantine equations mentioned above and give their solutions up to thirty flavors. For $SU(7)_{flavor}$, the possible supergroup is SU(15), while $SU(3)_{flavor}$ and $SU(6)_{flavor}$ lead to the correct number of generators for the exceptional groups F_4 and E_7, respectively. When the flavor group is equal to the identity or SU(3) $\times$ SU(3), G_2 and E_6 are seen to be possible supergroups, the former being presumably relevant only for illustrative purposes. In Sec. III, we explicitly construct the generators of G_2 and F_4 using the diquark charges to prove that these charges, together with the usual $q^\dagger q$ type, indeed constitute the generators of a larger group for these special cases. The cases of SU(15) and E_7 will be taken up in a future communication. In Sec. IV, we compare existing unified gauge theories based on exceptional groups with our classification scheme and present concluding remarks.

II. DIQUARK AND LEPTOQUARK CHARGES

We start with the following n^2+7 charges of the group $SU(3)_{color}\times SU(n)_{flavor}$

$$(8,1)=\int dv\,[q^{i\nu}q_{j\nu}-\tfrac{1}{3}\delta^i_j(q^{k\nu}q_{k\nu})], \tag{1}$$

$$(1,n^2-1)=\int dv\,[q^{i\alpha}q_{i\beta}-(1/n)\delta^\alpha_\beta(q^{k\nu}q_{k\nu})], \tag{2}$$

where latin and Greek indices denote color and flavor indices, respectively. Following Nambu's suggestion,[9] we can generate diquark charges from (1) and (2) using Pauli-Gürsey transformations expressed by

$$q_{i\alpha}\to a q_{i\alpha}+b\gamma_5(q_{i\alpha})^c,\quad |a|^2+|b|^2=1, \tag{3}$$

$$q_{i\alpha}\to \exp(i\alpha\gamma_5)q_{i\alpha}, \tag{4}$$

where q^c denotes the charge-conjugate field, given as $\gamma_2 q^*$ in the Dirac-Pauli representation of the γ matrices. Depending on whether (3) is applied singly or with (4), the following types of new charges

$$\int dv\, q^{i\alpha}\gamma_5\gamma_2 q^{j\beta}, \tag{5}$$

$$\int dv\, q^{i\alpha}\gamma_2 q^{j\beta}, \tag{6}$$

and their conjugates, in which covariant SU(3) $\times$ SU(n) indices appear, are found. In contrast to $q^\dagger q$ charges, some components of (5) and (6) vanish identically due to the symmetry properties of the product representations of two covariant or two contravariant quark fields. For example, in (5), the interchange of matrix indices of $\gamma_5\gamma_2$ gives a minus sign just as the interchange of two anticommuting quark fields does, hence only representations with color and flavor both antisymmetric or both symmetric survive. On the other hand, since γ_2 is symmetric, (6) leads to representations where color and flavor indices have opposite symmetry. If one assumes that only color-singlet states are observable, then only diquark charges with antisymmetric color content can couple to physical particles. Thus, as pointed out by Lee,[4] two quarks which are necessarily in a color-antisymmetric state in a proton can annihilate via a diquark current also antisymmetric in color, while the remaining quark decays to a lepton. This argument is naturally intended only to illustrate the group theory and cannot be taken literally unless a lepton with exactly the proton mass is found to exist. The important point is that the color-symmetric diquark charges cannot contribute to any observable process as long as color-nonsinglet states are excluded. *We therefore assume that only color-antisymmetric diquark operators are among the generators of the unifying supergroup and, in fact, that the number of generators of the supergroup consists of the n^2+7 in (1) and (2), plus the aforementioned number of diquarks.*

Now we can simply obtain quadratic expressions in n for the number of the supergroup generators: There are $2\times 3\times n(n-1)/2$ new charges of the type (5) and its Hermitian conjugates, and $2\times 3\times n(n+1)/2$ of the type (6), again with conjugates included. Thus, according to the assumptions presented in Sec. I, we have the following possibilities: (i) only $\gamma_5\gamma_2$ diquark charges, yielding $n^2+7+3n(n-1)$ generators for the supergroup, (ii) only γ_2 diquark charges, giving $n^2+7+3n(n+1)$ generators, (iii) both $\gamma_5\gamma_2$ and γ_2 charges, giving n^2+7+6n^2 generators. To see whether these numbers can correspond to the number of generators of SU(k)-, SO(k)-, or Sp(k)-type groups, we have to set them equal to k^2-1, $k(k-1)/2$, $k(2k+1)$, respectively. For $n\le 30$, the only solution of this kind turns out to be $n=7$, SU($k=15$). In this case it is not possible to fit all the quarks and leptons into a single fundamental representation, but this is not a serious objection to an SU(15) gauge theory.[15]

On the other hand, it is intriguing that the numbers of generators for all exceptional groups other than E_8 can be generated by the above method. These are the groups G_2, F_4, E_6, E_7, each having at least one SU(3) subgroup which Gürsey[16,17] has proposed to interpret as the color group. Thus, with just a color SU(3), no flavor group, and only the γ_2 diquark charges, one has $8+2\times 3=14$ generators, the correct number for G_2. Similarly, $SU(3)_{\rm color}\times SU(3)_{\rm flavor}$ extended by γ_2 diquark charges gives 52 generators, as is needed for F_4. $SU(3)_{\rm color}\times SU(6)_{\rm flavor}$ together with $\gamma_5\gamma_2$ diquark charges leads to 133 generators which E_7 requires. E_6 corresponds to a flavor group of a different structure: an $SU(3)\times SU(3)$ flavor subgroup with γ_2-type diquark charges antisymmetric in all SU(3) spaces gives 78 generators, which is the number of generators for E_6. This seems to suggest that diquark charges, especially in association with SU(3)- or SU(6)-type flavor groups, have an intimate connection with exceptional groups which is not enjoyed by any other simple Lie group. Exceptional groups also have the comforting property that the fundamental representation of an $SU(3)_{\rm color}\times SU(3)_{\rm flavor}$ group can be fitted into the fundamental representation of the corresponding exceptional group, unlike the situation with $SU(3)\times SU(7)\subset SU(15)$ already commented upon.[15]

In the following section we will present explicit generators for the exceptional groups G_2 and F_4 in terms of diquark/leptoquark charges, and verify that they obey the required commutation relations.

III. QUARK-CHARGE GENERATORS FOR G_2 AND F_4

We will start with the simpler case of G_2. As already described applying (3) and (4) together on the "color" charges

$$T^i_j=\int dv\,[q^i q_j-\tfrac{1}{3}\delta^i_j(q^k q_k)] \tag{7}$$

yields

$$D^{[i,j]}=\int dv\, q^i\gamma_2 q^j, \tag{8}$$

$$[D^{[i,j]}]^\dagger=D_{[j,i]}=\int dv\, q_j\gamma_2 q_i, \tag{9}$$

where $[i,j]$ denotes an antisymmetrized combination, divided by a factor of two. Thus there are indeed 14 charges as needed. However, the fundamental representation of G_2 is 7-dimensional, where 6 of these can be interpreted as tricolored quarks and their antiquarks. This leaves another 2-component, SU(3)-singlet field p which we take as a Majorana "lepton." Clearly, this field must also appear in the generators. From the earlier arguments involving quark-lepton transitions and the group-theoretic fact that $q^*\gamma_2 q^*\sim q$, we expect to have generators of the form

$$Q^i=a\epsilon^{ijk}\int dv\, q_j\gamma_2 q_k+b\int dv\, q^i p, \tag{10}$$

$$Q_i=a^*\epsilon_{ijk}\int dv\, q_k\gamma_2 q_j+b^*\int dv\, p\gamma_2 q_i, \tag{11}$$

where a, b are complex numbers to be determined later. Note the change in the order of the dummy indices j and k in (11) and the presence of γ_2 in the last term, which comes from the Majorana character of the field. It is useful to collect the following properties of this field here:

$$p^c=\gamma_2 p^*=p,\quad p^\dagger=p\gamma_2, \tag{12}$$

$$p_r^*(\vec{x},0)p_s(\vec{x}',0)+p_s(\vec{x}',0)p_r^*(\vec{x},0)=\delta_{rs}\,\delta^3(\vec{x}-\vec{x}'), \tag{13}$$

$$p_r(\vec{x},0)p_s(\vec{x}',0)+p_s(\vec{x}',0)p_r(\vec{x},0)=[\gamma_2]_{rs}\delta^3(\vec{x}-\vec{x}'), \tag{14}$$

$$:\bar{p}\gamma_\mu p:=0. \tag{15}$$

In the above equations r, s are 4-spinor indices. Equation (14) follows from Eqs. (13) and (12). The well-known Eq. (15) with (13) gives the result

$$p_r^*(\vec{x},0)p_r(\vec{x},0)=2\delta^3(\vec{0}). \tag{16}$$

Equation (16) must be kept in mind while trying to verify Eq. (18) below. In addition, p anticommutes with all quark fields since it is an SU(3) singlet.

With

$$a=e^{i\pi/3}/2\sqrt{3},\quad b=(\tfrac{2}{3})^{1/2}e^{i\pi/3}$$

one obtains the G_2 commutation relations given in Eqs. (7.2) in Günaydin and Gürsey,[18] which we reproduce below for a self-contained presentation:

$$[Q^i, Q^j] = -\frac{2}{\sqrt{3}}\epsilon^{ijk}Q_k, \qquad (17)$$

$$[Q^i, Q_j] = T^i_j, \qquad (18)$$

$$[T^i_j, Q_k] = -\delta^i_k Q_j + \tfrac{1}{3}\delta^i_j Q_k. \qquad (19)$$

The remaining commutation relations are usual ones of SU(3), plus Hermitian conjugates of (17)–(19). In comparing the above set with Eq. (7.2) of Ref. 18, it should be observed that the Q ($Q^\dagger$) charges there transform like antitriplets (triplets), contrary to what the notation first suggests and the correspondence given at the bottom left of p. 1657 of Ref. 18.

In the case of F_4, we first introduce the color- and flavor-octet charges

$$C^i_j = \int dv[q^{i\alpha}q_{j\alpha} - \tfrac{1}{3}\delta^i_j(q^{k\nu}q_{k\nu})], \qquad (20)$$

$$F'^{\alpha}_{\beta} = \int dv[q^{i\alpha}q_{i\beta} - \tfrac{1}{3}\delta^\alpha_\beta(q^{k\nu}q_{k\nu})]. \qquad (21)$$

According to our hypothesis of universality between quark and lepton flavor currents, an octet of leptonic charges should be added on to (21) to form the complete flavor-octet generators. Since the fundamental representation contains 26 two-component fields of which 18 are the quarks and antiquarks, we must construct the leptonic currents from 8 Majorana leptons. It is perhaps worth pointing out here that since we are not addressing ourselves to the question of whether F_4 can be a realistic supergroup (the answer to which is probably no, as will be argued in the next section) in this discussion, we will not attempt to identify any combinations of the Majorana leptons with observed leptons. The leptons are now represented by a traceless octet l^α_β, with the properties

$$l^\alpha_\delta(\vec{x},0)\gamma_2 l^\beta_\rho(\vec{x}',0) + l^\beta_\rho(\vec{x}',0)\gamma_2 l^\alpha_\delta(\vec{x},0) = \delta^\beta_\delta\delta^\alpha_\rho\delta^3(\vec{x}-\vec{x}'), \qquad (22)$$

$$l^\alpha_\delta(x)\gamma_2 l^\beta_\rho(x) - l^\beta_\rho\gamma_2 l^\alpha_\delta \equiv 2:l^\alpha_\delta\gamma_2 l^\beta_\rho:. \qquad (23)$$

Thus the correct set of flavor generators are the charges F^α_β defined by

$$F^\alpha_\beta = F'^\alpha_\beta + L^\alpha_\beta, \qquad (24)$$

where

$$L^\alpha_\beta = \int dv[l^\alpha_\tau\gamma_2 l^\tau_\beta - \tfrac{1}{3}\delta^\alpha_\beta(l^\mu_\nu l^\nu_\mu)]. \qquad (25)$$

Now we extend (20) and (21) to obtain the γ_2-type diquark charges as in $\underline{\underline{G}}_2$. These belong to the representations $(3,\bar{6})$ and $(\bar{3},6)$ as given below:

$$(3,\bar{6}) \sim \mathfrak{D}^{\{\alpha\beta\}}_k = \int dv\,\epsilon_{kij}q^{i\{\alpha}\gamma_2 q^{\beta\}j}, \qquad (26)$$

$$(\bar{3},6) \sim \mathfrak{D}^k_{\{\alpha\beta\}} = \int dv\,\epsilon^{kij}q_{j\{\alpha}\gamma_2 q_{\beta\}i}. \qquad (27)$$

Again, $\{\alpha\beta\}$ represents a symmetrized product divided by two. The accompanying leptoquark charges are

$$(3,\bar{6}) \sim \mathfrak{L}^{\{\alpha\beta\}}_k = \int dv\, l^{\{\alpha}_\delta\gamma_2 q_{k\mu}\epsilon^{\beta\}\delta\mu}, \qquad (28)$$

$$(\bar{3},6) \sim \mathfrak{L}^k_{\{\alpha\beta\}} = \int dv\, q^{k\mu}l^\delta_{\{\alpha}\epsilon_{\beta\}\delta\mu}. \qquad (29)$$

Hence we expect the proper diquark/leptoquark generators to be of the form

$$Q^{\alpha\beta}_k = a\mathfrak{D}^{\{\alpha\beta\}}_k + b\mathfrak{L}^{\{\alpha\beta\}}_k, \qquad (30)$$

and their Hermitian conjugates. With $a=\frac{1}{2}$, $b=1$, we have the commutation relations of F_4:

$$[Q^{\alpha\beta}_k, Q^{\rho\sigma}_l] = \epsilon_{klj}\epsilon^{\rho\nu\{\beta}\epsilon^{\alpha\}\sigma\mu}Q^j_{\mu\nu}, \qquad (31)$$

$$[Q^{\alpha\beta}_k, Q^l_{\rho\sigma}] = \delta^l_k\delta^{\{\beta}_{\{\rho}F^{\alpha\}}_{\sigma\}} - \delta^{\{\beta}_{\{\rho}\delta^{\alpha\}}_{\sigma\}}C^l_k, \qquad (32)$$

$$[C^l_k, Q^{\alpha\beta}_j] = -\delta^l_j Q^{\alpha\beta}_k + \tfrac{1}{3}\delta^l_k Q^{\alpha\beta}_j, \qquad (33)$$

$$[F^\mu_\nu, Q^{\alpha\beta}_j] = 2\delta^{\{\alpha}_\nu Q^{\beta\}\mu}_j - \tfrac{1}{3}\delta^\mu_\nu Q^{\alpha\beta}_j, \qquad (34)$$

$$[C^l_k, F^\mu_\nu] = 0, \qquad (35)$$

where we have left out the well-known SU(3)-octet commutators of color and flavor charges and the Hermitian conjugates of (31)–(34). It is tedious but straightforward to verify the above relations if one is careful about the following points: (i) Each component of the Majorana lepton octet obeys an equation like (16), and these infinite c-numbers cancel with those arising from interchanging the order of q and q^* in quark-octet charges, (ii) for two *different* Majorana fields l and l', the combination $l\gamma_2 l' + l'\gamma_2 l$ vanishes identically because of the anticommutation relations of the spinor field.

IV. DISCUSSION AND COMPARISON WITH EXCEPTIONAL GAUGE THEORIES

We have shown that with a number of plausible assumptions about the properties of the supergroup, the choice of possible flavor and supergroups becomes severely restricted. However, it must be immediately added that these assumptions are not at all meant to be unique, in spite of their rather economic character. Hence we will attempt to list below some general alternatives that are not covered by our assumptions and discuss whether our methods can be modified to handle these cases as well.

First of all, instead of an SU(3) color group and color-singlet leptons, one could construct models in which leptons correspond to a fourth color. A completely integral-charged version of this has been proposed by Pati and Salam.[19] A unified gauge theory constructed along these lines has been shown[3,5,20] to lead to the groups SO(10) and SO(14). Secondly, both the flavor and/or the supergroup may be semisimple rather than simple.[21] Then, for example, the former could be any product of simple groups, presumably with one SU(3) factor to represent the light quarks; the latter would have to consist of products of a single simple group so that one coupling constant could suffice. In this case one has, in principle, infinitely many possibilities. However, if a small subset of these is picked out on the basis of some physical considerations, one can again set up a system of Diophantine equations, possibly larger than those considered here. For instance, taking an $SU(3) \times SU(n)$-type flavor group and a simple supergroup would yield a small multiple of the number of equations described in Sec. II. Similarly, p times the equations of Sec. II would cover the possibility of a supergroup G^P, where G is simple. Also, as we have mentioned already, the case where an arbitrary number of axial-vector currents is allowed should also be amenable to our general treatment. In this paper, we have only restricted ourselves to the simplest alternatives of the above type. It is interesting and encouraging that the same features, i.e., a simple flavor group, flavor universality between quark and lepton currents, and use of diquark/leptoquark currents have recently also been obtained[22] as a result of the following set of requirements not all of which are identical to ours: (i) the supergroup is simple, (ii) only color-singlet leptons and color-triplet and antitriplet quarks appear in the fermion representation, (iii) only a minimum number of color-singlet vector bosons are used for a given number of fermions.

The fact that the above diquark extensions of groups built of SU(3) and SU(6) factors leads uniquely to exceptional groups throws new light both on diquark/leptoquark charges and on gauge theories based on exceptional groups. It has already been pointed out[5,23] that such gauge theories necessarily lead to proton decay in second order through diquark/leptoquark currents. What is new here is that these diquark charges represent the inevitable Pauli-Gürsey symmetry of a massless quark Lagrangian and that this symmetry uniquely favors the exceptional groups when the explicit internal symmetry consists of familiar particle-physics groups such as SU(3) and SU(6). Of course, we are not referring here to the phenomenological quark model SU(6) but to recently proposed models[24,25] with six quarks which also happen to be the smallest set of quarks to incorporate a description of CP violation.[26] Also, in view of the connection between exceptional groups and octonions and the very attractive proposal of Güsey[17,27] that internal symmetries correspond to a natural generalization of quantum mechanics, employing octonions instead of the usual complex numbers, we find unified gauge theories bases on E_6 (Ref. 28) and E_7 (Refs. 5, 23) more appealing candidates than the SU(15) possibility also found here. The other exceptional groups are not likely to be relevant for realistic gauge theories for the following reasons: (i) G_2 does not admit a flavor group, (ii) F_4 only admits three flavors of quarks and one new charged lepton in addition to the electronic and muonic leptons, and therefore cannot account for the new (presumably) hadronic degrees of freedom represented by the ψ family (Refs. 29, 30) and the high value of $R = \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$, (iii) E_8 involves color-octet quarks.[5]

The relation of E_7 with Pauli-Gürsey transformations and its commutation relations will be given in a forthcoming paper.

ACKNOWLEDGMENTS

This work owes its origins to Dr. Y. Nambu, who suggested a general investigation of diquark currents in connection with Pauli-Gürsey symmetry. Also, it could not have been completed without the crucial suggestion of introducing a Majorana field in the case of G_2 by Dr. F. Gürsey, and his later generous correspondence on exceptional groups. I would like to thank Dr. A. Ashtekar, Dr. L. Hughston, Dr. N. Woodhouse, and Dr. R. Penrose for a discussion on black holes and baryon-lepton number violation, with special thanks to Dr. A. Ashtekar for guidance to the literature on the subject. Finally, I am grateful to Dr. J. C. Taylor for a critical reading of the manuscript, to all members of the High Energy Theory group at Oxford University for encouraging discussions, and to the Scientific and Technical Research Council of Turkey for partial support of this work during its final stages.

*Present address: Middle East Technical University, Ankara, Turkey.
†Work supported in part by the Scientific and Technical Research Council of Turkey.

[1]H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).
[2]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).
[3]H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) 93, 193 (1975).
[4]B. W. Lee, summary talk presented at the Orbis Scientiae Conference at Coral Gables, 1976; Fermilab Report No. Fermilab-Conf-76 20-THY/EXP, 1976 (unpublished).
[5]P. Ramond, Nucl. Phys. B110, 214 (1976).
[6]F. C. Michel, Comments Astrophys. Space Phys. 3,

163 (1971); J. D. Bekenstein, Phys. Rev. D 5, 2403 (1972); C. Teitelboim, *ibid.* 5, 2941 (1972).

[7]S. W. Hawking, Nature (London) 248, 30 (1974); Commun. Math. Phys. 43, 199 (1975).

[8]A. Ashtekar and R. Geroch, Rep. Prog. Phys. 37, 1211 (1974).

[9]Y. Nambu, lectures given at the International Summer School in High Energy Physics, Erice, Italy, 1972 (unpublished).

[10]W. Pauli, Nuovo Cimento 6, 204 (1957).

[11]F. Gürsey, Nuovo Cimento 7, 411 (1958).

[12]Note that this does not necessarily imply a parity-conserving neutral current if the supergroup is exceptional. See Ref. 5.

[13]Equivalently, this implies the presently unchallenged hypothesis that there are no purely leptonic interaction types in which hadrons never participate.

[14]Asymptotic freedom (Ref. 31) is also destroyed when the number of quark flavors exceeds 16.

[15]In fact, a similar situation occurs in the SU(5) model (Ref. 1); fitting four flavors of tricolored quarks together with muonic and electronic leptons into SU(5) representations requires the use of $\underline{5} \oplus \underline{5} \oplus \underline{10} \oplus \underline{10}$ rather than just a single fundamental representation. Analogously, for SU(15) we can use $\underline{105}$, i.e., the antisymmetric part of $\underline{15} \oplus \underline{15}$, which decomposes into seven flavors of tricolored quarks and 84 color-singlet leptons, $\underline{84}$ being the completely symmetric part of $\underline{7} \oplus \underline{7} \oplus \underline{7}$. However, the fact that such a large number of new leptons are required perhaps also makes this model phenomenologically less appealing than those based on the considered exceptional groups, which use at most ten Dirac leptons altogether.

[16]M. Günaydin and F. Gürsey, Phys. Rev. D 9, 3387 (1974).

[17]F. Gürsey, in *Kyoto Symposium on Mathetical Problems in Theoretical Physics*, edited by H. Araki (Springer, Berlin, 1975).

[18]M. Günaydin and F. Gürsey, J. Math. Phys. 14, 1651 (1973).

[19]J. C. Pati and A. Salam, Phys. Rev. Lett. 31, 661 (1973).

[20]H. Georgi, in *Particles and Fields—1974*, proceedings of the Williamsburg meeting of the Division of Particles and Fields of the American Physical Society, edited by Carl E. Carlson (AIP, New York, 1975).

[21]That interesting solutions of this type can exist is evidenced by the example $SU(3) \times SU(3) \times SU(3)_{color} \subset E_6$ found at the end of Sec. II. The flavor group could also have U(1) factors which would increase the number of color-singlet bosons. An SU(n) flavor group has been chosen here since it corresponds to the most direct physical interpretation and affords the simplest treatment with a system of Diophantine equations. General criteria favoring simple flavor groups have been derived in Ref. 22 and summarized in Ref. 5.

[22]M. Gell-Mann, P. Ramond, and R. Slansky, work in preparation; the results of the above work mentioned in the present article are quoted in Ref. 5, where it is referred to as Ref. 14.

[23]F. Gürsey and P. Sikivie, Phys. Rev. Lett. 36, 775 (1976).

[24]R. M. Barnett, Phys. Rev. Lett. 34, 41 (1975); H. Harari, Ann. Phys. (N.Y.) 94, 391 (1975).

[25]H. Fritzsch, M. Gell-Mann, and P. Minkowski, Phys. Lett. 59B, 293 (1975).

[26]M. Kobayashi and K. Maskawa, Prog. Theor. Phys. 49, 652 (1973).

[27]F. Gürsey, talk presented at the Orbis Scientiae Conference at Coral Gables, 1976; Yale Report No. COO-3075-136 (unpublished)

[28]F. Gürsey, P. Ramond, and P. Sikivie, Phys. Lett. 60B, 177 (1976).

[29]J. E. Augustin *et al.*, Phys. Rev. Lett. 33, 1406 (1974).

[30]J. J. Aubert *et al.*, Phys. Rev. Lett. 33, 1453 (1974).