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Trace anomalies and the Hawking effect*

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The general spherically symmetric, static solution of $\nabla_\nu T^\nu_\mu = 0$ in the exterior Schwarzschild metric is expressed in terms of two integration constants and two arbitrary functions, one of which is the trace of $T_{\mu\nu}$. One constant is the magnitude of T_{tr} at infinity, and the other is determined if the physically normalized components of $T_{\mu\nu}$ are finite on the future horizon. The trace of the stress tensor of a conformally invariant quantum field theory may be nonzero (anomalous), but must be proportional (here) to the Weyl scalar, $48M^2r^{-6}$; we fix the coefficient for the scalar field by indirect arguments to be $(2880\pi^2)^{-1}$. In the two-dimensional analog, the magnitude of the Hawking blackbody effect at infinity is directly proportional to the magnitude of the anomalous trace (a multiple of the curvature scalar); a knowledge of either number completely determines the stress tensor outside a body in the final state of collapse. In four dimensions, one obtains instead a relation constraining the remaining undetermined function, which we choose as $T^\theta_\theta - T^\alpha_\alpha/4$. This, plus additional physical and mathematical considerations, leads us to a fairly definite, physically convincing qualitative picture of $\langle T_{\mu\nu}\rangle$. Groundwork is laid for explicit calculations of $\langle T_{\mu\nu}\rangle$.

I. INTRODUCTION

The stress tensor (energy-momentum tensor), $T_{\mu\nu}$, carries much of the physical content of a quantum field theory in a curved background space-time.[1,2] Even in a theory with only external interactions (i.e., linear field equations), and even if the field operator has been expanded in normal modes, calculating expectation values of the stress tensor is difficult. One must do integrations involving functions which usually are not known in closed form and cannot even be approximated uniformly in the integration parameter. These technical complications are made still worse by the need to regularize the integrals in some way in preparation for eliminating their infinite parts by renormalization—not to mention any ambiguities or questions of principle in the renormalization procedure itself. Therefore, any information about the stress tensor in a particular problem which can be obtained from general principles without detailed calculations is of great interest.

Such principles include geometrical symmetries, the covariant conservation law $\nabla_\nu T^\nu_\mu = 0$, and, in some cases, a restriction on the form of the trace, T^α_α. In a classical theory with a conformally invariant Lagrangian the trace vanishes. However, recent work (Refs. 15–25) has shown that in the corresponding quantized theory the stress-tensor operator may acquire a trace during renormalization. (This is called a conformal anomaly[3] or trace anomaly.) The trace is still a c number and a local geometrical scalar, and for dimensional reasons the number of derivatives of the metric tensor which appear in it must equal the dimension of space-time. Thus, in two dimensions, T^α_α can only be proportional to R, and in four dimensions, it is a linear combination of $\Box R$, R^2, $R^{\alpha\beta}R_{\alpha\beta}$, and $C^2 \equiv C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$. Here the dimensionless coefficients can depend on the particular type(s) of conformally invariant field considered (e.g., on spin), but can be calculated for each case, at least in principle. (A calculation of the trace in a particular space-time model gives universally valid information about the anomaly coefficients.) Knowledge of the trace is sufficient to restrict the form of $T_{\mu\nu}$ considerably. The fact that the trace is not zero has qualitative consequences; in particular, we shall argue that it is intimately related to particle production processes in certain geometries.

The model which has attracted the most attention in recent years is that of a spherically symmetric body of mass M undergoing gravitational collapse.[4] Outside the body, space-time has the Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \qquad (1.1)$$

and the region to which this metric applies extends across the future horizon ($r = 2M$, $t = +\infty$). (The reaction of the quantized fields on the geometry is neglected.) At late values of the retarded time (and not too close to the body), in the so-called steady-state region, the expectation value of $T_{\mu\nu}$ in the initial vacuum state is expected to become nearly independent of t (see Ref. 1), and to take at spatial infinity the form characteristic of radiation at temperature $kT=(8\pi M)^{-1}$ from a body with emission/absorption characteristics determined by the optical properties of the exterior Schwarzschild space-time. As a step toward understanding the local physics of this problem, therefore, we study the form of the stress tensor in the background (1.1) under the assumption that it is independent of time and is spherically symmetric. The conservation equations, $\nabla_\nu T^\nu_\mu=0$, then force $T_{\mu\nu}$ to be a sum of four terms [Eqs. (2.8)], each of which separately is conserved, spherically symmetric, and time-independent, and each of which satisfies all but one of the following conditions: (1) tracelessness, (2) time-reversal invariance (specifically, $T_{tr}=0$), (3) tangential pressure equal to one-fourth of the trace: $\Theta\equiv T^\theta_\theta-\frac{1}{4}T^\alpha_\alpha=0$, (4) finiteness of the tensor components with respect to a local orthonormal frame on the future horizon. The term with a trace is completely determined when the trace is known; for conformally invariant fields, as pointed out above, the trace is independent of the state and falls out of general theory relatively easily. The terms violating (2) and (4) involve arbitrary constants which are fixed by the behavior of $T_{\mu\nu}$ at infinity and at the horizon. The remaining arbitrary function, $\Theta(r)$, is not known, but some constraints can be put on it by physical reasoning.

The static solutions of the conservation law are globally applicable to the full Schwarzschild black-hole manifold, where there is no central body and the metric (1.1) extends all the way to a past horizon. (We shall not need to consider directly regions of space-time beyond the horizons.) In this model there are several time-independent quantum states which have some of the properties ordinarily expected of a vacuum state.[5] The Unruh vacuum[6] (labeled by ξ) is defined by the absence of incoming flux at the past horizon ($\mathcal{H}^-$) and past null infinity ($\mathcal{I}^-$), and exhibits an outward flux of radiation at infinity (toward $\mathcal{I}^+$). The state is believed to reproduce rather closely the late-time conditions in the (initial) vacuum state of the Hawking collapse model. The Israel-Gibbons-Perry vacuum[7,8] (υ) represents the black hole in thermal equilibrium with a gas of massless particles at infinity. The Boulware vacuum[9] (η) is constructed in the traditional way by requiring "particles" to have positive energy with respect to the Schwarzschild Killing vector. It represents a situation in which the space is empty near infinity, but physical conditions become implausible near the horizon. This happens because the Killing trajectories are geodesics at infinity but have arbitrarily large accelerations near the horizon. (The Greek letters, ξ, υ, and η, were assigned in Refs. 6 and 5.) Each of these states corresponds to different conditions on the constants of integration and the function $\Theta(r)$ in the stress tensor.

The paper proceeds as follows. In Sec. II the general solution of the conservation law is found and decomposed as described above. In Sec. III the physical requirement of finiteness on the horizon is discussed and the two-dimensional analog of the problem is treated. In that case, one finds that the strength of the Hawking flux is determined by the trace of the stress tensor, so that for two-dimensional massless fields either the Hawking effect or the conformal anomaly could have been deduced from a knowledge of the other; the expectation values of the stress tensor in the various states of interest are then completely determined. Section IV reviews what is known so far about the magnitudes of the conformal anomalies of various field theories. By piecing together information, we conclude that the trace of $T_{\mu\nu}$ for a conformally coupled massless scalar field in the four-dimensional Schwarzschild metric is $(2880\pi^2)^{-1}C^2 =(60\pi^2)^{-1}M^2r^{-6}$, and that the trace is probably progressively larger for neutrino and electromagnetic fields. In Secs. V and VI, by studying the asymptotic behavior of the tensor components, we fix the integration constants and gain information about the function Θ in the $\langle T_{\mu\nu}\rangle$'s of various states. We find a strong dependence of Θ on the spin of the field, for which a physical explanation is offered in Sec. VII. Section VI includes precise definitions, for the scalar field, of the three vacuum states mentioned, and the differences in their expectation values are reduced to convergent sums over the normal modes of the field in the Schwarzschild metric. This section constitutes a major step toward the direct calculation of $\langle T_{\mu\nu}\rangle$.

We use the Misner-Thorne-Wheeler sign conventions. We set $c=\hbar=1$ and attribute the dimensions of length to the radial and time coordinates and to the black-hole "mass" M. The standard notation

$$r^*=r+2M\ln\left(\frac{r}{2M}-1\right),\quad \frac{dr}{dr^*}=1-\frac{2M}{r},\tag{1.2}$$

$$v=t+r^*,\quad u=t-r^*\tag{1.3}$$

is used. $T_{\mu\nu}$ does not include the factor $g^{1/2}$.

II. GENERAL FORM OF THE STRESS TENSOR

In this section we will find the most general solution of the covariant conservation equations,

$$\nabla_\nu T^\nu_\mu = 0 \ , \tag{2.1}$$

in the Schwarzschild background (1.1), under the assumption that T^ν_μ is independent of time, as it would be in the steady-state region outside of a collapsing body. Since the quantum states of basic interest are spherically symmetric, the only possibly nonzero components of T^ν_μ are T^r_r, T^t_t, T^t_r, T^r_t, T^θ_θ, and T^ϕ_ϕ. Spherical symmetry also implies that $T^\phi_\phi = T^\theta_\theta$ and that T^ν_μ is a function only of r. Under these requirements, the conservation equations take the form

$$\partial_r T^r_r + \left[\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} + \frac{2}{r}\right] T^r_r - \frac{1}{r}T^\theta_\theta - \frac{1}{r}T^\phi_\phi - \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} T^t_t = 0 \ , \tag{2.2a}$$

$$\partial_r T^r_t + \frac{2}{r} T^r_t = 0 \ , \tag{2.2b}$$

and

$$T^\phi_\phi = T^\theta_\theta \ . \tag{2.2c}$$

The last of these is simply a repetition of the spherical symmetry condition.

Equation (2.2b) may be solved immediately:

$$T^r_t = -\frac{K}{M^2 r^2} \tag{2.3}$$

where K is a constant of integration.

The T^t_t component is to be determined from the trace equation,

$$T^t_t = -T^r_r - 2T^\theta_\theta + T^\alpha_\alpha \ . \tag{2.4}$$

Substituting Eqs. (2.2c) and (2.4) into Eq. (2.2a) and solving the differential equation, we obtain

$$T^r_r = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1} \times \left\{\frac{Q-K}{M^2} + \int_{2M}^{r} \left[M T^\alpha_\alpha(r') + 2(r'-3M)T^\theta_\theta(r')\right] dr'\right\} \ . \tag{2.5}$$

Equations (2.2c), (2.3), (2.4), and (2.5) constitute a complete solution of $\nabla_\nu T^\nu_\mu = 0$, under conditions of time independence and spherical symmetry, in terms of two constants, K and Q, and two functions, $T^\alpha_\alpha(r)$ and $T^\theta_\theta(r)$.

In terms of the coordinate r^* defined in Eq. (1.2), we have

$$T^{r^*}_{r^*} = T^r_r, \quad T^{r^*}_t = \left(1-\frac{2M}{r}\right)^{-1} T^r_t, \quad T^t_{r^*} = -T^{r^*}_t \ .$$

When r^* is used, the mixed tensor components are equal to the quantities of local physical significance (the components with respect to an orthonormal frame).

Now if we define

$$\Theta(r) \equiv T^\theta_\theta(r) - \tfrac{1}{4}T^\alpha_\alpha(r) \ , \tag{2.6}$$

$$H(r) \equiv \tfrac{1}{2}\int_{2M}^{r} (r'-M)\, T^\alpha_\alpha(r')\, dr' \ . \tag{2.7a}$$

and

$$G(r) \equiv 2\int_{2M}^{r} (r'-3M)\,\Theta(r')\, dr' \ , \tag{2.7b}$$

we may write T^ν_μ in the form

$$T^\nu_\mu = T^{(1)\nu}_\mu + T^{(2)\nu}_\mu + T^{(3)\nu}_\mu + T^{(4)\nu}_\mu \ ,$$

where (in t, r^*, θ, ϕ coordinates)

$$T^{(1)\nu}_\mu = \begin{bmatrix} -\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1} H(r) + \frac{1}{2}T^\alpha_\alpha(r) & 0 & 0 & 0 \\ 0 & \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1} H(r) & 0 & 0 \\ 0 & 0 & \frac{1}{4}T^\alpha_\alpha(r) & 0 \\ 0 & 0 & 0 & \frac{1}{4}T^\alpha_\alpha(r) \end{bmatrix} \ , \tag{2.8a}$$

$$T^{(2)\nu}_\mu = \frac{K}{M^2 r^2}\left(1-\frac{2M}{r}\right)^{-1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \tag{2.8b}$$

and

$$T_\mu^{(3)\nu}=\begin{bmatrix} -\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1}G(r)-2\Theta(r) & 0 & 0 & 0 \\ 0 & \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1}G(r) & 0 & 0 \\ 0 & 0 & \Theta(r) & 0 \\ 0 & 0 & 0 & \Theta(r) \end{bmatrix}, \tag{2.8c}$$

$$T_\mu^{(4)\nu}=\frac{Q}{M^2r^2}\left(1-\frac{2M}{r}\right)^{-1}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.8d}$$

All four of these tensors satisfy $\nabla_\nu T_\mu^\nu=0$. Only $T_\mu^{(1)\nu}$ has a nonzero trace, only $T_\mu^{(2)\nu}$ has off-diagonal (flux) components, and only $T_\mu^{(3)\nu}$ has a traceless part $(T_\mu^\nu-\frac{1}{4}T_\alpha^\alpha g_\mu^\nu)$ whose $\theta\theta$ component is not zero. Thus only the last of the four conditions listed in Sec. I remains to be investigated. One can show (see Appendix and Ref. 13) that T_μ^ν, as measured in local frames on the future horizon, will be finite if T_θ^θ, T_{vv}, and $T_t^t+T_{r^*}^{r^*}$ are finite as $r\to 2M$ and

$$\lim_{r\to 2M}(r-2M)^{-2}\,|T_{uu}(r)|<\infty\,. \tag{2.9}$$

Geometrically, one of these factors of $(r-2M)^{-1}$ comes from the normalization of the basis vectors ("infinite time dilation"), and the other comes from the Lorentz transformation from a frame aligned with the Killing vector to the frame of an observer crossing the horizon with finite velocity ("infinite red-shift"). We find easily that

$$\begin{aligned} T_{uu}&=\tfrac{1}{4}(T_{tt}+T_{r^*r^*}-2T_{tr^*}) \\ &=\frac{1}{2r^2}\left(\frac{Q}{M^2}+G+H\right)+\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\Theta-\tfrac{1}{4}T_\alpha^\alpha\right) \\ &=\frac{1}{2r^2}\left[\frac{Q}{M^2}+O((r-2M)^2)\right], \end{aligned} \tag{2.10}$$

the final step following quickly from the definitions of G and H. Therefore, the condition (2.9) is equivalent to $Q=0$; that is, $T_\mu^{(4)\nu}$ is singular on the future horizon, but the others are not. [Regularity of T_μ^ν on the *past* horizon requires that $(r-2M)^{-2}T_{vv}$ remain finite there. Since $T_{vv}=T_{uu}-K/M^2r^2$, this condition translates into $Q=2K$.]

Finally, note that the equations of this section apply to any kind of quantum field. Only when we specify $T_\alpha^\alpha(r)$ (Sec. IV) will we need to specialize to a particular field theory.

III. THE TWO-DIMENSIONAL CASE

The two-dimensional Hawking problem for a massless scalar field and a collapsing body with exterior metric

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2 \tag{3.1}$$

has been solved exactly.[10] Here we shall show that the results of those calculations can also be obtained from some general physical arguments plus one piece of numerical information, which could be either the coefficient in the Hawking temperature formula, $kT=(8\pi M)^{-1}$, or the coefficient in the general formula for the trace anomaly of the two-dimensional scalar field, $T_\alpha^\alpha=(24\pi)^{-1}R$.

The analysis of Sec. II is easily repeated for the metric (3.1). The conservation equations for time-independent T_μ^ν are

$$\partial_r T_t^r=0\,, \tag{3.2a}$$

$$\partial_r T_r^r=\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}(T_t^t-T_r^r)\,. \tag{3.2b}$$

Equation (3.2b) is equivalent to

$$\partial_r\left[\left(1-\frac{2M}{r}\right)T_r^r\right]=\frac{M}{r^2}T_\alpha^\alpha\,.$$

With

$$H_2(r)\equiv M\int_{2M}^r (r')^{-2}\,T_\alpha^\alpha(r')\,dr'\,, \tag{3.3}$$

the general solution, in the style of Eqs. (2.8), is

$$T_\mu^\nu=T_\mu^{(1)\nu}+T_\mu^{(2)\nu}+T_\mu^{(3)\nu}\,,$$

where (in coordinates t, r^*)

$$T_{\mu}^{(1)\nu} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right)^{-1} H_2(r) + T^{\alpha}_{\alpha}(r) & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} H_2(r) \end{pmatrix}, \tag{3.4a}$$

$$T_{\mu}^{(2)\nu} = \frac{K}{M^2}\left(1-\frac{2M}{r}\right)^{-1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \tag{3.4b}$$

and

$$T_{\mu}^{(3)\nu} = \frac{Q}{M^2}\left(1-\frac{2M}{r}\right)^{-1}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4c}$$

There is no arbitrary function Θ in this case.

To interpret these expressions, we begin by recalling that, in our present sign convention, $T_t^{\,t}$ is negative (T^{tt} is positive) for classical matter, and $T_r^{\,t}$ is positive for matter moving in the positive r direction. In two-dimensional flat space, the energy density of a gas of massless bosons (without spin, charge, or internal degrees of freedom) in thermal equilibrium is

$$T_{tt}^{(e)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|p|\,dp}{e^{|p|/kT}-1} = \frac{\pi}{6}(kT)^2 .$$

The energy density of a beam of massless blackbody radiation moving in the positive r direction is just the $p>0$ half of this integral, and is accompanied by a flux of the same magnitude. Thus the stress tensor of the equilibrium gas is

$$T_{\mu}^{(e)\nu} = \frac{\pi}{12}(kT)^2\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \tag{3.5}$$

and that of the radiation is

$$T_{\mu}^{(r)\nu} = \frac{\pi}{12}(kT)^2\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}. \tag{3.6}$$

Hawking's argument (Ref. 4) that a collapsing body must radiate with the effective temperature at infinity given by $kT = (8\pi M)^{-1}$ remains valid in two dimensions. Moreover, because of conformal invariance and conformal flatness, there is no scattering of massless particles or waves by the geometry in two dimensions, and consequently the stress tensor at infinity is given precisely by Eq. (3.6), without the complication of emission-absorption coefficients different from unity. Comparing with Eqs. (3.4) in the limit $r\to\infty$, we see that the stress tensor for the Hawking problem in the steady-state region must have

$$K = (768\pi)^{-1} . \tag{3.7}$$

However, $T_{\mu}^{(2)\nu}$ by itself is not the full stress tensor, since the diagonal components have the wrong sign. Indeed, $T_{\mu}^{(2)\nu}$ with $K>0$ would represent *negative* energy flowing *into* the black hole. (We have $T_{uu}^{(2)}=0$, $T_{vv}^{(2)} = -K/M^2$.)

If $T^{\alpha}_{\alpha}(r)=0$ everywhere, the only way to recover the asymptotic form (3.6) is to take $Q=2K$. (Then $T_{uu}=K/M^2$, $T_{vv}=0$.) This would be the only way out, if the Hawking flux consisted of classical massless radiation (which would necessarily be produced inside or very near the collapsing body). The intensity of that radiation would have to become infinite at the horizon, since Eq. (2.9) would be violated. It is most unlikely that the expectation value of the quantum $T_{\mu}^{\,\nu}$ in the initial vacuum state behaves in that way, since both the initial conditions defining the vacuum, and the geometry on, and in the past of, the horizon are perfectly smooth. The vagaries of renormalization hinder one in making this argument rigorous, and the point has been controversial: According to one school of thought, the renormalized $\langle T_{\mu\nu}\rangle$ does become singular on the horizon,[11] while another holds that $\langle T_{\mu\nu}\rangle$ has no invariant meaning at all.[12] However, the calculations in two-dimensional models confirm the expectation that $\langle T_{\mu\nu}\rangle$ is well defined and nonsingular,[13] and we adopt this point of view in the present paper.

The tensor with $Q=2K\neq 0$ and $T^{\alpha}_{\alpha}=0$ is also unacceptable for Unruh's ξ vacuum, since, contrary to the intended definition of that state, T_{uu} does not vanish on the past horizon.

The resolution of these difficulties is that $T_{\mu}^{\,\nu}$ does have a trace, even if the field equations are conformally invariant. With $Q=0$, the total energy density [from Eqs. (3.4a) and (3.4b)] then approaches $H_2(\infty) - T^{\alpha}_{\alpha}(\infty) - K/M^2$ as $r\to\infty$. For almost any kind of field one expects this quantity to be larger than or equal to the asymptotic flux, K/M^2. (Since the radiation reaching $\mathscr{I}^+$ is in a truly thermal state, there are no quantum correlations which could disrupt the classical energy conditions.) Thus $T^{\alpha}_{\alpha}(r)$ cannot be identically zero.

For a field describing massless particles [to which Eq. (3.6) applies], the density and the flux are actually equal, so that

$$K = \tfrac{1}{2}M^2\left[H_2(\infty) - T^{\alpha}_{\alpha}(\infty)\right] = \tfrac{1}{2}M^3\int_{2M}^{\infty}(r')^{-2}\,T^{\alpha}_{\alpha}(r')\,dr' , \tag{3.8}$$

since $T^{\alpha}_{\alpha}(\infty)$ must vanish in this case. The equation relates the Hawking effect to the trace of $T_\mu^{\ \nu}$. In the case of a two-dimensional conformally invariant field, as explained in Sec. I and in more detail in Sec. IV, one expects the trace to be proportional to the curvature scalar, which does not vanish for the reduced Schwarzschild metric (3.1):

$$T_\alpha^{\ \alpha} = \alpha R = 4\alpha M r^{-3} \ . \qquad (3.9)$$

Thus one has

$$H_2(r) = \frac{\alpha}{16M^2} - \frac{\alpha M^2}{r^4} \ , \qquad (3.10)$$

hence $\alpha = 32K$. With Eq. (3.7) this yields $\alpha = (24\pi)^{-1}$, in agreement with the general theory (see Sec. IV). Thus we have derived the conformal anomaly from the Hawking effect.

Conversely, knowing $T_\alpha^{\ \alpha}$, one can calculate the Hawking flux, K. Indeed, even the qualitative nature of the effect could have been predicted from a knowledge of the trace anomaly. That conditions on $\mathscr{I}^+$ asymptotically approach a steady state is suggested by the classical time-dilation argument for black-hole effects. Then one would argue as above that $T_\mu^{\ \nu} = T_\mu^{(1)\,\nu} + T_\mu^{(2)\,\nu}$, where $T_\mu^{(1)\,\nu}$ now is known but K must be determined. If $K \neq \alpha/32$, then $T_{vv} \neq 0$ in the steady-state region. At large r, $T_\mu^{\ \nu}$ is almost trace-free, so the conservation law implies that T_{vv} is independent of u. This means that an ingoing flux can be traced all the way back to $\mathscr{I}^-$, which contradicts the initial conditions of the problem.

The generality of the Hawking effect and of the foregoing arguments implies that all two-dimensional conformally invariant boson field theories have the same conformal anomaly coefficient, α, except for a factor to account for the distinct types of particles in the theory and their spin and charge states. [This factor enters in Eq. (3.6).] For fermions one must use the Fermi-Dirac distribution, instead of the Bose-Einstein distribution, to calculate $T_{tt}^{(e)}$.

Finally, we can extract from Eqs. (3.4) the stress tensors of the Israel and Boulware vacuum states (see Sec. I) of the full two-dimensional Schwarzschild black hole. The former is regular on both the past and future horizons ($2K = Q = 0$), hence is just equal to $T_\mu^{(1)\,\nu}$. At infinity it approaches the equilibrium form (3.5). The physical meaning of the negative flux $T_{vv}^{(2)}$ is now clear: It represents the *absence* from the ξ vacuum, relative to the v vacuum, of the inward-moving half of the radiation in the equilibrium distribution.

The η-vacuum stress must vanish at infinity. Clearly, it must be $T_\mu^{(1)\,\nu} + T_\mu^{(3)\,\nu}$ with $Q = -M^2 H_2(\infty) = -\alpha/16 = -(384\pi)^{-1}$. Since $Q \neq 0$, this tensor is singular on the horizon. However, it does provide the expectation value of the stress in the vacuum outside a stable body of mass M and radius greater than $2M$. Note that $-T_\mu^{(3)\,\nu}$, the difference between stresses of the Israel and Boulware states, is the classical extension of the equilibrium distribution (3.6) throughout the static space-time; the effective local temperature is $T(g_{tt})^{-1/2}$.

The expressions deduced here for the stress tensors of the various states are in complete agreement with those obtained by direct calculation in Ref. 5. Our goal is to generalize the considerations of this section to four dimensions insofar as possible.

IV. TRACE ANOMALIES

In order to explain how we shall fix the form of $T_\alpha^{\ \alpha}(r)$ for a particular field theory, we need to review the subject of trace anomalies in general. Consider an action functional, $S[\phi]$, which is invariant under general coordinate transformations and under conformal transformations of the form

$$g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu} \ ,$$
$$\phi(x) \to \Omega^n(x)\, \phi(x) \ ,$$

where $\Omega(x)$ is some arbitrary space-time function and n is a number depending on the field ϕ being considered. Invariance under general coordinate transformations requires that $\nabla_\nu T_\mu^{\ \nu} = 0$, while conformal invariance forces $T_\alpha^{\ \alpha} = 0$.

When quantizing the field ϕ on a fixed curved background, one finds that the vacuum expectation value $\langle T_{\mu\nu}(x)\rangle$ is infinite. The natural next step is to choose some method of regularization which isolates the divergences in some physically meaningful manner. There are two regularization-renormalization techniques designed to be applied to curved-space problems—dimensional regularization and covariant geodesic point separation. The first of these maintains the requirement $\nabla_\nu T_\mu^{\ \nu} = 0$ throughout. However, the process of analytically continuing to arbitrary complex dimensions and expanding about the pole at dimension 4 causes the stress tensor for a conformally invariant theory to have a nonzero trace. The point-separation procedure, which involves expanding the $\langle T_{\mu\nu}(x)\rangle$ in powers of the tangent vector to a geodesic through the point x, also gives a trace once direction-dependent terms are discarded. A term which arises in a renormalized current operator in this way, in violation of a classical identity, is called an anomaly. Anomalies appear in various quantum field theories and have had experimentally verified consequences.[14]

Relatively little work has been done to fix the exact values of the trace anomalies for various theories, but we shall see that just enough information now exists for us to find their form in the conformally invariant massless scalar theories in two and four dimensions.

Deser, Duff, and Isham[15] have shown, in the context of dimensional regularization, that the general form of the anomalies in the renormalized stress tensor is

$$T^{\alpha}_{\alpha} = k_1 C^2 + k_2(R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2) + k_3\Box R + k_4 R^2 \quad (4.1)$$

in four dimensions and

$$T^{\alpha}_{\alpha} = \alpha R \quad (4.2)$$

in two dimensions. Their general arguments do not determine the dimensionless constants k_i and α. The following argument suggests that $k_4 = 0$. In a Robertson-Walker space-time model it is possible to define a vacuum state whose stress tensor must be, by construction, homogeneous and isotropic. A study[16] of the conservation law under those conditions (analogous to Sec. II of this paper) reveals that there is no conserved, homogeneous, isotropic tensor function of the type which can arise in point-separation calculations whose trace is a linear combination of $R^{\alpha\beta}R_{\alpha\beta}$ and R^2 other than a multiple of $R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2$. Since the k_i are the same for all quantum states and all space-time models, the R^2 term should not appear in Eq. (4.1).

At this point one might ask whether these anomalies are an artifact of the regularization scheme chosen. Fortunately, various results have been obtained by different methods and are consistent. The situation for the two-dimensional massless scalar field is quite well established and allows us to draw an important inference about the four-dimensional case. The integration over mode functions in the definition of $T_{\mu\nu}$, regularized by covariant point separation, has been done explicitly, for an arbitrary two-dimensional metric.[17,18] The result indicates that in Eq. (4.2)

$$\alpha = (24\pi)^{-1} \; ; \quad (4.3)$$

in fact, any other value is inconsistent with the conservation law. This number also appears in the two-dimensional analog[19] of Christensen's general calculation of $\langle \text{out, vac} | T_{\mu\nu} | \text{in, vac} \rangle$ for a *massive* scalar field by point separation.[20] Christensen's calculations are not directly applicable to the massless case because they involve an expansion in inverse powers of the mass. However, there is an unambiguous term *independent* of the mass, and of order 2ω (the dimension of space-time) in derivatives of the metric, which has a nonvanishing trace. This term is

$$-\frac{1}{24\pi} R \frac{\sigma_\mu \sigma_\nu}{\sigma^\rho \sigma_\rho} \quad (4.4)$$

in the two-dimensional case, and

$$-\frac{1}{2880\pi^2}(C^2 + R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2 + \Box R)\frac{\sigma_\mu \sigma_\nu}{\sigma^\rho \sigma_\rho} \quad (4.5)$$

in the four-dimensional case, where σ_μ is the tangent vector to the geodesic used in the point separation. In the calculations this term arises as the sum of a contribution of the type $g_{\mu\nu}/2\omega$ from the term in $T_{\mu\nu}$ which depends explicitly on the mass $(-\frac{1}{2}m^2\phi^2 g_{\mu\nu})$, and a contribution of the type

$$\frac{\sigma_\mu \sigma_\nu}{\sigma^\rho \sigma_\rho} - \frac{1}{2\omega} g_{\mu\nu} \quad (4.6)$$

from the term in $T_{\mu\nu}$ which has the same form as in the massless case $(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\partial_\rho\phi\partial^\rho\phi g_{\mu\nu}$ in two dimensions, for example). The general philosophy of point separation, that the terms of physical significance are those independent of σ_μ, then suggests that the terms associated with the $-g_{\mu\nu}/2\omega$ in Eq. (4.6) are the source of the trace anomaly. We therefore conclude that

$$k_1 = k_2 = k_3 = (2880\pi^2)^{-1} \,. \quad (4.7)$$

This interpretation is supported by an explicit mode-sum calculation for a massive field in a two-dimensional Robertson-Walker space-time,[21] where the anomalous term in the corresponding massless theory (Ref. 18) is seen to be canceled by a term from $-\frac{1}{2}m^2\phi^2 g_{\mu\nu}$ in the manner described.

In four dimensions there are two calculations by dimensional regularization which verify parts of Eq. (4.7). The first is the calculation of the stress tensor for a conformally invariant massless scalar field in a de Sitter space of radius a by Dowker and Critchley.[22] They find that

$$T^{\alpha}_{\alpha} = -\frac{1}{240\pi^2 a^4} = \frac{1}{2880\pi^2}(R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2) \,,$$

which confirms k_2 in Eq. (4.7), since $C^2 = \Box R = 0$ in de Sitter space.

Second, in momentum-space dimensional regularization, the infinite counterterm that must be added to the original Lagrangian to eliminate divergences in the scalar, neutrino, and photon contributions to the graviton self-energy has the form

$$\Delta\mathcal{L}(\omega) = \frac{1}{2-\omega} \frac{N(s)g^{1/2}}{24(4\pi)^2(4\omega^2-1)} \times [\tfrac{1}{2}\omega R^2 - (2\omega - 1)R^{\alpha\beta}R_{\alpha\beta}] \,,$$

where $N(s) = 1$, 3, and 12 for spin $(s) = 0$, $\frac{1}{2}$, and 1, respectively. From this Capper and Duff[23] obtain

$\Box R$ anomalies in the following way: Expand the ω-dependent part of the coefficient of $(2-\omega)^{-1}$ about $\omega=2$. The finite term which results will have a piece which is not conformally invariant, $\Delta\mathcal{L}_{\text{anom}}$. Form the action S_{anom} corresponding to $\Delta\mathcal{L}_{\text{anom}}$ and construct the stress tensor

$$T^{\mu\nu}_{\text{anom}} = 2g^{-1/2}\frac{\delta S_{\text{anom}}}{\delta g_{\mu\nu}},$$

where $\delta/\delta g_{\mu\nu}$ represents functional differentiation with respect to the metric. The trace of $T^{\mu\nu}_{\text{anom}}$ gives the $\Box R$ anomaly. Following this procedure, one finds that

$$T^{\alpha}_{\alpha} = \frac{N(s)}{2880\pi^2}\Box R,$$

so that $k_3=(2880\pi^2)^{-1}$ for a scalar field, as expected. The number is 3 times larger for a neutrino field and 12 times larger for the electromagnetic field.

In a four-dimensional Schwarzschild background, where $R_{\alpha\beta}=R=0$, we will therefore have

$$T^{\alpha}_{\alpha}(r) = \frac{1}{2880\pi^2}C^2 = \frac{1}{60\pi^2}\frac{M^2}{r^6} \tag{4.8}$$

for a neutral conformally invariant massless scalar field. Since the full (nonlocal) effective Lagrangian probably involves $N(s)$ as an overall factor, it seems likely that the k_1 coefficients for scalar, neutrino, and electromagnetic fields are related in the same 1 : 3 : 12 ratio as the k_3 coefficients.[24]

Finally, it should be noted that the coefficients in the anomaly are expected to be unique, since there are no divergent terms in $T_{\mu\nu}$ with traces of the same form, which would introduce an ambiguity.[25] Also, when there are several noninteracting fields in the theory, the coefficients (and the stress tensor as a whole) are additive.

The arguments presented in this section are preliminary. Detailed calculations will be needed to determine the values of all anomaly coefficients firmly and to make the relationship between massive and massless theories totally clear. In any event, the principal conclusions of this paper are independent of the precise values of the coefficients.

V. PHYSICAL REQUIREMENTS ON THE VACUUM STRESS TENSORS

In this section we shall match the expressions (2.8) with the properties expected for $\langle T^{\nu}_{\mu}\rangle$, defined relative to the three types of static vacuum state for the Schwarzschild space-time without a collapsing body. As mentioned previously, the ξ vacuum is expected to be a good approximation at late times to the $\langle T^{\nu}_{\mu}\rangle$ outside a body collapsing from a relatively static configuration in which it was initially surrounded by empty space.

It is immediately obvious that the relation between the trace anomaly and the Hawking flux will not be as tight as in two dimensions because of the unknown function Θ in the solution (2.8). We shall take both the anomaly coefficient and the asymptotic flux as independently calculable and use them to obtain information about Θ.

The tensor $T^{(1)\nu}_{\mu}$ [Eq. (2.8a)] will be the same for all vacuum states of a given massless field. From Eqs. (4.8) and (2.7a), we have

$$\begin{aligned} T^{(1)t}_{t} &= \frac{\beta}{M^2r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(-\frac{3}{640}+\frac{5M^4}{8r^4}-\frac{11M^5}{10r^5}\right),\\ T^{(1)r}_{r} &= \frac{\beta}{M^2r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{3}{640}-\frac{M^4}{8r^4}+\frac{M^5}{10r^5}\right),\\ T^{(1)r}_{r}-\tfrac{1}{4}T^{\alpha}_{\alpha} &= \frac{\beta}{M^2r^2}\left(1-\frac{2M}{r}\right)^{-1}\\ &\quad\times\left(\frac{3}{640}-\frac{3M^4}{8r^4}+\frac{3M^5}{5r^5}\right),\\ T^{(1)\theta}_{\theta}-\tfrac{1}{4}T^{\alpha}_{\alpha} &= \frac{\beta}{M^2r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{M^4}{4r^4}-\frac{M^5}{2r^5}\right), \end{aligned} \tag{5.1}$$

where $\beta^{-1}=60\pi^2$ for the neutral scalar field and (as a working hypothesis, used only in Sec. VII) $20\pi^2$ for neutrinos and $5\pi^2$ for photons, according to the discussion in Sec. IV.

The expectation value of T^{ν}_{μ} in the ξ vacuum must be of the form

$$\begin{aligned} T^{(\xi)\nu}_{\mu} &\equiv \langle\xi|T^{\nu}_{\mu}|\xi\rangle\\ &= T^{(1)\nu}_{\mu}+T^{(2)\nu}_{\mu}[K_\xi]+T^{(3)\nu}_{\mu}[\Theta_\xi], \end{aligned} \tag{5.2}$$

since this state is regular at the future horizon ($Q_\xi=0$). The luminosity (total power radiated by the black hole) is

$$\begin{aligned} L &= \int_{\text{large } r} r^2\sin\theta\, d\theta\, d\phi\,\langle T^{t}_{r^*}\rangle\\ &= \frac{4\pi K_\xi}{M^2}. \end{aligned} \tag{5.3}$$

The only matter present in the asymptotic region is the outward flux of massless radiation, so the leading terms (of order r^{-2}) of T^{r}_{r} and $-T^{t}_{t}$ must be equal to $T^{t}_{r^*}$ there. By inspection of the rr or tt components of the three terms of Eq. (5.2), we see that

$$M^2G_\xi(\infty) = 2K_\xi - \frac{3\beta}{640}, \tag{5.4}$$

where $G_\xi(r)$ is defined from $\Theta_\xi(r)$ by Eq. (2.7b). This equation is the analog of Eq. (3.8) and is one condition that Θ_ξ must satisfy. We shall relate it to some calculated values of K_ξ in Sec. VII.

What else can we say about Θ_ξ at this point? Asymptotically, it must be the transverse pressure, T_θ^{θ}, of the radiation. This quantity is not zero for radiation from a source of finite effective size. A particle with four-momentum p^μ makes a contribution to $T_{\mu\nu}$ proportional to $p_\mu p_\nu$. In flat space, a radiating body of radius b appears to an observer at distance r as a disk. Particles reaching the observer from the edge of the disk have $p_\theta/p_r \approx b/r$, and hence

$$\frac{T_\theta^{\theta}}{T_r^{r}} \approx \left(\frac{b}{r}\right)^2 \quad \left(T_r^{r} \sim \frac{L}{4\pi r^2}\right).$$

For particles coming from nearer the center of of the disk the ratio is smaller. This argument can be extended to black holes: A careful analysis of massless particle orbits in the Schwarzschild metric shows that a radiating body of radius slightly greater than $2M$ presents a disk of effective radius $\sqrt{27}M$ to a distant observer.[26] This conclusion applies also to the Hawking radiation, whose form *at infinity* is determined by the same emission coefficients which govern the propagation of classical waves from $r \approx 2M$ to infinity (see Refs. 4 and 1). Therefore, we anticipate that as $r \to \infty$,

$$\Theta_\xi(r) \sim \lambda K_\xi / r^4 \quad (0 < \lambda \lesssim 27)\,. \tag{5.5}$$

(The coefficient is independent of M, because the decrease in angular spread as $M \to 0$ is compensated by the increase in temperature and hence luminosity.)

Let us now turn to the η vacuum. Since this state is defined in the conventional way relative to a Killing vector which generates geodesic trajectories at infinity, it should represent essentially empty space for sufficiently large r. One therefore expects the components of $T_\mu^{(\eta)\nu}$ to fall off faster than those of $T_\mu^{(\xi)\nu}$ as $r \to \infty$, and also to vanish as $M \to 0$. By analogy with the two-dimensional case, in fact, it seems likely that all the components decrease as rapidly as the trace does—viz., as $M^2 r^{-6}$. On the other hand, $T_\mu^{(\eta)\nu}$ is presumably not regular on the future horizon. Indeed, one must anticipate the possibility that $\Theta_\eta(r)$ is so badly behaved as $r \to 2M$ that the integral (2.7b) defining $G_\eta(r)$ does not converge. Comparing Eqs. (2.8c) and (2.8d), one sees that changing the lower limit of that integral simply amounts to a redefinition of Q. Let us, therefore, use the definition

$$G_\eta(r) = 2\int_\infty^r (r' - 3M)\Theta_\eta(r')\,dr' \tag{5.6}$$

in this case. With this change in notation understood, we have

$$T_\mu^{(\eta)\nu} = T_\mu^{(1)\nu} + T_\mu^{(3)\nu}[\Theta_\eta] + T_\mu^{(4)\nu}[Q_\eta]$$

(where $T_\mu^{(3)\nu}[\Theta_\eta]$ does *not* satisfy the condition (2.9)). Since $T_r^{(\eta)r}$ and $T_t^{(\eta)t}$ must vanish at infinity, we have

$$Q_\eta = -3\beta/640 \tag{5.7}$$

Finally, the v vacuum should be regular on the horizon and invariant under time reversal:

$$T_\mu^{(v)\nu} = T_\mu^{(1)\nu} + T_\mu^{(3)\nu}[\Theta_v]\,. \tag{5.8}$$

Furthermore, the physical conditions at spatial infinity in this state should be thermal equilibrium at the Hawking temperature. Consequently, $T_\theta^{(v)\theta}$, $T_r^{(v)r}$, and $-\frac{1}{3}T_t^{(v)t}$ must all approach

$$\begin{aligned}\Theta_v(\infty) &= \tfrac{1}{3}\times\frac{1}{2\pi^2}\int_0^\infty \frac{p^3\,dp}{e^{p/kT}-1}\\ &= \frac{\pi^2}{90}(kT)^4\\ &= \tfrac{1}{90}(8^4\pi^2)^{-1}\frac{1}{M^4}\end{aligned} \tag{5.9}$$

as $r \to \infty$, in the case of a neutral scalar field. In other cases one must multiply by the number of independent charge and helicity states (e.g., by 2 for the electromagnetic field). For fermions there is an additional factor of $\frac{7}{8}$, since

$$\int_0^\infty \frac{p^3\,dp}{e^{p/kT}+1} = \frac{7\pi^4}{120}\,(kT)^4\,. \tag{5.10}$$

To ensure the proper behavior for all the components of $T_\mu^{(v)\nu}$, it suffices to assume it for Θ_v, since

$$\begin{aligned}\lim_{r\to\infty}\left[T_r^{(v)r}(r) - \tfrac{1}{4}T_\alpha^{\alpha}(r)\right] &= \lim_{r\to\infty} r^{-2}G_v(r)\\ &= \Theta_v(\infty)\,.\end{aligned} \tag{5.11}$$

In fact, in the nearly flat region far from the black hole, where the quantum effects of local space-time curvature are negligible, one would expect the energy-momentum tensor to take the classical equilibrium form, characterized by the position-dependent temperature

$$T(r) = T(g_{tt})^{-1/2} = T\left(1 - \frac{2M}{r}\right)^{-1/2}\,. \tag{5.12}$$

This supposition is consistent in the sense that if one assumes that

$$T_\theta^{(v)\theta}(r) = \Theta_v(\infty)\left(1 - \frac{2M}{r}\right)^{-2}\left[1 + O\!\left(\frac{M^3}{r^3}\right)\right], \tag{5.13a}$$

then one finds that

$$T_r^{(\upsilon)r}(r)=\Theta_\upsilon(\infty)\left(1-\frac{2M}{r}\right)^{-2}\left[1+O\left(\frac{M^2}{r^2}\right)\right]. \tag{5.13b}$$

How the stress tensors should behave near the horizon is less clear physically than at infinity. However, analogies with the two-dimensional case and with the situation at infinity do suggest a pattern, which will be confirmed by the considerations in Sec. VI.

In two dimensions, the traceless part of the Israel vacuum stress (3.4a) goes to zero linearly as r approaches $2M$:

$$\begin{aligned} T_r^{(\upsilon)r}-\tfrac{1}{2}T_\alpha^\alpha &= \left(1-\frac{2M}{r}\right)^{-1}H_2(r)-\tfrac{1}{2}T_\alpha^\alpha \\ &= \alpha\left(1-\frac{2M}{r}\right)^{-1}\left(-\frac{1}{16M^2}+\frac{2M}{r^3}-\frac{3M^2}{r^4}\right) \\ &= -\frac{\alpha}{16M^2}\left(1-\frac{2M}{r}\right)\left(1+\frac{4M}{r}+\frac{12M^2}{r^2}\right). \end{aligned} \tag{5.14}$$

Furthermore, the components in an orthonormal frame on (or off) the horizon vanish as one approaches the point of intersection of the past and future horizons (see Appendix). These results are in keeping with the intuitive idea (see Refs. 7 and 5) that the υ vacuum is, by construction, the state which is as empty as possible in the vicinity of that point; the conservation law and the curvature of space force the state to become nonempty as one moves away from the point. Hence it is reasonable to expect that in four dimensions all components of $T_\mu^{(\upsilon)\nu}-\frac{1}{4}T_\alpha^\alpha g_\mu^\nu$ will be proportional to $1-2M/r$ to lowest order, as $r\to 2M$. In the Appendix it is shown that if $\Theta_\upsilon\equiv T_\theta^{(\upsilon)\theta}-\frac{1}{4}T_\alpha^\alpha$ satisfies this condition, then all components do.

By analogy with the two-dimensional results, the traceless parts of the Unruh and Boulware vacuum stresses should be negative near the horizon. Also, for reasons which will become clear in the next section, we expect Θ_ξ to approach a constant in the limit, $T_r^{(\xi)r}-\frac{1}{4}T_\alpha^\alpha$ to go as $(1-2M/r)^{-1}$, and all components of $T_\mu^{(\eta)\nu}-\frac{1}{4}T_\alpha^\alpha g_\mu^\nu$ to go as $(1-2M/r)^{-2}$. It can easily be checked that these assertions are all consistent with the structure of Eqs. (2.8), and that an analog of Eq. (5.11) holds for the terms of order $(1-2M/r)^{-2}$ in $T_\mu^{(\eta)\nu}$.

The anticipated asymptotic forms (including conjectured signs, but no numerical coefficients) are summarized in Table I. Note that $1-2M/r$ plays a role at the horizon precisely analogous to that of r^{-2} at infinity, with the roles of $|\upsilon\rangle$ and $|\eta\rangle$ interchanged. This symmetry will arise naturally in the calculation of Sec. VI.

VI. DIFFERENCES OF VACUUM EXPECTATION VALUES

Here we begin the task of actually calculating $\langle T_\mu^\nu\rangle$ from the solution of the scalar field equation in the Schwarzschild metric. We set the problem up and pursue it far enough to obtain support for the statements which were made in Sec. V on the basis of abstract arguments of varying degrees of physical cogency. We concentrate on the differences among the expectation values $T_\mu^{(\upsilon)\nu}$, $T_\mu^{(\xi)\nu}$, and $T_\mu^{(\eta)\nu}$.

The formal expression (symmetrized) for the energy-momentum tensor of a conformally invariant massless Hermitian scalar field, in a background metric with $R_{\mu\nu}=0$, is

$$\begin{aligned} T_{\mu\nu}= & \tfrac{1}{3}(\nabla_\mu\phi\nabla_\nu\phi+\nabla_\nu\phi\nabla_\mu\phi) \\ & -\tfrac{1}{6}(g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi)g_{\mu\nu} \\ & -\tfrac{1}{6}[(\nabla_\mu\nabla_\nu\phi)\phi+\phi\nabla_\mu\nabla_\nu\phi]. \end{aligned} \tag{6.1}$$

If the field is expanded in a complete set of positive-norm mode functions u_j and their complex conjugates, and the coefficients are treated as annihilation and creation operators, then the expectation value in the corresponding "vacuum" state is

$$\langle T_{\mu\nu}\rangle=\sum_j T_{\mu\nu}[u_j,u_j^*], \tag{6.2a}$$

$$\begin{aligned} T_{\mu\nu}[u,u^*]= & 2\,\mathrm{Re}[\tfrac{1}{3}\nabla_\mu u\nabla_\nu u^*-\tfrac{1}{6}(\nabla_\mu\nabla_\nu u)u^*] \\ & -\tfrac{1}{6}(\nabla^\alpha u\nabla_\alpha u^*)g_{\mu\nu}. \end{aligned} \tag{6.2b}$$

For the exterior Schwarzschild metric a set of properly normalized basis functions is[27]

$$\begin{aligned} \vec{u}_{plm}(x) &= (4\pi p)^{-1/2}r^{-1}\vec{R}_l(p|r)\,Y_{lm}(\theta,\phi)\,e^{-ipt}, \\ \overleftarrow{u}_{plm}(x) &= (4\pi p)^{-1/2}r^{-1}\overleftarrow{R}_l(p|r)\,Y_{lm}(\theta,\phi)\,e^{-ipt}, \end{aligned} \tag{6.3}$$

where p is positive, the Y_{lm} are the standard

TABLE I. Anticipated asymptotic behavior of the expectation values (numerical factors omitted).

State	$r\to\infty$, $r^*\to\infty$: $T_r^r-\frac{1}{4}T_\alpha^\alpha$	$r\to\infty$, $r^*\to\infty$: $T_\theta^\theta-\frac{1}{4}T_\alpha^\alpha$	$r\to 2M$, $r^*\to-\infty$: $T_r^r-\frac{1}{4}T_\alpha^\alpha$	$r\to 2M$, $r^*\to-\infty$: $T_\theta^\theta-\frac{1}{4}T_\alpha^\alpha$
$\lvert\upsilon\rangle$	$+1$	$+1$	$+\left(1-\frac{2M}{r}\right)$	$+\left(1-\frac{2M}{r}\right)$
$\lvert\xi\rangle$	$+r^{-2}$	$+r^{-4}$	$-\left(1-\frac{2M}{r}\right)^{-1}$	-1
$\lvert\eta\rangle$	$-r^{-6}$	$-r^{-6}$	$-\left(1-\frac{2M}{r}\right)^{-2}$	$-\left(1-\frac{2M}{r}\right)^{-2}$

spherical harmonics normalized so that

$$\sum_{m=-l}^{l} |Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi},$$

and the radial functions satisfy

$$-\frac{d^2R}{dr^{*2}} + \left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right]R = p^2 R . \tag{6.4}$$

Since Eq. (6.4) has the form of a Schrödinger equation with a positive potential tending to zero both as $r^* \to \infty$ $(r\to\infty)$ and as $r^* \to -\infty$ $(r \to 2M)$, one can rely *for technical mathematical purposes* on the concepts of ordinary quantum scattering theory. We choose the "incoming" basis, in which the functions have the asymptotic forms

$$\vec{R}_l(p|r) \sim \begin{cases} e^{ipr^*} + \vec{A}_l(p)e^{-ipr^*}, & r \to 2M \\ B_l(p)e^{ipr^*}, & r\to\infty; \end{cases} \tag{6.5a}$$

$$\overleftarrow{R}_l(p|r) \sim \begin{cases} B_l(p)e^{-ipr^*}, & r \to 2M \\ e^{-ipr^*} + \overleftarrow{A}_l(p)e^{ipr^*}, & r\to\infty. \end{cases} \tag{6.5b}$$

The η vacuum is constructed by taking the scattering-theory interpretation seriously on the *physical* level. One writes

$$\phi(t,x) = \sum_j (a_j u_j + a_j^\dagger u_j^*) \tag{6.6}$$

and interprets the $a_j^\dagger$ as creation operators for real particles. The trouble with this theory is that t and r^* are not even approximately Cartesian coordinates near the horizon, so the identification of solutions of positive p with particle wave functions has no physical justification in that region. (This is an example of a conformal anomaly in the sense of Ref. 3.)

The definition of Israel's υ vacuum can be motivated as follows. To treat sensibly the local physics at the point of intersection of the past and future horizons, $\mathcal{H}^\pm$, one must consider a full space-time neighborhood around that point, and this brings in regions not covered by the exterior Schwarzschild coordinates. In the Kruskal analytic extension one can construct basis functions which on the light cone of the point have the analytic properties of positive-frequency plane waves. These are of the forms

$$\begin{aligned} w_j &= (2\sinh 4\pi M p)^{-1/2}(e^{2\pi M p}u_j + e^{-2\pi M p}v_j^*), \\ \bar{w}_j &= (2\sinh 4\pi M p)^{-1/2}(e^{-2\pi M p}u_j^* + e^{2\pi M p}v_j), \end{aligned} \tag{6.7}$$

where v_j are functions analogous to u_j on the second sheet of the manifold. (See Refs. 5–7 for details.) Substituting w_j and $\bar{w}_j$ for u_j in the general formula (6.2a) and simplifying, one obtains (for points in the original region)

$$\begin{aligned} T_{\mu\nu}^{(\upsilon)} &= \sum_j T_{\mu\nu}[w_j, w_j^*] + \sum_j T_{\mu\nu}[\bar{w}_j, \bar{w}_j^*] \\ &= \sum_j T_{\mu\nu}^{(\upsilon)}[u_j, u_j^*], \end{aligned} \tag{6.8a}$$

where

$$T_{\mu\nu}^{(\upsilon)}[u_j,u_j^*] = \coth(4\pi M p)\, T_{\mu\nu}^{(\eta)}[u_j,u_j^*]. \tag{6.8b}$$

Here $T_{\mu\nu}^{(\eta)}[u_j,u_j^*]$ is the expression appropriate to the previous case, formed from the functions (6.3) according to Eq. (6.2b). [In equations such as (6.8a), the sum over j stands for an integral over p and sums over l, m, and the direction of the arrow.]

Unruh's ξ definition of the vacuum is based on the idea that "positive frequency" ought to be defined on the initial surfaces, $\mathcal{H}^-$ and $\mathcal{I}^-$. Therefore, the construction (6.7) is applied only to the waves $\vec{u}$, coming from $\mathcal{H}^-$, and not to the wave $\overleftarrow{u}$, which come from $\mathcal{I}^-$. The result for the stress tensor is

$$T_{\mu\nu}^{(\xi)} = \int_0^\infty dp \sum_{l,m} \{T_{\mu\nu}^{(\eta)}[\overleftarrow{u},\overleftarrow{u}^*] + \coth(4\pi M p)\, T_{\mu\nu}^{(\eta)}[\vec{u},\vec{u}^*]\}. \tag{6.9}$$

The ξ vacuum is the only one of the three whose definition depends essentially on the use of incoming basis modes. The outgoing basis would yield a state in which radiation converges onto the black hole instead of flowing away from it. This time-reversed ξ vacuum must be distinguished from the state obtained by interchanging in the definition of $|\xi\rangle$ the roles of the incoming $\vec{u}_j$ and $\overleftarrow{u}_j$, whose stress tensor is

$$\int_0^\infty dp \sum_{l,m}\{\coth(4\pi M p)\, T_{\mu\nu}^{(\eta)}[\overleftarrow{u},\overleftarrow{u}^*] + T_{\mu\nu}^{(\eta)}[\vec{u},\vec{u}^*]\}. \tag{6.10}$$

To understand the difference among these states, note that a thermal distribution of particles outside the black hole consists, roughly speaking, of three classes of particles: those coming from the direction of the hole, those falling into the hole, and those which will miss the hole entirely because they are moving tangentially. The ξ vacuum contains only the first class, the time-reversed ξ vacuum contains only the second, and the state with the stress (6.10) contains both the second and the third. The υ vacuum contains all three classes, and the η vacuum contains no particles at all at infinity.

We now concentrate our attention on the pairwise differences among the expectation values (6.2), (6.8), and (6.9):

$$T_\mu^{(\xi)\nu} - T_\mu^{(\eta)\nu} = 2\int_0^\infty \frac{dp}{e^{8\pi Mp}-1} \sum_{l,m} T_\mu^\nu[\vec{u},\vec{u}^*], \qquad (6.11)$$

$$T_\mu^{(\upsilon)\nu} - T_\mu^{(\xi)\nu} = 2\int_0^\infty \frac{dp}{e^{8\pi Mp}-1} \sum_{l,m} T_\mu^\nu[\overleftarrow{u},\overleftarrow{u}^*], \qquad (6.12)$$

$$T_\mu^{(\upsilon)\nu} - T_\mu^{(\eta)\nu} = 2\int_0^\infty \frac{dp}{e^{8\pi Mp}-1} \sum_{l,m} \{T_\mu^\nu[\vec{u},\vec{u}^*] + T_\mu^\nu[\overleftarrow{u},\overleftarrow{u}^*]\}. \qquad (6.13)$$

We have dropped the superscript (η) from $T_\mu^\nu[u,u^*]$, since there is no longer a chance of confusion. These integrals are convergent. Since no regularization or renormalization is required in defining them, they retain the tracelessness of their integrands [see Eq. (6.2b)]. (This is why T_α^α is the same for all quantum states.)

Let us evaluate $T_\mu^\nu[u,u^*]$ and perform the sum over m. We substitute Eqs. (6.3) into Eq. (6.2b) and obtain, after a long but routine calculation,

$$\sum_m T_\mu^\nu[u,u^*] = F_1\begin{bmatrix}-1&0&0&0\\0&\frac{1}{3}&0&0\\0&0&\frac{1}{3}&0\\0&0&0&\frac{1}{3}\end{bmatrix} + F_2\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + F_3\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \qquad (6.14)$$

$$F_1 = \frac{2l+1}{32\pi^2 pr^2}\left\{\frac{1}{r^2}[3l(l+1)+1]|R|^2 - \tfrac{1}{2}\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2|R|^2}{dr^{*2}}\right\}, \qquad (6.15)$$

$$F_2 = \frac{2l+1}{96\pi^2 pr^2}\left\{\left[6p^2\left(1-\frac{2M}{r}\right)^{-1} - \frac{9l(l+1)}{r^2} - \frac{2}{r^2}\left(1+\frac{3M}{r}\right)\right]|R|^2 - \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1}\left(1-\frac{3M}{r}\right)\frac{d|R|^2}{dr^*} + 2\left(1-\frac{2M}{r}\right)^{-1}\frac{d^2|R|^2}{dr^{*2}}\right\}, \qquad (6.16)$$

$$F_3 = -i\,\frac{2l+1}{32\pi^2 r^2}\left(1-\frac{2M}{r}\right)^{-1}\left(R^*\frac{dR}{dr^*} - R\frac{dR^*}{dr^*}\right). \qquad (6.17)$$

[The R's above are those of Eqs. (6.4) and (6.5).] If the tensor $T_\mu^{(\xi)\nu} - T_\mu^{(\eta)\nu}$, for example, is to be decomposed as $T_\mu^{(2)\nu} + T_\mu^{(3)\nu} + T_\mu^{(4)\nu}$, the terms having the forms (2.8), then one will have

$$K_{\xi-\eta} = K_\xi = M^2 r^2\left(1-\frac{2M}{r}\right)2\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_{l=0}^\infty \vec{F}_3(p,l), \qquad (6.18)$$

$$\Theta_{\xi-\eta} = 2\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_{l=0}^\infty \tfrac{1}{3}\vec{F}_1(p,l). \qquad (6.19)$$

However, we may not identify $Q_{\xi-\eta}$ with the corresponding integral over F_2+F_3, since the F_1 term is not necessarily of the form $T_\mu^{(3)\nu}$. The F_1 term and the F_2 term in Eq. (6.14) need not satisfy $\nabla_\nu T_\mu^\nu = 0$ separately, although their sum does.

The F_3 term is the easiest to evaluate. The Wronskian in Eq. (6.17) is a constant, so it can be calculated for all r from the asymptotic expressions (6.5). One obtains

$$T_{r^*}^{(\xi)t} - T_{r^*}^{(\eta)t} = -[T_{r^*}^{(\upsilon)t} - T_{r^*}^{(\xi)t}] = T_{r^*}^{(\xi)t} = \frac{1}{8\pi^2 r^2}\left(1-\frac{2M}{r}\right)^{-1}\int_0^\infty \frac{p\,dp}{e^{8\pi Mp}-1}\sum_{l=0}^\infty (2l+1)|B_l(p)|^2. \qquad (6.20)$$

So far, this formula is exact. DeWitt[28] has estimated the integral by a geometrical optics approximation (see below), obtaining

$$K_\xi = \tfrac{9}{40}(8^4\pi^2)^{-1}. \qquad (6.21)$$

However, the most important point is the dependence of the expression (6.20) on r, which is valid everywhere from $2M$ to ∞. Returning to Eqs. (2.8) for the case $T_\mu^{(\xi)\nu}$, we see that as $r\to 2M$ the $T_\mu^{(2)\nu}$ contributions dominate in $T_t^{(\xi)t}$ and $T_r^{(\xi)r}$. Thus there is a flux of negative energy over the horizon, as in the two-dimensional model; this is necessary to satisfy the conservation law and simultaneously keep the stress tensor physically finite on the horizon.

We now present some preliminary, quick approximations to the other parts of the difference tensors. We shall replace the radial functions in Eqs. (6.15) and (6.16) by their asymptotic approximations, Eqs. (6.5). Furthermore, we make the geometrical optics approximation: In each of the one-dimensional partial-wave scattering problems [Eq. (6.4)], a wave is totally transmitted if p^2 is greater than the maximum value of the effective potential

$$V_l(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right], \qquad (6.22)$$

but it is totally reflected if p^2 is less than that value. For each l, $V_l(r)$ has a peak very near $r=3M$ (see Fig. 1), and the peak value of $V_l(r)$ is thus very close to $(l+\frac{1}{2})^2/27M^2$. Thus we take $|B_l(p)|=1$ for

$$l+\tfrac{1}{2}<\sqrt{27}\,Mp\ , \qquad (6.23)$$

and $|B_l(p)|=0$ for $l+\frac{1}{2}>\sqrt{27}\,Mp$.

We can now estimate $T_\mu^{(\xi)\,\nu}-T_\mu^{(\eta)\,\nu}$ near infinity and $T_\mu^{(\upsilon)\,\nu}-T_\mu^{(\xi)\,\nu}$ near the horizon, since only transmitted waves contribute to these quantities:

$$\left.\begin{matrix} 2\displaystyle\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_l \tfrac{1}{3}\overrightarrow{\mathrm{F}}_1 \quad (r\approx\infty) \\ 2\displaystyle\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_l \tfrac{1}{3}\overleftarrow{\mathrm{F}}_1 \quad (r\approx 2M)\end{matrix}\right\}\sim \tfrac{243}{160}\,(8^4\pi^2)^{-1}\frac{1}{r^4} \qquad (6.24\mathrm{a})$$

$$\left.\begin{matrix} 2\displaystyle\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_l \overrightarrow{\mathrm{F}}_2 \quad (r\approx\infty) \\ 2\displaystyle\int_0^\infty \frac{dp}{e^{8\pi Mp}-1}\sum_l \overleftarrow{\mathrm{F}}_2 \quad (r\approx 2M)\end{matrix}\right\}\sim \tfrac{9}{40}\,(8^4\pi^2)^{-1}\frac{1}{M^2r^2}\times\left(1-\frac{2M}{r}\right)^{-1}. \qquad (6.24\mathrm{b})$$

Here we have used

$$\sum_{l=0}^{n}(l+\tfrac{1}{2})\approx\tfrac{1}{2}n^2\ ,$$
$$\sum_{l=0}^{n} l(l+1)(l+\tfrac{1}{2})\approx\sum_{l=0}^{\infty}(l+\tfrac{1}{2})^3\approx\tfrac{1}{4}n^4\ , \qquad (6.25)$$

and the Planck integral which appears in Eq. (5.9). Equation (6.24a) gives the leading asymptotic behavior of the $\theta\theta$ component of the tensor, and Eq. (6.24b) that of the rr component. Since we are con-

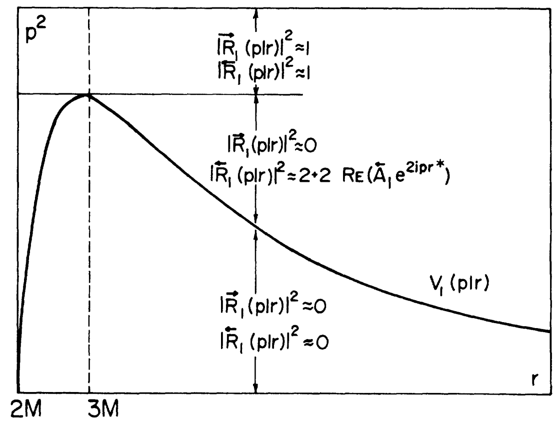

FIG. 1. The effective potential for the partial wave $l=1$ and the division of the (r,p^2) plane caused by it. The nature of the mode functions near a point $r>3M$ is indicated.

fident that $T_\mu^{(\upsilon)\,\nu}$ is relatively small near the horizon and that $T_\mu^{(\eta)\,\nu}$ is relatively small at large r, the results (6.24) are in complete agreement with all the statements about $T_\mu^{(\xi)\,\nu}$ summarized in the middle row of Table I. Note also that the coefficient in Eq. (6.24b) is the same as the one in Eq. (6.21) for the flux, as it should be in order to reproduce the outward massless radiation at infinity and the negative flux over the horizon.

To calculate $T_\mu^{(\xi)\,\nu}-T_\mu^{(\eta)\,\nu}$ near the horizon and $T_\mu^{(\upsilon)\,\nu}-T_\mu^{(\xi)\,\nu}$ near infinity, we have to deal with incident and reflected waves, since the point r of interest is on the side of the potential peak on which the relevant mode solutions are incident. The reflection coefficients satisfy

$$|\overrightarrow{A}_l(p)|^2=|\overleftarrow{A}_l(p)|^2=1-|B_l(p)|^2\ . \qquad (6.26)$$

For a fixed l we must divide the energies p into three intervals (see Fig. 1). Unless

$$p^2>V_l(r)\approx\frac{1}{r^2}\left(1-\frac{2M}{r}\right)(l+\tfrac{1}{2})^2\ , \qquad (6.27)$$

r is in a classically forbidden region where $R_l(p|r)$ is exponentially decaying, and hence we take $R_l(p|r)=0$ in the geometrical optics approximation. If

$$V_l(r)<p^2<(l+\tfrac{1}{2})^2/27M^2\ ,$$

we can take the wave to be completely reflected ($|A|=1$). For larger p the wave is completely transmitted as before ($|A|=0$). The contribution of modes in the third interval is the same as in the previous calculation, while in the second interval we get contributions of the same kind from both the incident and reflected waves, plus a rapidly oscillating interference term which can be neglected when $r^*\to\pm\infty$. Thus it is more convenient to calculate the *sum* of the present quantities and those calculated previously—i.e., to calculate $T_\mu^{(\nu)\nu} - T_\mu^{(\eta)\nu}$, from which $T_\mu^{(\xi)\nu} - T_\mu^{(\eta)\nu}$ near the horizon and $T_\mu^{(\nu)\nu} - T_\mu^{(\xi)\nu}$ near infinity can be obtained by subtracting the quantities (6.24). The summation and integration required (in the geometrical optics approximation) are now identical with those involved in Eqs. (6.24), except for a factor of 2 and the change of the lower limit on p (equivalently, the upper limit on l for a fixed p) from Eq. (6.23) to Eq. (6.27). The results are

$$2\int_0^\infty \frac{dp}{e^{8\pi M p}-1} \sum_l \tfrac{1}{3}(\vec{\mathrm{F}}_1 + \overleftarrow{\mathrm{F}}_1) \sim \tfrac{1}{240}(8^4\pi^2)^{-1}\frac{1}{M^4}\left(1-\frac{2M}{r}\right)^{-2},$$

$$2\int_0^\infty \frac{dp}{e^{8\pi M p}-1} \sum_l (\vec{\mathrm{F}}_2 + \overleftarrow{\mathrm{F}}_2) \sim \tfrac{1}{240}(8^4\pi^2)^{-1}\frac{1}{M^4}\left(1-\frac{2M}{r}\right)^{-2}.$$

However, these expressions are not correct because they do not yield the isotropic equilibrium stress of Eq. (5.9). The crudest geometrical optics approximation is not adequate at this point, and it is necessary to turn to the WKB approximation of next higher order. That is, we replace each wave $e^{\pm ipr^*}$ in Eqs. (6.5) by

$$[1+V_l(r)/p^2]^{-1/4}e^{\pm S_l(r^*)},$$

$$\frac{dS_l}{dr^*} = p[1+V_l(r)/p^2]^{1/2}.$$

However, we continue to use the geometrical optics ansatz for the A and B coefficients. With this approximation one obtains, as $r\to\infty$ or $r\to 2M$,

$$2\int_0^\infty \frac{dp}{e^{8\pi M p}-1} \sum_l \tfrac{1}{3}(\vec{\mathrm{F}}_1 + \overleftarrow{\mathrm{F}}_1) \sim \tfrac{1}{90}(8^4\pi^2)^{-1}\frac{1}{M^4}\left(1-\frac{2M}{r}\right)^{-2}, \tag{6.28a}$$

$$2\int_0^\infty \frac{dp}{e^{8\pi M p}-1} \sum_l (\vec{\mathrm{F}}_2 + \overleftarrow{\mathrm{F}}_2) \sim O\left(\frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\right). \tag{6.28b}$$

This is in complete agreement with our expectations regarding $T_\mu^{(\nu)\nu}$ near infinity and $T_\mu^{(\eta)\nu}$ near the horizon (see Sec. V and Table I). The disagreement with the simpler approximation can be traced to the nonuniformity of the limit

$$V_l(r)\to 0 \text{ as } r^*\to\pm\infty$$

for l and p near the boundary of the domain (6.27). When the WKB approximation is used for the integrals over the domain (6.23), the integrand can simply be expanded in powers of V_l and the result (6.24b) is recovered to the lowest asymptotic order.

We suspect that

$$T_\mu^{(\nu)\nu} - T_\mu^{(\eta)\nu} = \tfrac{1}{30}(8^4\pi^2)^{-1}\frac{1}{M^4}\left(1-\frac{2M}{r}\right)^{-2}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3}\end{bmatrix} \tag{6.29}$$

precisely. This [see Eqs. (5.12) and (5.13)] is the stress tensor of a classical gas in thermal equilibrium at temperature $kT=(8\pi M)^{-1}$ in a static space-time with $g_{tt}=1-2M/r$. Also, this is the only *isotropic* tensor with all the required properties.

Since the quantity (6.28a) dominates those in Eqs. (6.24) and (6.28b) in the asymptotic limits, the leading terms in $T_\mu^{(\xi)\nu} - T_\mu^{(\eta)\nu}$ near the horizon and $T_\mu^{(\nu)\nu} - T_\mu^{(\xi)\nu}$ near infinity are also given by Eq. (6.28a).

It is clear that the approximations used here are rather crude. In particular, they do not accurately represent the contribution to the integrals from small values of p, which are significant because of the exponential cutoff at large p. Our intention here is only to extract the qualitative asymptotic behavior of the stress tensors. However, we want to emphasize that the quantities (6.11)–(6.13) are well defined and finite and could be calculated accurately, for all values of r, by a judicious combination of analytical and numerical techniques. We hope that this will be done reasonably soon. Such a calculation is technically much easier than calculating a regularized $\langle T_\mu^{\nu}\rangle$ for a single state. (In the latter case low-order WKB methods appear to be useless, because the errors in the divergent terms swamp the finite terms one is trying to isolate.)

Anticipating, therefore, that the differences between the expectation values will be available before the expectation values themselves, we suggest the following method of constructing a reasonable guess for the expectation values. Assume that near $r=2M$,

$$T_\theta^{(\nu)\theta} - \tfrac{1}{4}T_\alpha^{\alpha} = A(r-2M) + B(r-2M)^2,$$

$$T_r^{(\nu)r} - \tfrac{1}{4}T_\alpha^{\alpha} = E(r-2M) + F(r-2M)^2,$$

and that for large r,

$$T_\theta^{(\eta)\theta} - \tfrac{1}{4}T_\alpha^{\alpha} = Cr^{-6} + Dr^{-7},$$

$$T_r^{(\eta)r} - \tfrac{1}{4}T_\alpha^{\alpha} = Gr^{-6} + Hr^{-7}.$$

From Eqs. (2.7b) and (2.8) one can find E, F, G, and H in terms of A, B, C, and D. Assume also that the regions of validity of these approximations overlap at some point r_0 (probably slightly less than $3M$). Since $T_\mu^{(\upsilon)\nu} - T_\mu^{(\eta)\nu}$ is assumed known, one can obtain a system of equations determining A, B, C, and D by requiring that the two expressions for $T_\theta^{(\upsilon)\theta}$, $T_r^{(\upsilon)r}$, and their derivatives match at r_0. (We have calculated the determinant of this system and it does not vanish.) This construction provides expressions for $T_\mu^{(\upsilon)\nu}$, $T_\mu^{(\eta)\nu}$, and (since the other difference tensors are also assumed known) $T_\mu^{(\xi)\nu}$. It will be very interesting to see how closely the $T_\theta^{(\xi)\theta}$ thus obtained satisfies the consistency relation (5.4).

VII. LUMINOSITY CALCULATIONS AND THE TANGENTIAL PRESSURE FUNCTION

In this section we confront the formula (5.4) with calculated values of the Hawking flux, K_ξ, for massless free fields of spins 0, $\frac{1}{2}$, and 1. We find that consistency with the trace anomaly coefficients listed below Eq. (5.1) and with the asymptotic properties of $\Theta_\xi(r)$ established in Secs. V and VI requires the function $\Theta_\xi(r)$ to have a rather definite general shape. Furthermore, this result depends on spin in a physically understandable way.

Page[29] has numerically calculated the neutrino, photon, and graviton luminosities [see Eq. (5.3)] of a Schwarzschild black hole. He finds that the luminosity decreases rapidly with spin. The values of K_ξ implied by his results for a neutrino field[30] and the electromagnetic field are given in Table II, along with the anomaly term in Eq. (5.4) and the values of $G_\xi(\infty)$ thereby implied. (We do not consider the quantized gravitational field, for which the stress tensor and the anomaly are gauge-dependent.) Page does not provide a value of L for the neutral scalar field, so we shall use DeWitt's geometrical optics estimate, Eq. (6.21). The latter is directly related (see remarks in Ref. 29) to the cross section for scattering of classical particles by a hard sphere of radius $\sqrt{27}\,M$ [cf. discussion preceding our Eq. (5.5)]. The geometrical optics figures for other fields differ only by the necessary factors to account for helicity states and statistics ($\frac{7}{4}$ for neutrinos, 2 for photons), and they are included in Table II.

The principal feature of these numbers is that the anomaly coefficient increases with the spin of the field, but the magnitude of the Hawking flux decreases. As a consequence, although $G_\xi(\infty)$ may be positive for the scalar field, it is negative for neutrinos, and even more so for photons. Referring to the definition of G_ξ, Eq. (2.7b), we see that a positive $\Theta_\xi(r)$ contributes positively to $G_\xi(\infty)$ if $r > 3M$, but negatively if $2M < r < 3M$. We know [Eq. (5.5)] that $\Theta_\xi(r)$ eventually approaches λr^{-4}, where λ is a positive constant; thus there is certainly a positive contribution to $G_\xi(\infty)$ from large r. If $\Theta_\xi(r) = \lambda r^{-4}$ everywhere, then $G_\xi(\infty)$ is exactly zero. However [see Eqs. (6.12) and (6.24a)], we have concluded that Θ_ξ approaches a *negative* constant as $r \to 2M$; this modification at the smallest values of r can only make $G_\xi(\infty)$ more positive. How, then, can $G_\xi(\infty)$ be negative? There are only two possibilities: Either Θ_ξ is negative over substantial intervals outside $r = 3M$, or Θ_ξ has a high positive peak inside $r = 3M$ before plummeting to a negative value at $r = 2M$ (Fig. 2).

The second alternative is more plausible. In the gently curved exterior region, where quantum particle-creation effects are locally negligible, one would not expect a transition between a beam of ordinary thermal radiation and a highly correlated state with negative expectation values of energy density and pressure. But a positive enhancement of $T_\theta^{(\xi)\theta}(r)$ just inside $r = 3M$ is what one should expect from the behavior of massless particle orbits there. A particle with considerable angular momentum takes a long time to crawl over the centrifugal potential barrier, if it succeeds in doing so at all. This is the quantum-mechanical manifestation of the fact that many classical orbits are tightly wound spirals as they

TABLE II. Trace anomalies, Hawking fluxes, and predicted integrated tangential pressures according to Eq. (5.4).

	Scalar	Neutrino	Electromagnetic
Anomaly, $10^5\pi^2 \times 3\beta/640$	7.81	23.4	93.7
Flux, $10^5\pi^2 K_\xi = 10^5\pi^2 \times M^2L/4\pi$			
Geometrical optics	5.50	9.62	11.00
Numerical calculation		6.42	2.65
$10^5\pi^2 M^2 G_\xi(\infty)$ [Eq. (2.7b)]			
Geometrical optics	+ 3.17	− 4.16	−71.7
Numerical calculation		−10.6	−88.4

cross over (or approach) $r=3M$. (See Ref. 26, especially Fig. 1.) Since such particles are moving primarily tangentially, they make a large contribution to T^{θ}_{θ}. Now, according to Ref. 29, the Hawking flux of a field with spin is small primarily because the lowest values of orbital angular momentum are cut out of the spectrum. This means that the spiral orbits (classes III and IV in Ref. 26, and also orbits confined inside $r=3M$) are relatively more important for higher spin, compared to orbits which are more nearly radial (class V). This effect accounts qualitatively for the phenomenon displayed in Table II and Fig. 2.

The picture suggested is thus the following. A description of the Hawking radiation in traditional particle language is physically adequate all the way back to somewhere inside the centrifugal barrier. At large r the stress tensor is like that of radiation from a source of radius $\sqrt{27}\,M$ in flat space, and at $r\approx 3M$ it is modified by purely classical geometrical effects, which are more pronounced for fields with spin. At r close to $2M$, quantum correlations result in violations of the classical energy conditions; in particular, at the future horizon there must be a net negative inward flux to satisfy the law of conservation of energy in the static space-time (cf. Ref. 17). (These conclusions do not hinge on the anomaly ratio $1:3:12$; we need only assume that the C^2 anomaly does not *decrease* with spin as fast as the flux does.)

VIII. CONCLUSION

The present work sheds light on two subjects, conformal anomalies and the question of vacuum energy in space-times with horizons.

With regard to the first, we have been able to piece together previously unrelated and fragmentary calculations into a coherent picture. We have found evidence of consistency of different regularization methods and have obtained values for the anomaly coefficients of the conformally coupled scalar field.

Second, we have used this information about the trace of the renormalized stress tensor to show how the Hawking radiation effect for black holes is consistent with the existence of a covariantly conserved stress-tensor operator whose expectation value in a physically nonsingular state is reasonably behaved on the horizon. The forms of the stress tensors for various states in regions of time independence have been rather narrowly circumscribed, without detailed calculations. (This, of course, was the original aim of the research.) The qualitative results have been shown to be physically reasonable.

We have also presented a means by which one could approximate the vacuum expectation value of the stress tensor in the states discussed. The method circumvents the problem of calculating the divergences and allows one to estimate, using standard approximation techniques, the finite, physically interesting terms.

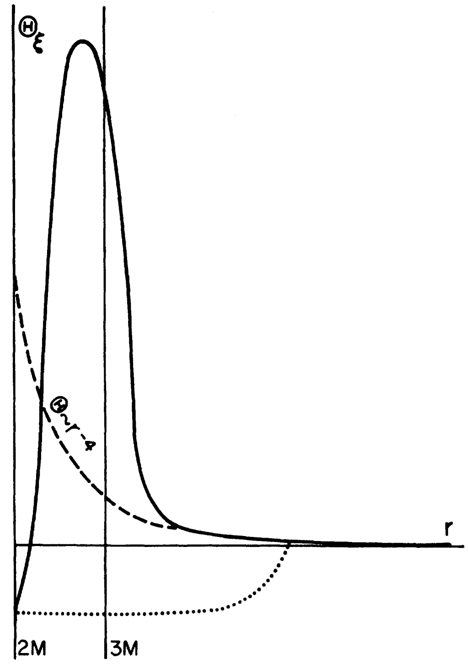

FIG. 2. Qualitative behavior of $\Theta_{\xi}(r)$ for a field with spin. The peak is attributed to particles spiraling slowly outward across the angular momentum barrier, or else up its inner slope and back down again. The alternative indicated by the dotted line is rejected as unphysical.

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APPENDIX

A tensor is finite on the horizon in the local, physical sense if and only if its components are finite with respect to a coordinate system which is regular there, such as the Kruskal null coordinates,

$$V=4Me^{v/4M},\quad U=-4Me^{-u/4M},$$

which satisfy

$$UV=-8Me^{r/2M}(r-2M)$$

and hence

$$T_{VV}=16M^2V^{-2}\,T_{vv}\,,$$

$$T_{UV}=-16M^2(UV)^{-1}\,T_{uv}=2Me^{-r/2M}(r-2M)^{-1}T_{uv}\,,$$

$$T_{UU}=16M^2U^{-2}\,T_{uu}=\tfrac{1}{4}\,e^{-r/M}V^2(r-2M)^{-2}\,T_{uu}\,.$$

It follows that $T_{\mu\nu}$ is physically finite on the future horizon ($U=0$) if and only if, as $r\to 2M$,

(1) $|T^{\theta}_{\theta}|<\infty$,

(2) $|T_{vv}|=\frac{1}{4}|T_{tt}+T_{r^*r^*}+2T_{tr^*}|<\infty$,

(3) $(r-2M)^{-1}|T_{uv}|=\frac{1}{4}|T^t_t+T^{r^*}_{r^*}|<\infty$,

(4) $(r-2M)^{-2}|T_{uu}|<\infty$.

The tensors (2.8a), (2.8b), and (2.8c) are constructed to satisfy these conditions, if $\Theta(2M)$ is finite.

Note that T_{UU} is of order V^2, and hence [see Eqs. (1.2) and (1.3)] of order $r-2M$ if t is fixed. Similarly, T_{VV} is $O(U^2)$ if $T_{\mu\nu}$ is finite on $\mathcal{H}^-$. Furthermore, since $g_{uv}=-\frac{1}{2}(1-2M/r)$, we find with the aid of Eqs. (2.4) and (2.6) that

$$T_{UV}-\tfrac{1}{4}\,T^{\alpha}_{\alpha}\,g_{UV}=Mr^{-1}e^{-r/2M}\Theta\,.$$

It follows that if $T_{\mu\nu}$ is finite on both $\mathcal{H}^+$ and $\mathcal{H}^-$ and if $\Theta(r)$ is of order $r-2M$, then all components of $T^{\nu}_{\mu}-\frac{1}{4}T^{\alpha}_{\alpha}g^{\nu}_{\mu}$ vanish to that order as one approaches the point of intersection of $\mathcal{H}^+$ and $\mathcal{H}^-$ along a surface of constant t. This is true of $T^{r^*}_{r^*}-\frac{1}{4}T^{\alpha}_{\alpha}$, etc., as well as the Kruskal components, because no infinite red shift is involved in approaching the horizon with t fixed. In the two-dimensional model of Sec. III, one has $T_{UV}-\frac{1}{2}T^{\alpha}_{\alpha}g_{UV}=0$, and all other statements of this appendix apply unchanged.

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[27]We follow DeWitt's notation (Ref. 1) except for the spherical harmonics.

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[30]That is, for two types of neutrinos, either ν_e and $\bar{\nu}_e$ or ν_μ and $\bar{\nu}_\mu$.