Gravitational Lagrangian and internal symmetry

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The field equations of general relativity, when interpreted as a set of equations in a Riemannian manifold endowed with internal space at each point, are shown to be derived from a variational principle. The failures of other approaches to recover the full general-relativity theory are traced to the absence in their Lagrangians of direct coupling between space-time and internal structure. Examination of this observation sheds some light on the fundamental implications of the intimate relation between space-time covariance and internal symmetry.

I. INTRODUCTION

In the classical formulation of Einstein's theory of gravitation the dynamical variables are the components of the metric field $g_{\mu\nu}(x)$ corresponding to the Riemannian manifold of space-time, together with the physical fields ("matter field") present. The field equations are derived from a principle of least action

$$\delta I=0, \quad I=\int L\sqrt{-g}\,d^4x \ ,$$

with a Lagrangian L. This Lagrangian decomposes into two parts: the gravitational Lagrangian, L_0, which is equal to the Ricci scalar R, and the matter Lagrangian L_M, viz.,

$$L=L_0+\kappa L_M=R+\kappa L_M$$

(κ is a coupling constant). Variation with respect to the metric field in this action integral leads to the Einstein field equation

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu} \ .$$

It has been found, however, that from a theoretical as well as a computational point of view, it is desirable to recast the theory as a formalism in space-time with internal structure, posessing SL(2, C) symmetry. This is the conceptual background for the two-component spinor theory in curved space-time.[1] This approach is also the basis for the tetrad calculus and the Newman-Penrose spin-coefficients formalism,[2] and is useful for the analysis and classification of gravitational fields. Furthermore, it has been shown by Carmeli[3] that, using such a framework, the theory of gravitation can be cast into a Yang-Mills-type formalism. Yang in his recent integral formalism[4] again relates gauge fields and gravitation when the latter is expressed as an SL(2, C)-covariant theory. This is one step in the current tendency to bring general relativity more in line with the prevalent trends in particle physics.

It is therefore of importance to find out whether in formulations of that type the Einstein field equations can be derived from an action principle *under the variation of the fundamental quantities appearing in the particular formulation*. More specifically, the question arises whether an action principle can yield the field equations without adding in the process *ad hoc* quantities. A partial answer to this question has been given in a few articles,[5] where it is demonstrated that an appropriately chosen Lagrangian leads to field equations which are related to Einstein equations. These equations, however, are weaker than the full Einstein equations, and must be augmented by additional conditions in order to recover general relativity. Furthermore, auxiliary quantities must be artificially introduced in order to facilitate the derivation. Carmeli's method[6] of first-order form suffers from the same latter deficiency. The merits of the aforementioned schemes lie in their Lagrangians being functions of gauge fields, an approach which is not attempted in the present article. Likewise, the equations obtained by Yang[4] from a variational principle constitute a vast generalization of general relativity, and admit solutions which are physically unacceptable.[7] It is the purpose of this paper to show that the full Einstein field equations are obtained from an SL(2, C)-invariant Lagrangian, which is considered as a functional of the dynamical variables alone, without any necessity to introduce any auxiliary quantities. This Lagrangian, in contrast to the Lagrangian considered by others, mixes up the space-time and internal structures.

In Sec. II the framework is set and the notation is established. A basic decomposition of the curvature is represented in Sec. III, and in Sec. IV the Lagrangian is introduced and the resulting Euler-Lagrange equations are derived. The equations as applied to vacuum and to space-time admitting a neutrino field are discussed in Sec. V. The last section is devoted to concluding remarks, and in the Appendix a few identities used in the article are stated.

II. RIEMANNIAN STRUCTURE IN TERMS OF SPINORS

In the standard spinor algebra in curved space-time[8] a one-to-one correspondence is set between vectors ξ_μ in the Riemannian manifold of space-time (endowed with metric tensor $g_{\mu\nu}$), and Hermitian spinors $\xi_{A\dot{B}}$, viz.,

$$\xi_{A\dot{B}} = \sigma^\mu_{A\dot{B}}\xi_\mu, \quad \xi^\mu = \sigma^\mu_{A\dot{B}}\xi^{A\dot{B}} \quad .$$

The connecting quantities $\sigma^\mu_{A\dot{B}}$ satisfy the relations

$$\epsilon^{\dot{C}\dot{D}}(\sigma^\mu_{A\dot{C}}\sigma^\nu_{B\dot{D}} + \sigma^\nu_{A\dot{C}}\sigma^\mu_{B\dot{D}}) = g^{\mu\nu}\epsilon_{AB} \quad , \tag{2.1}$$

$$\mu, \nu = 0, 1, 2, 3, \quad A, B = 0, 1 \quad .$$

The approach to be followed here is to take the spinor vecotrs $\sigma^\mu_{A\dot{B}}$ as the fundamental quantities of the theory, and to construct the Riemannian metric out of the spin structure. Consequently, Eq. (2.1) is the *definition* of the metric $g_{\mu\nu}$.

Covariant derivatives of spinors are defined[8] by

$$\eta^A{}_{|\mu} = \eta^A{}_{,\mu} + \Gamma^A{}_{B\mu}\eta^B \quad ,$$

where the spin connections $\Gamma^A{}_{B\mu}$ are determined by the usual requirements for differentiation (Leibnitz rule, etc.) together with

$$\epsilon_{AB|\mu} = 0 \quad , \tag{2.2}$$

$$\sigma^\nu{}_{A\dot{B}|\mu} = 0 \quad . \tag{2.3}$$

The spin curvature $R_{AB\mu\nu}$ is defined by the commutator of the covariant derivative

$$\eta^A{}_{|\mu\nu} - \eta^A{}_{|\nu\mu} = R^A{}_{B\mu\nu}\eta^B \quad ,$$

and is expressed in terms of the connections

$$R^A{}_{B\mu\nu} = \Gamma^A{}_{B\mu,\nu} - \Gamma^A{}_{B\nu,\mu} + \Gamma^A{}_{C\nu}\Gamma^C{}_{B\mu} - \Gamma^A{}_{C\mu}\Gamma^C{}_{B\nu}$$

The spin curvature is equivalent to the $F_{\mu\nu}$ matrices in Carmeli's gauge formulation,[3] and is related to the Riemann curvature tensor $R_{\lambda\tau\mu\nu}$ by

$$R_{AB\mu\nu} = -\tfrac{1}{2}S^{\lambda\tau}{}_{AB}R_{\lambda\tau\mu\nu} \quad , \tag{2.4}$$

$$R_{\lambda\tau\mu\nu} = -(S_{\lambda\tau}{}^{AB}R_{AB\mu\nu} + S_{\lambda\tau}{}^{\dot{A}\dot{B}}R_{\dot{A}\dot{B}\mu\nu}) \quad , \tag{2.5}$$

where

$$\begin{aligned} S_{\mu\nu}{}^{AB} &= \tfrac{1}{2}\epsilon_{\dot{C}\dot{D}}(\sigma_\mu{}^{A\dot{C}}\sigma_\nu{}^{B\dot{D}} - \sigma_\nu{}^{A\dot{C}}\sigma_\mu{}^{B\dot{D}}) \quad , \\ S_{\mu\nu}{}^{\dot{A}\dot{B}} &= \tfrac{1}{2}\epsilon_{CD}(\sigma_\mu{}^{C\dot{A}}\sigma_\nu{}^{D\dot{B}} - \sigma_\nu{}^{C\dot{A}}\sigma_\mu{}^{D\dot{B}}) \quad . \end{aligned} \tag{2.6}$$

III. DECOMPOSITION OF THE SPIN CURVATURE

The three bivectors $S_{\mu\nu}{}^{00}, S_{\mu\nu}{}^{01} = S_{\mu\nu}{}^{10}, S_{\mu\nu}{}^{11}$ and their complex conjugates are six independent complex bivectors (as can be shown with the aid of the "completeness relations" in the Appendix), and hence serve as a basis in the space of complex bivectors. The spin curvature $R_{AB\mu\nu}$, for fixed value of (A,B), is a complex bivector and therefore can be uniquely expanded in terms of the basis

$$R_{AB\mu\nu} = R_{ABCD}S_{\mu\nu}{}^{CD} + R_{AB\dot{C}\dot{D}}S_{\mu\nu}{}^{\dot{C}\dot{D}} \quad , \tag{3.1}$$

where the coefficients are given by

$$\begin{aligned} R_{ABCD} &= \tfrac{1}{2}S^{\mu\nu}{}_{CD}R_{AB\mu\nu} \quad , \\ R_{AB\dot{C}\dot{D}} &= \tfrac{1}{2}S^{\mu\nu}{}_{\dot{C}\dot{D}}R_{AB\mu\nu} \quad . \end{aligned} \tag{3.2}$$

Contracting (3.1) with $S^{\mu\nu AB}$ and using the "completeness relations" of the Appendix and (2.4), one finds

$$\Omega \equiv R_{AB}{}^{AB} = -\tfrac{1}{4}R \quad .$$

Furthermore, a similar but longer calculation shows that the coefficients $R_{AB\dot{C}\dot{D}}$ are related to the trace-free Ricci tensor:

$$\begin{aligned} R_{AB\dot{C}\dot{D}} &= \tfrac{1}{2}\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}}(R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R) \quad , \\ R_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R &= 2\psi_{\mu\nu}{}^{AB\dot{C}\dot{D}}R_{AB\dot{C}\dot{D}} \quad , \end{aligned} \tag{3.3}$$

where $\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}}$ is the totally symmetric spinor tensor defined in the Appendix.

In fact, $R_{AB\dot{C}\dot{D}}$ is equal to the spinor $-\phi_{AB\dot{C}\dot{D}}$ in the Newman-Penrose formalism.[2]

For further reference it is mentioned here that the symmetries of R_{ABCD} ($R_{ABCD} = R_{ABDC} = R_{CDAB}$) entail

$$R_{ACB}{}^C = \tfrac{1}{2}\Omega\epsilon_{AB} \quad . \tag{3.4}$$

IV. THE GRAVITATIONAL LAGRANGIAN AND ITS VARIATIONAL DERIVATIVE

If the theory is confined to quantities which are of, at most, second differential order in the $\sigma^\mu{}_{A\dot{B}}$, and linear in the second derivatives, then the only concomitant of the fundamental quantities $\sigma^\mu{}_{A\dot{B}}$ which is scalar under coordinate transformations in space-time and invariant under the action of the group SL(2, C) is (up to a constant, which contributes only a cosmological term)

$$L_0 = S^{\mu\nu AB}R_{AB\mu\nu} = 2\Omega = -\tfrac{1}{2}R \quad .$$

Variational principles of this type have been considered[9] from other points of view. Our calculation, which agrees with the aforementioned results, will be based on the decomposition scheme of Sec. III, thus gaining in simplicity and brevity. Thus let us consider the change of

$$\int L_0\sqrt{-g}\, d^4x = -\tfrac{1}{2}\int R\sqrt{-g}\, d^4x$$

under the variation

$$\sigma^\mu{}_{A\dot{B}} \to \sigma^\mu{}_{A\dot{B}} + \delta\sigma^\mu{}_{A\dot{B}} \quad .$$

For this purpose we have to express the variations of the quantities appearing under the integral sign in terms of $\delta\sigma^{\mu}{}_{A\dot{B}}$ or $\delta\sigma_{\mu}{}^{AB}$. Using (2.1) one finds

$$\delta g_{\mu\nu}=\sigma_{\mu A\dot{B}}\delta\sigma_{\nu}{}^{A\dot{B}}+\sigma_{\nu A\dot{B}}\delta\sigma_{\mu}{}^{A\dot{B}}\ ,$$

$$\delta\sqrt{-g}=\tfrac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}=\sqrt{-g}\ \sigma^{\mu}{}_{A\dot{B}}\delta\sigma_{\mu}{}^{A\dot{B}}\ ,$$

$$\delta\sigma_{\mu}{}^{A\dot{B}}=-\sigma_{\mu}{}^{C\dot{D}}\sigma_{\nu}{}^{A\dot{B}}\delta\sigma^{\nu}{}_{C\dot{D}}\ ,$$

$$\delta S^{\mu\nu AB}=\sigma^{[\mu A}{}_{\dot{C}}\delta\sigma^{\nu]B\dot{C}}+\sigma^{[\mu B}{}_{\dot{C}}\delta\sigma^{\nu]A\dot{C}}\ ,$$

$$S^{\mu\nu AB}\delta R_{AB\mu\nu}=V^{\mu}{}_{|\mu},\quad\text{with}\quad V^{\mu}=2S^{\mu\nu}{}_{A}{}^{B}\delta\Gamma^{A}{}_{B\nu}\ .$$

The last relation, which is coordinate- and SL(2, C)-invariant, can be most easily verified in a "geodetic" coordinate system and spin frame, adapted to the point under consideration. In such a coordinate system and spin frame the fundamental quantities $\sigma^{\mu}{}_{A\dot{B}}$ are constant (at the point) and reduce to the Pauli matrices, and $\Gamma^{A}{}_{B\mu}=0$ (at the point).

Taking these relations into account one finds

$$\begin{aligned}\delta(\sqrt{-g}L_0)&=\delta(\sqrt{-g}S^{\mu\nu AB}R_{AB\mu\nu})\\&=\sqrt{-g}S^{\mu\nu AB}\delta R_{AB\mu\nu}+\sqrt{-g}R_{AB\mu\nu}\delta S^{\mu\nu AB}+S^{\mu\nu AB}R_{AB\mu\nu}\delta\sqrt{-g}\\&=\sqrt{-g}\,V^{\mu}{}_{|\mu}+2\sqrt{-g}\,(R_{CD\nu\lambda}\sigma^{\mu\dot{C}}{}_{\dot{B}}\sigma^{\nu D\dot{P}}\sigma^{\lambda}{}_{A\dot{B}}+\Omega)\delta\sigma_{\mu}{}^{A\dot{B}}\ .\end{aligned}$$

Finally, using (A1), (3.4), and substituting (3.2), we conclude

$$\delta(\sqrt{-g}L_0)=\sqrt{-g}\,V^{\mu}{}_{|\mu}+2\sqrt{-g}\,(R_{AC\dot{B}\dot{D}}\sigma^{\mu C\dot{D}}+\tfrac{1}{2}\Omega\,\sigma^{\mu}{}_{A\dot{B}})\delta\sigma_{\mu}{}^{A\dot{B}}\ . \quad (4.1)$$

Now as usual we consider variations $\delta\sigma_{\mu}{}^{A\dot{B}}$ such that they and their first derivatives vanish on the boundary of integration. Under these assumptions the vector V^{μ} too vanishes on the boundary and by Gauss's theorem the divergence $\sqrt{-g}\,V^{\mu}{}_{|\mu}=(\sqrt{-g}\,V^{\mu})_{,\mu}$ does not contribute to the Euler-Lagrange equations. Thus (4.1) leads to the field equations

$$2R_{AC\dot{B}\dot{D}}\sigma^{\mu C\dot{D}}+\Omega\,\sigma^{\mu}{}_{A\dot{B}}=-\kappa T^{\mu}{}_{A\dot{B}}\ , \quad (4.2)$$

where

$$T^{\mu}{}_{A\dot{B}}=\frac{1}{\sqrt{-g}}\ \frac{\delta(\sqrt{-g}L_M)}{\delta\sigma_{\mu A\dot{B}}}\ .$$

That (4.2) is indeed equivalent to the Einstein equations is confirmed with the aid of (3.3), whereby we find that the left-hand side of (4.2) is equal to

$$(R^{\mu}{}_{\nu}-\tfrac{1}{2}\delta^{\mu}{}_{\nu}R)\sigma^{\nu}{}_{A\dot{B}}\ .$$

V. APPLICATION TO VACUUM AND NEUTRINO

In the case of a vacuum, the field equations (4.2) reduce to

$$2R_{AC\dot{B}\dot{D}}\sigma^{\mu C\dot{D}}+\Omega\sigma^{\mu}{}_{A\dot{B}}=0\,.$$

Contracting with $\sigma_{\mu}{}^{A\dot{B}}$ we find this equation to be equivalent to the pair

$$\Omega=0\ ,$$

$$R_{AC\dot{B}\dot{D}}=0\ ,$$

which in turn are equivalent to

$$R=0\ ,$$

$$R_{\mu\nu}-\tfrac{1}{4}g_{\mu}R=0\ ,$$

and these are the Einstein vacuum field equations $R_{\mu\nu}=0$.

For a space admitting a neutrino field η^{A}, the matter Lagrangian L_M is taken to be

$$L_M=i(\eta^{A}{}_{|A\dot{B}}\eta^{\dot{B}}-\eta^{A}\eta^{\dot{B}}{}_{|A\dot{B}})\,.$$

Variation of η^{A} and $\eta^{\dot{A}}$ in the action integral leads to the Euler-Lagrange equations

$$\eta^{A}{}_{|A\dot{B}}=0\quad(\text{and c.c.})\ ,$$

i.e., the Weyl equations for the neutrino field. This equation can now be used to find the "energy-momentum spinor vector" $T^{\mu}{}_{A\dot{B}}$, to be substituted in the right-hand side of (4.2), namely in order to calculate the variational derivative $\delta(\sqrt{-g}L_M)/\delta\sigma_{\mu}{}^{A\dot{B}}$. The variational variables $\sigma_{\mu}{}^{A\dot{B}}$ appear in $\sqrt{-g}$ (the contribution of this term is known from Sec. IV), and in the spin connections $\Gamma^{A}{}_{B\mu}$ in L_M. Thus

$$\delta L_M=i(\eta^{A}{}_{|\mu}\eta^{\dot{B}}-\eta^{A}\eta^{\dot{B}}{}_{|\mu})\delta\sigma^{\mu}{}_{A\dot{B}}+i\eta^{A}\eta^{\dot{B}}(\sigma^{\mu}{}_{C\dot{B}}\delta\Gamma^{C}{}_{A\mu}-\sigma^{\mu}{}_{A\dot{C}}\delta\Gamma^{\dot{C}}{}_{\dot{B}\mu})\,. \quad (5.1)$$

First observe that the indentity

$$0=\sigma_{\mu}{}^{A\dot{B}}{}_{|\nu}=\sigma_{\mu}{}^{A\dot{B}}{}_{,\nu}-\Gamma^{\lambda}{}_{\mu\nu}\sigma_{\lambda}{}^{A\dot{B}}+\Gamma^{A}{}_{C\nu}\sigma_{\mu}{}^{C\dot{B}}+\Gamma^{\dot{B}}{}_{\dot{C}\nu}\sigma_{\mu}{}^{A\dot{C}}$$

yields the following expressions for the variation of the spin connections:

$$\begin{aligned}\delta\Gamma^{A}{}_{B\mu}&=-\tfrac{1}{2}\sigma^{\nu}{}_{B\dot{C}}[(\delta\sigma_{\nu}{}^{A\dot{C}})_{|\mu}-\sigma_{\lambda}{}^{A\dot{C}}\delta\Gamma^{\lambda}{}_{\mu\nu}]\\&=+\tfrac{1}{2}\sigma_{\nu}{}^{A\dot{C}}[(\delta\sigma^{\nu}{}_{B\dot{C}})_{|\mu}+\sigma^{\lambda}{}_{B\dot{C}}\delta\Gamma^{\nu}{}_{\mu\lambda}]\,.\end{aligned}$$

(Again this relation is most easily verified in a "geodetic" coordinate system and spin frame.) Substituting in (5.1), the terms containing $\delta\Gamma^{\lambda}{}_{\mu\nu}$ cancel, and we have

$$\delta L_M=-i(\eta_{C|A\dot{B}}\eta_{\dot{D}}-\eta_{C}\eta_{\dot{D}|A\dot{B}})\sigma^{\mu C\dot{D}}\delta\sigma_{\mu}{}^{A\dot{B}}+\tfrac{1}{2}i(S^{\mu\nu}{}_{AC}\eta_{\dot{B}}\eta^{C}-S^{\mu\nu}{}_{\dot{B}\dot{C}}\eta_{A}\eta^{\dot{C}})(\delta\sigma_{\mu}{}^{A\dot{B}})_{|\nu}\,.$$

After some manipulations the following expression is obtained:

$$\delta L_M = V^\mu{}_{|\mu} - \tfrac{1}{2}i(\eta_{A|C\dot{D}}\eta_{\dot{B}} - \eta_A\eta_{\dot{B}|C\dot{D}} + \eta_{C|A\dot{B}}\eta_{\dot{D}} - \eta_C\eta_{\dot{D}|A\dot{B}})\sigma^{\mu C\dot{D}}\delta\sigma_\mu{}^{A\dot{B}}, \tag{5.2}$$

with

$$V^\mu = -\tfrac{1}{2}i(S^{\mu\nu}{}_{AC}\eta_{\dot{B}}\eta^C - \eta^{\mu\nu}{}_{\dot{B}\dot{C}}\eta_A\eta^{\dot{C}})\delta\sigma_\nu{}^{A\dot{B}}.$$

Again, the divergence term does not contribute to the variational equations.

The contribution of the variation of $\sqrt{-g}$, i.e., $L_M\delta\sqrt{-g}$, vanishes also due to the neutrino equation. Hence we conclude from (5.2)

$$T^\mu{}_{A\dot{B}} = \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_M)}{\delta\sigma_\mu{}^{A\dot{B}}} = -\tfrac{1}{2}i(\eta_{A|C\dot{D}}\eta_{\dot{B}} - \eta_A\eta_{\dot{B}|C\dot{D}} + \eta_{C|A\dot{B}}\eta_{\dot{D}} - \eta_C\eta_{\dot{D}|A\dot{B}})\sigma^{\mu C\dot{D}}. \tag{5.3}$$

From the field equations thus obtained, viz.,

$$2R_{AC\dot{B}\dot{D}}\sigma^{\mu C\dot{D}} + \Omega\sigma^\mu{}_{A\dot{B}} = -\kappa T^\mu{}_{A\dot{B}}, \tag{5.4}$$

with $T^\mu{}_{A\dot{B}}$ given by (5.3), the familiar form can be recovered. Contracting (5.4) with $\sigma_\mu{}^{A\dot{B}}$ yields

$$\Omega = 0,$$

and therefore

$$R_{AC\dot{B}\dot{D}} = \tfrac{1}{4}i\kappa(\eta_{A|C\dot{D}}\eta_{\dot{B}} - \eta_A\eta_{\dot{B}|C\dot{D}} + \eta_{C|A\dot{B}}\eta_{\dot{D}} - \eta_C\eta_{\dot{D}|A\dot{B}}).$$

VI. CONCLUSION

It has been demonstrated that general relativity can be viewed as a theory involving the fundamental quatities $\sigma_\mu{}^{A\dot{B}}$ alone (along with external physical fields). The field equations are derived from a variational principle, the gravitational Lagrangian being

$$L_0 = S^{\mu\nu AB}R_{AB\mu\nu}.$$

It will be instructive to compare this Lagrangian to other Lagrangians considered recently,[4,5,6] which have not led directly to the full general-relativistic field equations, but have nevertheless been adopted by proponents of the gauge-field approach to gravitation. A typical Lagrangian proposed by those authors is (in our notation) $R_{AB\mu\nu}R^{AB\mu\nu}$. In such a Lagrangian the space-time indices (μ, ν) are contracted with space-time indices, and internal-group indices (A, B) are contracted with internal-group indices, and no mixing occurs. The only way to couple directly indices of one kind with indices of the other kind is to recast $R_{AB\mu\nu}$ into a form where all the indices appear on the same footing. This is accomplished by the usual way of changing tensorial indices into spinorial indices, viz.,

$$R_{AB\mu\nu} \to R_{ABP\dot{Q}X\dot{Y}} = \sigma^\mu{}_{P\dot{Q}}\sigma^\nu{}_{X\dot{Y}}R_{AB\mu\nu}.$$

Now the coupling emerges naturally:

$$\epsilon^{AP}\epsilon^{BX}\epsilon^{\dot{Q}\dot{Y}}R_{ABP\dot{Q}X\dot{Y}} = S^{\mu\nu AB}R_{AB\mu\nu},$$

and this is precisely the Lagrangian of the present work, which indeed leads to the full general-relativity field equations. (We mention in passing that the program outlined here can be carried out, *mutatis mutandis,* within the framework of spinor calculus in the five-dimensional unified theory of relativity,[10] thus tying up the electromagnetic field with internal degrees of freedom.)

The last observation may provide more insight into the centrality in general relativity of the inseparability of space-time aspects and internal group, and into the fundamental role played by the intimate relation between general covariance and internal symmetry.

APPENDIX: ALGEBRAIC PROPERTIES OF THE $S_{\mu\nu}{}^{AB}$

Several properties of the basic quantities $S_{\mu\nu}{}^{AB}$, which are used in this work, are listed here. All of them can be verified from the definition (2.6) and the relations (2.1).

"Completeness relations" are

$$S^{\mu\nu}{}_{AB}S_{\mu\nu}{}^{CD} = \delta_A{}^C\delta_B{}^D + \delta_A{}^D\delta_B{}^C,$$

$$S^{\mu\nu}{}_{AB}S_{\mu\nu}{}^{\dot{C}\dot{D}} = 0,$$

$$S^{\mu\nu}{}_{AB}S_{\lambda\tau}{}^{AB} = \tfrac{1}{2}(\delta^\mu{}_\lambda\delta^\nu{}_\tau - \delta^\mu{}_\tau\delta^\nu{}_\lambda) - \tfrac{1}{2}ie^{\mu\nu}{}_{\lambda\tau}$$

($e_{\mu\nu\lambda\tau}$ is the alternating tensor $e_{\mu\nu\lambda\tau} = \sqrt{-g}\,\epsilon_{\mu\nu\lambda\tau}$, $\epsilon_{1234} = 1$).

Single-index contraction yields

$$S^{\mu\lambda}{}_{AB}S^\nu{}_{\lambda CD} = -\tfrac{1}{2}(S^{\mu\nu}{}_{A(C}\epsilon_{D)B} + S^{\mu\nu}{}_{B(C}\epsilon_{D)A} + \epsilon_{A(C}\epsilon_{D)B}g^{\mu\nu}),$$

$$S^{\mu\lambda}{}_{AB}S^\nu{}_{\lambda\dot{C}\dot{D}} = -\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}},$$

where $\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}}$ is the totally symmetric spinor tensor

$$\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}} = \sigma^{(\mu}{}_{A(\dot{C}}\sigma^{\nu)}{}_{B\dot{D})}$$

($\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}} = \psi^{\nu\mu}{}_{AB\dot{C}\dot{D}} = \psi^{\mu\nu}{}_{BA\dot{C}\dot{D}} = \psi^{\mu\nu}{}_{AB\dot{D}\dot{C}}$, $g_{\mu\nu}\psi^{\mu\nu}{}_{AB\dot{C}\dot{D}} = 0$). Another useful identity relates products of σ's with S's:

$$\sigma^\mu{}_{A\dot{B}}\sigma^\nu{}_{C\dot{D}} - \sigma^\nu{}_{A\dot{B}}\sigma^\mu{}_{C\dot{D}} = S^{\mu\nu}{}_{AC}\epsilon_{\dot{B}\dot{D}} + S^{\mu\nu}{}_{\dot{B}\dot{D}}\epsilon_{AC}. \tag{A1}$$

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